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# Adding Involution to Residuated Structures

**Abstract.** Two constructions for adding an involution operator to residuated ordered monoids are investigated. One preserves integrality and the mingle axiom $x^2 \leq x$ but fails to preserve the contraction property $x \leq x^2$. The other has the opposite preservation properties. Both constructions preserve commutativity as well as existent nonempty meets and joins and self-dual order properties. Used in conjunction with either construction, a result of R.T. Brady can be seen to show that the equational theory of commutative distributive residuated lattices (without involution) is decidable, settling a question implicitly posed by P. Jipsen and C. Tsinakis. The corresponding logical result is the (theorem-) decidability of the *negation-free* axioms and rules of the logic **RW**, formulated with fusion *and* the Ackermann constant $t$. This completes a result of S. Giambrone whose proof relied on the absence of $t$.

*Keywords*: Residuation, residuated lattice, involution, negation, contraction, mingle, expansion, **RW**

## 1. Introduction

We develop here two simple constructions for extending any residuated ordered monoid to a bounded one with an involution, in such a way that if the original structure is a residuated lattice then so is the containing involutive structure. Both constructions preserve finiteness, commutativity, existent meets and joins of nonempty subsets, distributivity and the semi-contraction and semi-expansion properties $x^2 \leq x^3$ and $x^3 \leq x^2$. The first construction, which also preserves integrality, greatest elements (if they exist) and the mingle axiom $x^2 \leq x$, fails to preserve the contraction axiom $x \leq x^2$, while the second has just the reverse properties.

The first construction generalizes one that appears in A. Wroński's work on BCK-algebras. The second generalizes a construction from relevance logic that is due to R.K. Meyer.

Either of the constructions given here can be used to show that the relevant logic **RW** is a strongly conservative extension of its positive axioms and rules. This allows us to deduce from a theorem of R.T. Brady that the variety CDRL of commutative distributive residuated lattices has a decidable equational theory, filling a gap in the table of decidability results displayed in

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[18]. This contrasts with the fact, obtained in [14], that the *quasi*-equational theories of a range of varieties including CDRL are undecidable, as well as with some undecidability results of A. Urquhart [37] for the theorems of logics with a weak form of contraction.

## 2. Residuated Pomonoids

A *pomonoid* is a structure $\langle A; \cdot, t; \leq \rangle$ such that $\langle A; \cdot, t \rangle$ is a monoid, $\leq$ is a partial order of $A$ and for all $a, b, c, d \in A$, if $a \leq b$ and $c \leq d$ then $a \cdot c \leq b \cdot d$.

Given a monoid $\langle A; \cdot, t \rangle$ and a partial order $\leq$ of $A$, we say that the structure $\langle A; \cdot, t; \leq \rangle$ is *right residuated* provided that whenever $a, c \in A$, then the set

$$\{b \in A : a \cdot b \leq c\}$$

has a *greatest* element. This element is usually denoted by $a \backslash c$ and the binary operation $\backslash$ on $A$ is called *right residuation*. Dually, $\langle A; \cdot, t; \leq \rangle$ is *left residuated* if for any $a, c \in A$, the set $\{b \in A : b \cdot a \leq c\}$ has a greatest element, which we then denote by $c / a$. Let us call $\langle A; \cdot, t; \leq \rangle$ *residuated* if it is both right and left residuated. In this case, it is necessarily a pomonoid.

Accordingly, from now on, by a *right* [or *left*] *residuated pomonoid* — or briefly, a *RRP* [or a *LRP*] — we mean a structure $\mathbf{A} = \langle A; \cdot, \backslash$ [or $/$], $t; \leq \rangle$ such that $\langle A; \cdot, t; \leq \rangle$ is a pomonoid and for all $a, b, c \in A$,

$$a \cdot b \leq c \text{ iff } b \leq a \backslash c \quad (\textit{right residuation})$$

$$[\text{or } b \cdot a \leq c \text{ iff } b \leq c / a \quad (\textit{left residuation})].$$

And by a *residuated pomonoid* — or briefly, a *RP* — we mean a structure $\mathbf{A} = \langle A; \cdot, \backslash, /, t; \leq \rangle$ whose $\{/\}$-free and $\{\backslash\}$-free reducts are a RRP and a LRP, respectively.

For any elements $a, b, c$ of a RP, the statements

$a \leq b$ iff $t \leq a \backslash b$ iff $t \leq b / a$;

$t \backslash a = a = a / t$;

$t \leq a$ iff $a \backslash a \leq a$ iff $a / a \leq a$;

$a \backslash (b \backslash c) = (b \cdot a) \backslash c$ and $(c / b) / a = c / (a \cdot b)$;

$(b \backslash a) / c = b \backslash (a / c)$;

$a \leq b \backslash (b \cdot a)$ and $a \leq (a \cdot b) / b$;

$a \backslash b \leq (c \backslash a) \backslash (c \backslash b)$ and $b / a \leq (b / c) / (a / c)$;

$a / b \leq (c / a) \backslash (c / b)$ and $b \backslash a \leq (b \backslash c) / (a \backslash c)$;

$a \leq (b / a) \backslash b$ and $a \leq b / (a \backslash b)$

are true and the binary operation $\backslash$ is isotone in its second argument and antitone in its first, while the reverse is true for $/$. A RRP satisfies all of the above properties (and their parts) that do not mention $/$.

In fact, every right residuated pomonoid is a substructure of the right residuation reduct of a residuated pomonoid: see, e.g., [13]. By symmetry we get a similar statement for LRPs. The symmetry just mentioned can be used to get analogues for LRPs of any statements about RRPs made in the sequel.

A RP or RRP is called *commutative* if its monoid operation $\cdot$ is commutative. In a commutative RP we denote the common value of $a \backslash b$ and $b / a$ by $a \to b$. We regard a commutative RP as a structure with signature $\langle \cdot, \to, t; \leq \rangle$.

The least and greatest elements of a RP or RRP, if they exist, are usually denoted by $\bot$ and $\top$, respectively. In fact, if $\bot$ exists for a RRP $\mathbf{A}$ then (extending our language temporarily to include $\bot, \top$, for convenience), we find that $\mathbf{A}$ must satisfy $\bot \leq x \backslash \bot$, i.e., $x \cdot \bot \leq \bot$, whence also $x \cdot \bot \approx \bot$. This forces $\mathbf{A}$ to have a greatest element $\top$ also, with $\top \cdot \bot \approx \bot$. Accordingly, let us call a RP or RRP $\mathbf{A}$ *upper bounded* or *bounded* if $\langle A; \leq \rangle$ has a greatest element or both a least and a greatest element.

An upper bounded RP or RRP is called *integral* if $t$ is its greatest element. Let us define $x^0 = t$ and, for nonnegative integers $n$, also $x^{n+1} = x^n \cdot x$. Note that a RP need not satisfy any of the inequalities

$$x^{n+1} \leq x^n \quad (n\text{-}expansion) \tag{1}$$

$$x^n \leq x^{n+1} \quad (n\text{-}contraction). \tag{2}$$

A 0-contractive RP is trivial (since $t \leq a \backslash b$ forces $a \leq b$). 0-expansion is just *integrality*. The law $x^2 \leq x$ of 1-expansion is abbreviated as *expansion*, but it is better known as the *mingle* axiom. It may be recast as $x \backslash y \leq x \backslash (x \backslash y)$. Likewise, 1-contraction $x \leq x^2$ is abbreviated as *contraction* and is equivalent to the law $x \backslash (x \backslash y) \leq x \backslash y$. An integral contractive RP must be commutative — in fact, it is a Brouwerian semilattice.

## 3. Bounds and Rigorous Compactness

We have seen that a bounded (not necessarily integral) RP must satisfy

$$x \cdot \bot \approx \bot \cdot x \approx \bot\,. \tag{3}$$

If $\mathbf{A}$ is an upper bounded RRP then, for all $a \in A$, clearly $a \backslash \top = \top$; if $\mathbf{A}$ is a bounded RP then also $\bot \backslash a = \top$. By symmetry, an upper bounded RP

**A** must satisfy

$$x \backslash \top \approx \top / x \approx \top \tag{4}$$

and if it is bounded, also

$$\bot \backslash x \approx x / \bot \approx \top. \tag{5}$$

Note that if a RRP **A** is upper bounded and not lower bounded, we can extend it by a new least element $\bot$, using the necessary identity $x \cdot \bot \approx \bot$ as well as the special one $\bot \cdot x \approx \bot$ to extend $\cdot$ to the enlarged structure, in which we must now have $a \backslash \bot = \bot$ whenever $a \neq \bot$. If **A** was a RP, the enlarged structure satisfies

$$x \not\approx \bot \;\Rightarrow\; x \backslash \bot \approx \bot / x \approx \bot. \tag{6}$$

Obviously this construction preserves integrality.

Regardless of whether a RRP **A** is upper bounded or bounded, we can extend it by two new extreme elements $\bot, \top$, defining the extension of $\cdot$ for $\bot$ as above and using in addition the definition

$$x \not\approx \bot \;\Rightarrow\; x \cdot \top \approx \top \cdot x \approx \top. \tag{7}$$

In the resulting structure, in addition to the above residuation properties, we also have $\top \backslash a = \bot$ whenever $a \neq \top$ and, in the case of RPs,

$$x \not\approx \top \;\Rightarrow\; \top \backslash x \approx x / \top \approx \bot \tag{8}$$

is satisfied. This construction obviously does not preserve integrality. Both constructions evidently do preserve commutativity, existent nonempty meets and joins, $n$-expansion and $n$-contraction for any $n \geq 1$.

It is convenient to re-state what we have just noted a little more generally.

LEMMA 3.1. *A bounded pomonoid* **A** *with partial right and left residuation operations must satisfy* (4), (5), (6) *and* (8) *if it satisfies* (3) *and* (7).

PROOF. We used nothing but (3) and (7) in proving (4), (5) and (8) but (6) requires fresh proof in the more general setting the present lemma. Note first that $\top \cdot \bot = \bot$, by (3). Let $\bot \neq a \in A$. Of course, this forces $\bot \neq \top$ and we also have $a \cdot \bot = \bot$, by (3). Suppose $\bot \neq b \in A$ and $a \cdot b = \bot$. Then $\top \cdot b = \top \cdot a = \top$, by (7). Using associativity of $\cdot$, we find that

$$\top = \top \cdot b = (\top \cdot a) \cdot b = \top \cdot (a \cdot b) = \top \cdot \bot = \bot,$$

a contradiction. It follows that $a \backslash \bot = \bot$ and, by symmetry, $\bot / a = \bot$. ■

Extending the terminology of R.K. Meyer [22] to the (fully) residuated case, we say that a bounded RP is *rigorously compact* if it has at least two elements and it satisfies (3)–(8). (Meyer's definition applies to RRPs as well as some weaker structures and requires exactly the properties from (3)–(8) that don't mention $/$. In our context this could naturally be called *right rigorous compactness*.) The above discussion can be construed as saying that every RP has a rigorously compact 2-point extension. Not every rigorously compact RP arises as the bounded 2-point extension of another RP, however. As Meyer observes, the $\{\wedge, \vee\}$-expansion of a lattice ordered rigorously compact RRP may be mapped homomorphically onto the $\{\sim\}$-free reduct of the 3-element Sugihara algebra, where $\backslash$ corresponds to the customary $\rightarrow$ of this algebra.

## 4. Residuated Lattices

A *residuated lattice* — briefly, a *RL* — is an algebra $\mathbf{A} = \langle A; \cdot, \backslash, /, \wedge, \vee, t\rangle$ whose $\{\wedge, \vee\}$-reduct is a lattice and whose $\{\cdot, \backslash, /, t; \leq\}$-reduct is a RP (where $\leq$ denotes the induced lattice order). In this case $\mathbf{A}$ is first order definitionally equivalent to the structure $\langle \mathbf{A}; \leq\rangle$, since $x \leq y$ is equationally definable as $x \wedge y \approx x$. A RL is called *distributive* if its lattice reduct is a distributive lattice.

Right and left residuated lattices (RRLs and LRLs) are defined similarly. It is important for most of what follows to note that the bounds $\perp$ and $\top$ are *not* included as constant symbols in the signature of a bounded RRP or RRL or RP or RL as defined here (despite the formal liberties taken in the previous section). The $\{\perp, \top, f\}$-expansion of a bounded RRL, where $f$ is an arbitrary distinguished element about which nothing is postulated, is the same thing as a *full Lambek algebra* in the terminology of [27].

For recent accounts of the algebraic theory of residuated lattices, see [8] and [18].

We shall present in Sections 6 and 9 two constructions for embedding RPs and RLs into richer structures. Readers working with structures that are residuated on one side only can still make some use of these results by combining them with the known facts contained in the next two results. (We shall make no further use of these results, except in our concluding remarks.)

PROPOSITION 4.1. *A right residuated lattice* $\mathbf{A}$ *can be embedded into the* $\{\cdot, \backslash, \wedge, \vee, t\}$*-reduct of a residuated lattice iff* $\mathbf{A}$ *satisfies*

$$(x \vee y) \cdot z \approx (x \cdot z) \vee (y \cdot z)\,. \tag{9}$$

This can be proved by generalizing to the non-integral case an embedding procedure developed by H. Ono and Y. Komori in [30] and noting that the procedure yields not only a RRL (as reported there) but actually a RL. (This was drawn to our attention by C.J. van Alten.) A finite integral RRL that fails to satisfy (9) appears in [38, p. 295] (in a dual notation). The next result is implicit in [26] and [13]:

PROPOSITION 4.2.

(i) *Every RRP is a subreduct (i.e., it is a substructure of the appropriate reduct) of a RL that is a distributive lattice.*

(ii) *Every contractive RRP is a subreduct of a contractive RL.*

*In both cases, if the original RRP is finite or commutative or integral or has any combination of these properties then the containing RL can be chosen to have the same attributes.*

## 5. Involution

A *(cyclic) involutive* RP [or RL] is a structure [or algebra] **A** with a fundamental unary operation $\sim$, whose $\{\sim\}$-free reduct is a RP [or RL], and which satisfies

$$\sim\sim x \approx x \quad \text{and} \quad x \setminus \sim y \approx (\sim x) / y. \tag{10}$$

We call the operation $\sim$ a *(cyclic) involution.* In an involutive RP let us define

$$f = \sim t.$$

Evidently, such a structure must satisfy

$$x \setminus f \approx (\sim x) / t \approx \sim x$$

and, similarly, $f / x \approx \sim x$, as well as

$$x \leq y \text{ iff } t \leq x \setminus y \text{ iff } t \leq x \setminus \sim\sim y \text{ iff } t \leq (\sim x) / \sim y \text{ iff } \sim y \leq \sim x.$$

We call $f$ the *(cyclic) involution constant.*

Note that the involution constant and the involution operation are interdefinable. An RP or RL is the reduct of an involutive RP or RL if and only if it contains an element $f$ for which its $\{f\}$–expansion satisfies $x \setminus f \approx f / x$ and $f / (f / x) \approx x$. The involution may then be defined as $\sim x = x \setminus f$.

The $\{f, \bot, \top\}$-expansions of involutive RLs are called *cyclic Grishin algebras* in [19]. Up to first order definitional equivalence, the involutive integral commutative RPs coincide with the '$L_0$-*algebras*' studied by V.N. Grishin in [17], while, up to term equivalence, C.C. Chang's *MV-algebras* [10] are examples of involutive integral commutative distributive RLs.

The following lemma establishes the connection between $\cdot$ and $\sim$.

LEMMA 5.1. *Every involutive RP satisfies*

$$x \cdot y \;\approx\; \sim (y \setminus \sim x) \;\approx\; \sim ((\sim y) \,/\, x).$$

PROOF. Let $\mathbf{A}$ be an involutive RP and let $a, b \in A$. We have:

$$\sim (a \cdot b) \;=\; (a \cdot b) \setminus f \;=\; b \setminus (a \setminus f) \;=\; b \setminus \sim a$$

so $a \cdot b \;=\; \sim\sim\; (a \cdot b) \;=\; \sim\; (b \setminus \;\sim a)$. The other equation follows by our definition of involution. ■

## 6. Adding Involution (I): Preserving Expansion

We shall now give a simple construction for embedding an arbitrary *upper bounded* RP into a bounded *involutive* RP. (We shall then discuss its antecedents in the literature.) The construction will preserve, among other properties, commutativity, the mingle axiom $x^2 \leq x$ and integrality, but not the contraction axiom $x \leq x^2$. The assumption of upper boundedness is merely a convenience in the proof; it can be dispensed with in view of Section 3, without sacrificing preservation of any of the properties mentioned above.

Let $\mathbf{A} = \langle A; \cdot, \rightarrow, t; \leq \rangle$ be a residuated pomonoid that is upper bounded. Let $A' = \{a' : a \in A\}$ be a disjoint bijective copy of $A$, and let $A^* = A \cup A'$. We extend $\leq$ to a partial order of $A^*$, also denoted by $\leq$, by defining that for any $a, b \in A$,

$$\text{(i)}\;\; a' < b \;\text{ and }\; \text{(ii)}\;\; a' \leq b' \;\text{ iff }\; b \leq a\,.$$

Thus, $\langle A^*; \leq \rangle$ is order isomorphic to the ordinal sum of the dual poset of $\langle A; \leq \rangle$ and $\langle A; \leq \rangle$ itself.

Let $\top$ be the greatest element of $\langle A; \leq \rangle$. We define $\bot = \top'$, and $f = t'$, so $\bot$ is the least element of $\langle A^*; \leq \rangle$. For each $a \in A$, we define $a'' = a$, so that $'$ becomes a total unary operation on $A^*$ and $A^*$ satisfies $x'' \approx x$.

We define an extension to $A^*$ of the monoid operation $\cdot$ of $A$, which we also denote by $\cdot$, as follows: if $a, b \in A$ then

$$a \cdot b' = (b \,/\, a)'\,; \quad b' \cdot a = (a \,\backslash\, b)' \quad \text{and} \quad a' \cdot b' = \bot.$$

In fact, since we hope to turn $A^*$ into an involutive RP, the first two equations in this definition of $\cdot$ are forced on us, because Lemma 5.1 implies that every such structure must satisfy $x \cdot \sim y \approx \sim (y \,/\, x)$ and $(\sim y) \cdot x \approx \sim (x \,\backslash\, y)$.

It follows immediately that $t \cdot a' = (a \,/\, t)' = a' = (t \,\backslash\, a)' = a' \cdot t$ for all $a \in A$, so $t$ is an identity element for $\cdot$ on all of $A^*$.

Note that whenever $a \in A$ then $a \cdot \bot = a \cdot \top' = (\top \,/\, a)' = \top' = \bot$ and, symmetrically, $\bot \cdot a = \bot$. It follows that the associative law $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$ will hold in $A^*$ whenever at least two of $x, y, z$ are elements of $A'$. To prove the full associative law for $A^*$, therefore, the following three calculations suffice. Let $a, b, c \in A$.

$$(a \cdot b) \cdot c' = (c \,/ (a \cdot b))' = ((c \,/\, b) \,/\, a)' = a \cdot (c \,/\, b)' = a \cdot (b \cdot c')\,;$$
$$(a \cdot b') \cdot c = (b \,/\, a)' \cdot c = (c \backslash (b \,/\, a))' = ((c \,\backslash\, b) \,/\, a)' = a \cdot (c \,\backslash\, b)' = a \cdot (b' \cdot c)\,;$$
$$(a' \cdot b) \cdot c = (b \,\backslash\, a)' \cdot c = (c \backslash (b \,\backslash\, a))' = ((b \cdot c) \,\backslash\, a)' = a' \cdot (b \cdot c)\,.$$

It is easy to see that $\langle A^*; \cdot, t; \leq \rangle$ is a pomonoid. Because all elements of $A$ dominate all elements of $A'$ in $A^*$, left and right residuals calculated in $\mathbf{A}$ continue to function as such in $\langle A^*; \cdot, t; \leq \rangle$, so there is scope to extend our use of the symbols $\backslash$ and $/$ unambiguously to $A^*$. We claim that $\langle A^*; \cdot, t; \leq \rangle$ is residuated as follows, where $a, b \in A$:

$$a \,\backslash\, b' = a' \,/\, b = (b \cdot a)'\,; \qquad b' \,\backslash\, a = a \,/\, b' = \top;$$
$$a' \,\backslash\, b' = a \,/\, b\,; \qquad b' \,/\, a' = b \,\backslash\, a.$$

[Again, note that if $\langle A^*; \cdot, {}', t; \leq \rangle$ is to become an involutive RP then all but one of these equations will be forced to hold, by Lemma 5.1 and the definition of involution.] To see that the equations are indeed true, let $a, b \in A$.

To show that $a \,\backslash\, b' = (b \cdot a)'$, let $c \in A^*$. If $c \in A$ then $a \cdot c \not\leq b'$. If $c = d' \in A^*$, where $d \in A$, then $a \cdot d' \leq b'$ iff $(d \,/\, a)' \leq b'$ iff $b \leq d \,/\, a$ iff $b \cdot a \leq d$ iff $d' \leq (b \cdot a)'$, as required.

To show that $b' \,\backslash\, a = \top$, note that $b' \cdot \top = (\top \,\backslash\, b)' \leq a$, because $(\top \,\backslash\, b)' \in A'$. Since $\top$ is the greatest element of $A^*$, we are done.

To show that $a' \,\backslash\, b' = a \,/\, b$, let $c \in A^*$. If $c = d' \in A'$, where $d \in D$, then $a' \cdot d' = \bot \leq b'$. Suppose $c \in A$. Then $a' \cdot c \leq b'$ iff $(c \,\backslash\, a)' \leq b'$ iff $b \leq c \,\backslash\, a$ iff $c \cdot b \leq a$ iff $c \leq a \,/\, b$, so the desired equation follows. The claims about left residuation follow symmetrically.

This shows that $\langle A^*; \cdot, \backslash, /, t; \leq\rangle$ is a bounded RP of which $\mathbf{A}$ is a substructure. The embedding preserves greatest elements but not least elements (if such exist). It clearly also preserves commutativity, integrality, $n$-expansion for all $n$, and $n$-contraction for $n \geq 2$. In particular, mingle is preserved. Trivially, all existing (even infinitary) nonempty meets and joins in $A$ are preserved in $A^*$, as are all self-dual order properties, so if $\mathbf{A}$ was the reduct of a RL $\mathbf{C}$ then the order $\leq$ on $A^*$ is a lattice order and if $\wedge, \vee$ denote its induced meet and join operations then $\mathbf{C}$ is a subalgebra of $\langle A^*; \cdot, \backslash, /, \wedge, \vee, t\rangle$. Moreover, if $\mathbf{C}$ was a distributive or complete RL (or both) then the same is true of $\langle A^*; \cdot, \backslash, /, \wedge, \vee, t\rangle$. Of course if $\mathbf{C}$ was merely semilattice-ordered then $\langle A^*; \leq\rangle$ will not be semilattice-ordered.

Note that the construction does *not* preserve contraction. For example, if $\top > a \in A$ then $a' > \bot = a' \cdot a'$.

The above characterizations of $\backslash$ and $/$ already show that for all $a, b \in A^*$, we have $a \backslash b' = b / a'$. For each $a \in A$, we define $\sim a = a'$. The algebra $\mathbf{A}^* = \langle A^*; \cdot, \backslash, /, \sim, t\rangle$ is thus an involutive RP. Note that it is not rigorously compact. The property

$$f \leq t$$

of $\mathbf{A}^*$ is not a necessary property of (even finite) bounded commutative distributive involutive RLs. [It is true in all involutive RPs that satisfy the mingle axiom: in such RPs we get from $f \cdot f \leq f$ and Lemma 5.1 that $t = \sim f \leq \sim (f \cdot f) = f \backslash \sim f = f \backslash t$, so $f \leq t$.]

We conclude:

THEOREM 6.1.

(i) *Every RP or RL is a subreduct of a bounded involutive one satisfying* $f \leq t$.

(ii) *For each nonnegative integer* $n$*, every* $n$*-expansive RP or RL is a subreduct of a bounded involutive* $n$*-expansive one satisfying* $f \leq t$.

(iii) *If the initial structure in* (i) *or in* (ii) *is finite or complete or commutative or distributive or* $m$*-contractive for some* $m \geq 2$*, or has any combination of these properties then the involutive structure can be chosen to have the same combination of properties.*

The above construction was inspired by one of A. Wroński which was formulated in [41] for 'BCK-algebras with condition (S)'. These algebras are, it turns out, just the $\{\rightarrow, t\}$-reducts of integral commutative RPs. The algebra $\langle A^*; \rightarrow, t\rangle$ constructed as above from the $\{\rightarrow, t\}$-reduct $\mathbf{A}^{\rightarrow}$ of an integral commutative RP $\mathbf{A}$ is called in [41] the *reflection* of $\mathbf{A}^{\rightarrow}$, except that Wroński's notation is dual to that used here. In [31] M. Pałasinski noted that the class of BCK-algebras with condition (S) is closed under reflection.

By a *congruence* of a RP we shall mean a congruence of the *algebra reduct* of the RP. We denote by $\mathrm{Con}\,\mathbf{A}$ and by $\mathbf{Con}\,\mathbf{A}$, respectively, the set and the lattice of congruences of a RP or RL $\mathbf{A}$. When $\mathbf{A}$ is an upper bounded RP or RL and $\theta \in \mathrm{Con}\,\mathbf{A}$, we set $\theta' = \{\langle a', b'\rangle : \langle a, b\rangle \in \theta\}$ and $\theta^* = \theta \cup \theta'$, where the elements $a', b'$ are calculated in the structure $\mathbf{A}^*$, constructed as above.

PROPOSITION 6.2. *Let* $\mathbf{A}$ *be an upper bounded RP or RL.*

(i) *If* $\mathbf{B}$ *is a substructure (or subalgebra) of* $\mathbf{A}$ *which contains the top element of* $\langle A; \leq\rangle$ *then the set* $B \cup \{b' : b \in B\} \subseteq A^*$ *is the universe of a substructure (or subalgebra) of* $\mathbf{A}^*$ *which is isomorphic to* $\mathbf{B}^*$.

(ii) *If* $\theta \in \mathrm{Con}\,\mathbf{A}$ *then* $\theta^* \in \mathrm{Con}\,\mathbf{A}^*$.

(iii) *The map* $\theta \mapsto \theta^*$ *defines a lattice isomorphism from* $\mathbf{Con}\,\mathbf{A}$ *onto an interval sublattice of* $\mathbf{Con}\,\mathbf{A}^*$.

(iv) *If* $\mathbf{A}$ *is a RL (or more generally, if the equivalence classes of all congruences of* $\mathbf{A}^*$ *are convex sets) then* $\mathbf{Con}\,\mathbf{A}^*$ *is isomorphic to the ordinal sum of* $\mathbf{Con}\,\mathbf{A}$ *and a one-element lattice.*

The proof is straightforward. With regard to (iv), use the fact that if $a, b \in A$ and a congruence of $\mathbf{A}^*$ identifies $a$ with $b'$, and therefore $a'$ with $b$, it must identify $\bot = a' \cdot b'$ with $b \cdot a \in A$ and also $\top = (a' \cdot b')'$ with $(b \cdot a)' \in A^*$.

Some applications of this proposition for integral commutative RPs appear in [32].

## 7. Commutative Distributive Residuated Lattices

A *CDRL* shall mean a *commutative distributive residuated lattice.* In this section and the next we shall be concerned almost entirely with CDRLs. The class of all CDRLs and the class of all involutive CDRLs are clearly varieties. Recall that in any CDRL we use $a \to b$ to denote the common value of $a \setminus b$ and $b \,/\, a$. The particular case of Theorem 6.1 (i), (iii) that we need here is:

COROLLARY 7.1. *The class of involution-free subreducts of involutive CDRLs is exactly the variety of all CDRLs.*

*Consequently, a $\{\sim\}$-free universal first order sentence — in particular a $\{\sim\}$-free quasi-identity — in the language of CDRLs holds in every CDRL iff it holds in every involutive CDRL.*

N. Galatos proved in [14] the undecidability of the quasi-equational theories of a range of varieties of (not necessarily commutative) distributive residuated lattices, including the class of all CDRLs. Combining this result with the above, we have:

COROLLARY 7.2. *The quasi-equational theory of the variety of involutive CDRLs is undecidable.*

The same clearly applies to the subvariety of involutive CDRLs satisfying $f \leq t$, and to the variety of $\{\bot, \top\}$-expansions of bounded involutive CDRLs, in view of the full statement of Theorem 6.1.

The survey paper [18] contains a table indicating the status of various kinds of decision problem for selected varieties of residuated lattices. In this table the entry corresponding to the decision problem for the *equational* theory of CDRLs is blank. We shall fill this gap here by combining Corollary 7.1 with some results from logic, which we explain below.

## 8. The Formal System RW

Throughout this section assume that a fixed infinite set of variables is given. The formal system **RW** has signature $L = \langle \cdot, \rightarrow, t, \wedge, \vee, \sim \rangle$. The ranks of these symbols are just as for involutive CDRLs. For convenience we define also a derived binary connective $\leftrightarrow$ by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The axioms (A1)–(A14) and postulated rules (R1), (R2) of **RW**, taken from [9], are as follows, where $x, y, z$ denote three distinct variables.

(A1) $x \rightarrow x$

(A2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$

(A3) $x \rightarrow ((x \rightarrow y) \rightarrow y)$

(A4) $(x \wedge y) \rightarrow x$

(A5) $(x \wedge y) \rightarrow y$

(A6) $((x \rightarrow y) \wedge (x \rightarrow z)) \rightarrow (x \rightarrow (y \wedge z))$

(A7) $x \rightarrow (x \vee y)$

(A8) $y \to (x \vee y)$

(A9) $((x \to z) \wedge (y \to z)) \to ((x \vee y) \to z)$

(A10) $(x \wedge (y \vee z)) \to ((x \wedge y) \vee (x \wedge z))$

(A11) $(\sim\sim x) \to x$

(A12) $(x \to \sim y) \to (y \to \sim x)$

(A13) $(x \to (y \to z)) \leftrightarrow ((x \cdot y) \to z)$

(A14) $x \leftrightarrow (t \to x)$

(R1) $x,\, x \to y \rhd y$ (modus ponens)

(R2) $x,\, y \rhd x \wedge y$ (adjunction)

To be accurate, the system we are calling **RW** here is called $\mathbf{RW}^{\circ t}$ in [9]. R.T. Brady's $\circ$ is our $\cdot$ and he reserves the name **RW** for the subsystem of our **RW** got by deleting $\cdot, t$ and the axioms (A13) and (A14).

In this logical context, it is natural to use the expression *L-formula* (or *formula of* **RW**) for what algebraists working with involutive CDRLs would call *L-terms*. For similar reasons, we refer here to $\sim$ as 'negation', etc. A *substitution* shall mean an endomorphism of the absolutely free $L$-algebra freely generated by our set of variables. For an $L$-formula $\varphi$, a *substitution instance* of $\varphi$ shall mean an $L$-formula of the form $\sigma(\varphi)$ for some substitution $\sigma$.

Recall that a *proof* of an $L$-formula $\varphi$ in **RW** is a finite nonempty sequence $\varphi_0, \varphi_1, \ldots, \varphi_{n-1} = \varphi$ such that for each $i < n$, one of the following is true:

(i) $\varphi_i$ is a substitution instance of one of the axioms (A1)–(A14) above;

(ii) for some $j, k < i$, $\varphi_k$ is $\varphi_j \to \varphi_i$;

(iii) $\varphi_i$ is $\varphi_j \wedge \varphi_k$ for some $j, k < i$.

Recall that $\varphi$ is called a *theorem* of **RW** if it has a proof in **RW**. The intended interpretation of (R1) is: whenever $\varphi$ and $\varphi \to \psi$ are *theorems* of **RW** then so is $\psi$; similarly for (R2).

It is well known that

(A3)′ $(x \to (y \to z)) \to (y \to (x \to z))$

is a theorem of **RW**.[1] In fact (A3)′ is interchangeable with (A3) in the presence of (A1), (A2) and (R1).

We shall make use of the following result of Brady.

THEOREM 8.1. (Brady) [9, Theorem 3] **RW** *is decidable.*

This means that there is an algorithm which decides whether arbitrary $L$-formulas are *theorems* of **RW**. Brady's proof of this result makes sophisticated use of sequent methods.

To clarify the relationship between **RW** and CDRLs, we need to consider also the *deducibility relation* of **RW**, which is the relation $\vdash_{\mathbf{RW}}$ between sets of $L$-formulas and single $L$-formulas defined as follows: $\langle \Gamma, \varphi \rangle \in \vdash_{\mathbf{RW}}$ iff there is a finite nonempty sequence $\varphi_0, \varphi_1, \ldots, \varphi_{n-1} = \varphi$ such that for each $i < n$, either $\varphi \in \Gamma$ or one of the conditions (i), (ii), (iii) in the earlier definition of proofs and theorems is true. We abbreviate $\langle \Gamma, \varphi \rangle \in \vdash_{\mathbf{RW}}$ as $\Gamma \vdash_{\mathbf{RW}} \varphi$ and if this is true we call $\langle \Gamma, \varphi \rangle$ a *derivable rule* of $\vdash_{\mathbf{RW}}$ (or of **RW**).

For any formal system **S**, i.e., any selection of axioms and postulated rules (with finitely many 'premisses' and a single 'conclusion') in a specified signature, we can define the deducibility relation of **S** in the obviously analogous way, allowing the axioms and postulated rules of **S** to play the roles of the **RW**-axioms, (R1) and (R2) in the definition.

The formal system without the connective $\sim$ which is axiomatized by (A1)–(A10), (A13), (A14), (R1) and (R2) will be denoted here by $\mathbf{RW}^+$. Its deducibility relation will be denoted by $\vdash_{\mathbf{RW}^+}$. Our $\mathbf{RW}^+$ is essentially the system called $\mathbf{RW}^{ot}_{+}$ by S. Giambrone in [15], i.e. it can be shown that the two systems have the same deducibility relation.[2]

The deducibility relations of formal systems such as **RW** and $\mathbf{RW}^+$ are 'finitary and structural consequence relations' in the sense, e.g., of [40]. That is to say they are relations $\vdash$ from the set of all formulas of a signature to single formulas of the same signature satisfying:

- $\varphi \in \Gamma$ implies $\Gamma \vdash \varphi$;
- $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash \varphi$;
- $\Gamma \vdash \varphi$ and $\Delta \vdash \gamma$ for all $\gamma \in \Gamma$ implies $\Delta \vdash \varphi$;
- (finitariness) $\Gamma \vdash \varphi$ implies $\Gamma' \vdash \varphi$ for some finite $\Gamma' \subseteq \Gamma$;
- (structurality) $\Gamma \vdash \varphi$ implies $\sigma[\Gamma] \vdash \sigma(\varphi)$ for every substitution $\sigma$.

The *positive fragment* of $\vdash_{\mathbf{RW}}$ is the set of all pairs $\langle \Gamma, \varphi \rangle \in \vdash_{\mathbf{RW}}$ such that $\Gamma \cup \{\varphi\}$ is a set of *negation-free* $L$-formulas.[3] It is easy to see that this is again a finitary and structural consequence relation. By a theorem of J. Łos and R. Suszko [20], any finitary and structural consequence relation is the deducibility relation of a formal system, although the problem of finding a transparent axiomatization may be difficult and finite axiomatizability is not guaranteed.

The positive fragment of $\vdash_{\mathbf{RW}}$ obviously contains $\vdash_{\mathbf{RW}^+}$ but in the literature on **RW** we have found no statement of the precise relationship between these two consequence relations, no effective axiomatization of the former and no indication of whether these relations have even the same *theorems*.[4] These questions will be settled here.

The principal relevance logic **R** is defined in [3] as the formal system that extends **RW** by the contraction axiom $(x \to (x \to y)) \to (x \to y)$, which is sometimes denoted by $W$. Thus **RW** really abbreviates $\mathbf{R} - W$.[5]

The varieties of CDRLs and involutive CDRLs model the deducibility relations of $\mathbf{RW}^+$ and **RW** in a sense that we shall now make precise. For this purpose we adopt:

DEFINITION 8.2. A quasivariety $\mathsf{K}$ of algebras whose signature includes the symbols $\to, t, \wedge$ will be called a *standard* (*algebraic*) *semantics* for a finitary and structural consequence relation $\vdash$ over the same signature provided that for any finite set $\Gamma \cup \{\varphi\}$ of formulas and any finite set $\Sigma \cup \{\alpha \approx \beta\}$ of equations of this signature,

(i) $\Gamma \vdash \varphi \quad$ iff $\quad \mathsf{K} \models (\bigwedge_{\gamma \in \Gamma} t \wedge \gamma \approx t) \;\Rightarrow\; t \wedge \varphi \approx t$

(ii) $\mathsf{K} \models (\bigwedge \Sigma) \Rightarrow \alpha \approx \beta \quad$ iff $\quad \{\xi \leftrightarrow \eta : \xi \approx \eta \in \Sigma\} \;\vdash\; \alpha \leftrightarrow \beta$

(iii) $x \;\vdash\; (t \wedge x) \leftrightarrow t \quad$ and $\quad (t \wedge x) \leftrightarrow t \;\vdash\; x\,.$

In (ii), if $\Sigma = \emptyset$, we interpret the condition preceding the 'iff' as asserting that $\mathsf{K}$ satisfies $\alpha \approx \beta$. The same convention applies to $\Gamma$ in (i).

The above definition is a specialization of the more abstract notion of '*equivalent* algebraic semantics' given in [5]. Since quasivarieties are determined by the quasi-identities that they model, it is easy to see that when a standard semantics $\mathsf{K}$ for $\vdash$ exists then it is unique.

The formal system got from $\mathbf{RW}^+$ by deleting the distribution axiom (A10) shall be denoted by $\mathbf{LL}'$. This notation is motivated by the system's connection with linear logic: see [36, p. 67]. The following result is essentially well known.

THEOREM 8.3. *The variety of commutative residuated lattices is the standard semantics for the deducibility relation of* $\mathbf{LL}'$.

A complete proof of Theorem 8.3, with some special features not needed here, can be found in [39]. The argument used in [39] relies on the absence of the distribution axiom (A10) in $\mathbf{LL}'$, so it does not immediately provide us with a standard semantics for $\vdash_{\mathbf{RW}^+}$. Nevertheless, we can establish standard semantics for both $\vdash_{\mathbf{RW}^+}$ and $\vdash_{\mathbf{RW}}$ by putting Theorem 8.3 together with the following result, which itself combines [5, Theorems 4.7, 2.17].

LEMMA 8.4. *Let* $\mathsf{K}$ *be a standard semantics for the deducibility relation of a formal system* $\mathbf{S}$ *and let* $\mathbf{S}'$ *be a formal system got by adding to* $\mathbf{S}$ *new axioms — possibly involving new connectives — but no new postulated rules.*

(i) *The deducibility relation* $\vdash_{\mathbf{S}'}$ *of* $\mathbf{S}'$ *also has a standard semantics, provided that for each new connective* $\rho$*, of rank* $n$*, say, we have*

$$x_1 \leftrightarrow y_1\,,\ \dots\,,\ x_n \leftrightarrow y_n\ \vdash_{\mathbf{S}'}\ \rho(x_1,\dots,x_n) \leftrightarrow \rho(y_1,\dots,y_n).$$

(ii) *If the condition in* (i) *holds then the standard semantics for* $\vdash_{\mathbf{S}'}$ *is the quasivariety in the signature of* $\mathbf{S}'$ *axiomatized, relative to* $\mathsf{K}$*, by the equations* $t \wedge \alpha \approx t$*, where* $\alpha$ *ranges over all the new axioms.*

We shall now denote the variety of all CDRLs as $\mathsf{CDRL}$ and the variety of all involutive CDRLs as $\mathsf{iCDRL}$.

Since $\mathbf{RW}^+$ is the extension of $\mathbf{LL}'$ (in the same signature) by just the distribution axiom (A10), it follows from the previous two results that the unique standard semantics for $\vdash_{\mathbf{RW}^+}$ is the variety of commutative RLs satisfying $t \wedge ((x \wedge (y \vee z)) \to ((x \wedge y) \vee (x \wedge z))) \approx t$, i.e., satisfying

$$x \wedge (y \vee z)\ \leq\ (x \wedge y) \vee (x \wedge z).$$

This is just the variety $\mathsf{CDRL}$.

It is easy to show, using (A11), (A12), (R1) and (R2), that

$$x \leftrightarrow y\ \vdash_{\mathbf{RW}}\ ({\sim}\, y) \leftrightarrow {\sim}\, x\,.$$

Since $\mathbf{RW}$ is the extension of $\mathbf{RW}^+$ by the connective $\sim$ and the axioms (A11) and (A12), the previous observation and Lemma 8.4 shows also that $\vdash_{\mathbf{RW}}$ has a standard semantics, namely the variety of $\{\sim\}$-expansions of CDRLs satisfying

$$t \wedge (({\sim}{\sim}\, x) \to x) \approx t \quad \text{and} \quad t \wedge ((x \to {\sim}\, y) \to (y \to {\sim}\, x)) \approx t,$$

i.e., satisfying

$${\sim}{\sim}\, x \leq x \quad \text{and} \quad x \to {\sim}\, y\ \leq\ y \to {\sim}\, x.$$

It is easy to see that this is just the variety $\mathsf{iCDRL}$. In summary, we have shown:

THEOREM 8.5. *For any finite set* $\Gamma \cup \{\varphi\}$ *of* $L$*-formulas, we have*

(A) $\quad \Gamma \vdash_{\mathbf{RW}} \varphi$ *iff* $\mathsf{iCDRL} \models (\bigwedge_{\gamma\in\Gamma} t \leq \gamma)\ \Rightarrow\ t \leq \varphi$;

(A)$^+$ $\quad \Gamma \vdash_{\mathbf{RW}^+} \varphi$ *iff* $\mathsf{CDRL} \models (\bigwedge_{\gamma\in\Gamma} t \leq \gamma)\ \Rightarrow\ t \leq \varphi$.

*In addition, for any finite set* $\Sigma \cup \{\alpha \approx \beta\}$ *of* $L$*-equations,*

(B) $\mathsf{iCDRL} \models (\bigwedge\Sigma) \Rightarrow \alpha \approx \beta$ *iff* $\{\xi \leftrightarrow \eta : \xi \approx \eta \in \Sigma\} \vdash_{\mathbf{RW}} \alpha \leftrightarrow \beta$;

$(\mathrm{B})^+$ $\mathsf{CDRL} \models (\bigwedge\Sigma) \Rightarrow \alpha \approx \beta$ *iff* $\{\xi \leftrightarrow \eta : \xi \approx \eta \in \Sigma\} \vdash_{\mathbf{RW}^+} \alpha \leftrightarrow \beta$.

Note that in $(\mathrm{A})^+$ and $(\mathrm{B})^+$, the formulas or equations considered are understood to be negation-free, since for expressions with negation, we can attach no meaning to $\vdash_{\mathbf{RW}^+}$ or to '$\mathsf{CDRL} \models$'. Of course, expressions such as $t \leq \varphi$ abbreviate formal equations $t \wedge \varphi \approx t$.

Putting Theorem 8.5 and Corollary 7.1 together, we infer that for any finite set $\Gamma \cup \{\varphi\}$ of negation-free $L$-formulas, we have $\Gamma \vdash_{\mathbf{RW}^+} \varphi$ iff $\Gamma \vdash_{\mathbf{RW}} \varphi$. In other words, the deducibility relation of $\mathbf{RW}^+$ is equal to the positive fragment of the deducibility relation of $\mathbf{RW}$. It is customary to express this as:

COROLLARY 8.6. **RW** *is a strongly conservative extension of* $\mathbf{RW}^+$.

In Brady's and Giambrone's notation, this reads: $\mathbf{RW}^{\circ t}$ is a strongly conservative extension of $\mathbf{RW}^{\circ t}_{+}$. In particular (in our notation again), **RW** and $\mathbf{RW}^+$ have the same negation-free *theorems*.

By Theorem 8.5, testing whether an *equation* $\alpha \approx \beta$ holds in iCDRL amounts exactly to testing whether $\alpha \leftrightarrow \beta$ is a *theorem* of **RW**. Similarly for CDRL and $\mathbf{RW}^+$. Applying Brady's Theorem 8.1 to the first of these equation-theorem equivalences, we obtain part (i) of the next corollary. Part (ii) follows from (i) using Corollary 7.1, while (iii) follows from Corollary 8.6 and Brady's Theorem.

COROLLARY 8.7.

(i) *The equational theory of* iCDRL *is decidable.*

(ii) *The equational theory of* CDRL *is decidable.*

(iii) $\mathbf{RW}^+$ *is decidable.*

The second of these assertions fills the previously mentioned gap in the table of decidability results in [18]. In Giambrone's notation, the last assertion reads: $\mathbf{RW}^{\circ t}_{+}$ is decidable. Note that in [15], Giambrone did not claim or prove the decidability of this system, although he did prove the decidability of its $\{t\}$-free subsystem got by deleting our (A14) (i.e. his R3, R4).[6]

Over CDRL or iCDRL, all quasi-identities can be brought into the form of the ones in Theorem 8.5 (A) and $(\mathrm{A})^+$ because each equation $\alpha \approx \beta$ is equivalent to the (abbreviated) equation $t \leq \alpha \leftrightarrow \beta$. The parts of Theorem 8.5 that deal with iCDRL therefore combine with Corollary 7.2 — or with A. Urquhart's [37, Theorem 5.1] — to yield (in our notation) the following:

COROLLARY 8.8. (Urquhart) *The deducibility problem for* **RW** *is unsolvable.*

In other words there is no algorithm that decides whether a given pair $\langle \Gamma, \varphi \rangle$ (where $\Gamma \cup \{\varphi\}$ is a *finite* set of $L$-formulas) belongs to $\vdash_{\mathbf{RW}}$ or not. (To get this from [37, Theorem 5.1], add as axioms enough theorems of Urquhart's '$L(V)$' involving $f, \rightarrow$ to force the desired definitions and properties of $\cdot\,, \sim$.)

The unsolvability of the deducibility problem for $\mathbf{RW}^+$ is also implied in [37]. The task of extracting it from [37] is complicated, in comparison with the case discussed above, by the absence of the connective $\cdot$ from Urquhart's formulation of '$\mathbf{R}_+ - W$' and the absence of $f$ from (our) $\mathbf{RW}^+$. The result can more easily be got directly from Theorem 8.5 and the undecidability result in [14] from which we inferred Corollary 7.2.

Corollary 8.7 (iii) contrasts with another of Urquhart's results in [37], viz. that the extension of (our) $\mathbf{RW}^+$ by the axiom $((x \rightarrow y) \wedge x \wedge t) \rightarrow y$ is undecidable.

If a variety with equationally definable principal congruences (EDPC) has a decidable equational theory then its quasi-equational theory must be decidable also. Thus, although CDRL and iCDRL are congruence distributive and have the congruence extension property, their decidability discrepancies corroborate the well known fact that they lack EDPC. Equivalently, they confirm that the deducibility relations of $\mathbf{RW}^+$ and $\mathbf{RW}$ have no deduction-detachment theorem in the general sense defined in [6] or [11].

## 9. Adding Involution (II): Preserving Contraction

A second construction for embedding an arbitrary (not necessarily commutative) RP into a bounded *involutive* RP will be set out briefly in this section. Unlike the construction in Section 6, the one given here preserves the contraction axiom $x \leq x^2$ and does not preserve integrality, mingle or existent greatest elements. All of the other properties whose preservation under the first construction was mentioned in Section 6 are preserved also by the second construction. In addition, the second construction produces a rigorously compact RP.

The method generalizes a construction called 'relevant enlargement' in [22] that was used by Meyer in the more special context of relevance logic. In Meyer's construction, however, $t$ was omitted from the signature.

Let $\mathbf{A} = \langle A; \cdot, \backslash, /, t; \leq \rangle$ be a residuated pomonoid. Let $A' = \{a' : a \in A\}$ be a disjoint bijective copy of $A$, and let $A^\dagger = A \cup A' \cup \{\bot, \top\}$, where $\bot, \top$ are distinct non-elements of $A \cup A'$. Extend $\leq$ to a partial order of $A^\dagger$, also denoted by $\leq$, by defining that for any $a, b \in A$,

$$\text{(i)} \;\; \bot < a < b' < \top \quad \text{and} \quad \text{(ii)} \;\; a' \leq b' \;\text{ iff }\; b \leq a\,.$$

Thus, $\langle A^\dagger; \leq \rangle$ is order isomorphic to the bounded 2-point extension of the ordinal sum of $\langle A; \leq \rangle$ and its dual poset. Define $f = t'$, $\bot' = \top$, $\top' = \bot$ and for each $a \in A$, $a'' = a$, so that $'$ becomes a total unary operation on $A^\dagger$. We define an extension to $A^\dagger$ of the monoid operation $\cdot$ of $A$, which we also denote by $\cdot$, as follows: if $a, b \in A$ and $c \in A^\dagger$ then

$$\begin{aligned} & a \cdot b' = (b \,/\, a)' \quad \text{ and } \quad b' \cdot a = (a \setminus b)'\,; \\ & a' \cdot b' = \top\,; \\ & \text{if } c \neq \bot \text{ then } c \cdot \top = \top \cdot c = \top\,; \\ & \bot \cdot c = c \cdot \bot = \bot. \end{aligned}$$

It follows that $t$ is an identity element for $\cdot$ on all of $A^\dagger$, that $\cdot$ is associative on $A^\dagger$ and that $\langle A^\dagger; \cdot, t; \leq \rangle$ is a pomonoid.

Note that if $z \in A' \cup \{\top\}$ and $a, b \in A$ then $a \cdot z, z \cdot a \in A' \cup \{\top\}$, so $a \cdot z, z \cdot a \not\leq b$. It follows that $a \setminus b$ and $b \,/\, a$ (calculated in $\mathbf{A}$) are, respectively, the greatest elements $z_1, z_2$ of $A^\dagger$ such that $z_1 \cdot a \leq b$ and such that $a \cdot z_2 \leq b$ in $A^\dagger$. So we can use $\setminus$ and $/$ unambiguously to denote right and left residuals wherever they exist in $A^\dagger$. It can be checked that they exist throughout $A^\dagger$ and that the extensions of $\setminus$ and $/$ to $A^\dagger$ are determined by (4), (5), (6) and (8) of Section 3, as well as

(I) $a \setminus b' = (b \cdot a)' = a' \,/\, b\,;$

(II) $b' \setminus a = \bot = a \,/\, b'\,;$

(III) $a' \setminus b' = a \,/\, b$ and $b' \,/\, a' = b \setminus a\,,$

where $a, b$ denote elements of $A$. Note that (4), (5), (6) and (8) are already justified by Lemma 3.1 together with the definitions of $\cdot$ and $\leq$ on $A^\dagger$, because $\langle A^\dagger; \cdot, t; \leq \rangle$ is a pomonoid.

We have established that $\langle A^\dagger; \cdot, \setminus, /, t; \leq \rangle$ is a bounded rigorously compact RP of which $\mathbf{A}$ is a substructure. It is easy to see that if $\mathbf{A}$ is commutative or $n$-contractive for some $n \geq 1$ or $n$-expansive for some $n \geq 2$ then $\langle A^\dagger; \cdot, \setminus, /, t; \leq \rangle$ has the same property. Trivially, all existing (even infinitary) meets and joins of *nonempty* sets in $A$ are preserved in $A^\dagger$, as are all self-dual order properties such as latticehood, complete latticehood and distributivity, so if $\mathbf{A}$ was the reduct of a RL $\mathbf{C}$ then the order $\leq$ on $A^\dagger$ is a lattice order and if $\wedge, \vee$ denote its induced meet and join operations then **C** is a subalgebra of $\langle A^\dagger; \cdot, \backslash, /, \wedge, \vee, t\rangle$. Note that the embedding preserves boundedness but not bounds themselves (whereas the construction in Section 6 preserved greatest elements also). Obviously, integrality is not preserved by the present construction. The mingle axiom is also not preserved: if $a \in A$ then $a' \cdot a' = \top \not\leq a'$.

The above characterizations of $\backslash$ and $/$ show that for all $a, b \in A^\dagger$, we have $a \backslash b' = a' / b$ and therefore also $x' \approx x \backslash f \approx f / x$. For each $a \in A$, we now define $\sim a = a'$. The algebra $\mathbf{A}^\dagger = \langle A^\dagger; \cdot, \backslash, /, \sim, t\rangle$ is thus an involutive RP, in which $t \leq f$.[7]

In summary, we can claim:

THEOREM 9.1.

(i) *Every RP is a* $\{\cdot, \backslash, /, t; \leq\}$*-subreduct of a bounded involutive rigorously compact RP satisfying* $t \leq f$.

(ii) *Every RL is a* $\{\cdot, \backslash, /, \wedge, \vee, t\}$*-subreduct of a bounded involutive rigorously compact RL satisfying* $t \leq f$.

(iii) *If the initial structure in* (i) *or in* (ii) *is finite or a complete lattice or a distributive lattice or is commutative or* $n$*-contractive for some* $n \geq 1$ *or* $n$*-expansive for some* $n \geq 2$*, or has any combination of these properties then the involutive structure can be chosen to have the same combination of properties.*

## 10. Functoriality

It is natural to ask whether either of the constructions in Sections 6 and 9 is *functorial*. Here we show first that the construction of Section 9 provides a functor *directly*. We shall then deal briefly with the situation for the first construction, which is more complex.

Denote by **RP** the category whose objects are just all RP's and whose morphisms are all order-preserving monoid homomorphisms between RP's that preserve the residuation operations. Also, let **RL** be the category of RL's and RL-homomorphisms. Similarly, we define the categories **iRP** and **iRL**. We do not require the morphisms of these categories to preserve the bounds $\bot, \top$, where these exist.

Note, however, that when **A** is an object of **RP** or **RL** then the bounds $\bot, \top$ of the structure $\mathbf{A}^\dagger$ are *termwise definable*: the construction of $\mathbf{A}^\dagger$ imposes the relations $\top = f^2$ and $\bot = t / \top$.

For **A** in **RP** (in **RL**, respectively), define $F(\mathbf{A}) = \mathbf{A}^\dagger$. Moreover, if $g : \mathbf{A} \to \mathbf{B}$ is a morphism in **RP** (**RL**, resp.), define $F(g) = g^\dagger : \mathbf{A}^\dagger \to \mathbf{B}^\dagger$, where $g^\dagger(a) = g(a)$ and $g^\dagger(a') = (g(a))'$, for all $a \in A$, while $g^\dagger(\bot) = \bot$ and $g^\dagger(\top) = \top$. It is easy to see that $g^\dagger$ is a morphism and that $F$ is a functor from **RP** to **iRP** (from **RL** to **iRL**, resp). Conversely, if $\mathbf{A}$ is in **iRP** (**iRL**, resp.), define $G(\mathbf{A})$ to be the involution-free reduct of $\mathbf{A}$. Moreover, if $g : \mathbf{A} \to \mathbf{B}$ is a morphism in **iRP** (**iRL**, resp.), let $G(g)$ be the same map as $g$, but considered as a morphism between $G(\mathbf{A})$ and $G(\mathbf{B})$. Then $G$ is a functor, known as the *forgetful functor*, between the categories **iRP** and **RP** (**iRL** and **RL**, resp.).

Although the functor $F$ is not onto the objects of the target category, it is both full and faithful. Recall that a functor $F : \mathbf{C} \to \mathbf{D}$ is said to be *onto* the objects of the category $\mathbf{D}$, if for every object $d$ of $\mathbf{D}$ there exists an object $c$ of $\mathbf{C}$ such that $F(c) = d$. It is called *full* if for any pair of objects $a, c$ of $\mathbf{C}$ and any morphism $h : F(a) \to F(c)$ of $\mathbf{D}$, there exists a morphism $g : a \to c$ of $\mathbf{C}$, such that $F(g) = h$. Also, $F$ is called *faithful*, if for every pair of objects $a, c$ and for any pair of morphisms $g, h : a \to c$ of $\mathbf{C}$, $F(h) = F(g)$ implies $h = g$.

To see that $F$ is full, let $h : \mathbf{A}^\dagger \to \mathbf{B}^\dagger$ be a morphism in **iRP** (**iRL**, resp.). We will show that $h[A] \subseteq B$. First note that, since the bounds are term definable, they are preserved under $h$. Assume that $h(a) \notin B$, for some $a \in A$. If $h(a) \in B'$, or $h(a) = \top$, then $h(a^2) = (h(a))^2 = \top$. Moreover, $h(t \,/\, a^2) = t \,/\, \top = \bot$. Thus, in all cases, there exists an element $c \in A$, such that $h(c) = \bot$. Consequently, $h(\bot \,/\, c) = \bot \,/\, \bot$, hence $h(\bot) = \top$, a contradiction. Since $G(h)$ is a morphism of **RP** (**RL**, resp.) between $G(\mathbf{A}^\dagger)$ and $G(\mathbf{B}^\dagger)$, $\mathbf{A}$ is a substructure of $G(\mathbf{A}^\dagger)$ and $h[A] \subseteq B$, we have that the restriction of $G(h)$ to $A$ is a morphism of **RP** (**RL**, resp.) from $\mathbf{A}$ to $\mathbf{B}$. Clearly, the image under $F$ of this restriction is $h$.

On the other hand, the forgetful functor $G$ is neither onto objects, nor full, nor faithful. It is not faithful, because a lattice-ordered group can be considered as an involutive RL in multiple ways (every element can serve as the constant $f$), where $x \backslash y = x^{-1}y$, and $y \,/\, x = yx^{-1}$. Consider such a lattice-ordered group with two different involutions; the two structures have the same RL-reduct. If $G$ were faithful, it would be bijective on objects.

To see that $G$ is not full, consider the same lattice-ordered group with two different involutions. The identity map on the common involution-free reduct is not induced by a morphism between the involutive structures. So, if $g : \mathbf{A} \to \mathbf{B}$ is a morphism in **RP** (**RL**, resp.), $G(\mathbf{C}) = \mathbf{A}$ and $G(\mathbf{D}) = \mathbf{B}$, there may be no morphism $h : \mathbf{C} \to \mathbf{D}$ such that $G(h) = g$. Nevertheless, under the same assumptions, in case $g$ is onto, there exist $\mathbf{E}$ in **iRP** (**iRL**, resp.) and $k : \mathbf{C} \to \mathbf{E}$ such that $G(\mathbf{E}) = \mathbf{B}$ and $G(k) = g$. This is true because the image $\mathbf{B}$ of the reduct $\mathbf{A}$ of an involutive structure $\mathbf{C}$, with involution constant $f$, under a morphism $g$ of **RP** (**RL**, resp.) is also the reduct of an involutive structure **E**, such that $g(f)$ is the involution constant of **E**. We use here the fact that the conditions defining the involution constant are *equations* (see Section 5), so they are preserved under homomorphisms.

The construction of Section 6 does not yield a functor directly. To obtain functoriality we have to modify the construction and in the process, we must give up the preservation of integrality and existent upper bounds. More precisely, we first add new bounds to the original structure, i.e., we take its rigorously compact 2-point extension as described in Section 3, even if the original structure is already bounded; then we apply the construction of Section 6. One may then argue similarly that the composition of these two embeddings defines a functor that has the same properties as the functor described above.

## 11. Concluding Remarks

The constructions leading to Theorems 6.1 and 9.1 make no essential use of the existence of $t$ and work equally well for residuated ordered *semigroups*. (Also, Lemma 5.1 can be proved without reference to $f$ or $t$.) Meyer's relevant enlargement construction in [22] applies to lattice-ordered commutative residuated semigroups satisfying $x \leq x \cdot (y \rightarrow y)$ and *contraction* and it produces an involution that also satisfies the contraction-implying condition $x \rightarrow {\sim} x \leq {\sim} x$.

Notice that a general construction for embedding commutative RPs or RLs into involutive ones cannot preserve both integrality and contraction, for such a construction would allow us to embed arbitrary Brouwerian semilattices into Boolean algebras. Of course this would be contradictory even at the level of pure residuation, since Peirce's Law

$$(x \rightarrow y) \rightarrow x \leq x$$

already serves to distinguish these classes. It would be of interest to find a construction for adding involution to (at least commutative) RPs and RLs that preserves both contraction and mingle (and therefore idempotence) while necessarily not preserving integrality. It may be better here to develop a construction that applies to residuated *semigroups*, since, for example, one cannot embed all involutive idempotent residuated commutative semigroups into monoids with all the same properties: see [21, Note 4], [4, p. 710 and Theorem I.18].

On the other hand, *not every idempotent CDRL can be embedded into an idempotent involutive CDRL.* This is shown by the following example.

EXAMPLE 11.1. It is known that every idempotent involutive CDRL satisfies

$$[(((x \to y) \to y) \to x) \to z] \cdot [(((y \to x) \to x) \to y) \to z] \leq z. \tag{11}$$

This inequality comes from a logical formula whose significance was discovered in [25]: it holds in all Sugihara algebras, and the $\{t\}$-free reduct of any idempotent involutive CDRL is a Sugihara algebra (see, e.g., [7] and its references).

Now consider two total orders on the set $E = \{-1, 0, \frac{1}{3}, \frac{1}{2}, 1\}$. The conventional order is denoted by $\leq$ and a second order $\preceq$ is given by

$$-1 \prec 1 \prec \tfrac{1}{2} \prec \tfrac{1}{3} \prec 0.$$

Let $x \cdot y$ denote the minimum of $\{x, y\}$ with respect to $\preceq$. This is necessarily an idempotent commutative monoid operation with identity 0. Our use of $\preceq$ to define $\cdot$ made this otherwise tedious step immediate; $\preceq$ shall play no further role in the example. The totally (conventionally) ordered pomonoid $\langle E; \cdot, t; \leq\rangle$ is clearly residuated. Letting $\to$ denote the residuation, we obtain an idempotent CDRL $\mathbf{E}$ on the same set. If we substitute $x = \frac{1}{2}$, $y = \frac{1}{3}$ and $z = 0$ into (11), however, we obtain $1 \cdot 1 \leq 0$, which is false. Thus $\mathbf{E}$ cannot be isomorphic to a subalgebra of an idempotent involutive CDRL.

The method used in Section 8 to obtain the equational decidability of CDRL is potentially applicable to other quasivarieties of RPs or RLs that are not contractive and that are characterized by properties preserved in the construction of Section 6, provided that a decidability result is for some reason more accessible in the involutive than in the involution-free case. If further useful applications of this kind exist, the corresponding logics could be expected to include fragments of noncommutative linear logic or the Lambek calculus $\mathbf{FL}$. In view of the first construction presented here, and Proposition 4.2 (i), Hilbert systems corresponding to several positive logics in this family can be extended in a strongly conservative manner by a single involutive negation (along with $/$), provided that the constants $f$ and $\perp$ are omitted from consideration and that the signature includes $\backslash$, $\cdot$ and both or neither of $\wedge, \vee$. There are two options for the treatment of negation in noncommutative linear logic: one may have a single involutive negation or two negations. See, e.g., [42], [1].

The second construction extends these possibilities to (quasi) varieties *with* contraction that do not satisfy the mingle axiom. The corresponding logics include extensions of the logic $\mathbf{FL_c}$, formulated without $f, \perp$ and $\top$. For example, Proposition 4.2 (ii) and the second construction can be used to show that the Hilbert-style axioms and rules for $\mathbf{FL_c}$ in the language consisting of $\cdot$ , $\backslash$ and possibly $t$ can be extended in a strongly conservative manner by a single involutive negation (along with $/$). It seems that nothing is known about the decidability of even the pure implication fragment of $\mathbf{FL_c}$. For information about $\mathbf{FL_c}$ and its neighbours, see [27], [28] and [29].

## NOTES

**1.** The proof is difficult to find in the literature. The following proof was extracted from [35]:

Definitions: $D$: $(y \to z) \to z$, $E$: $x \to (y \to z)$, $F$: $y \to (x \to z)$.

Proof:
- i. $y \to D$ [(A3)]
- ii. $(y \to D) \to ((D \to (x \to z)) \to F)$ [(A2)]
- iii. $(D \to (x \to z)) \to F$ [i, ii, (R1)]
- iv. $E \to (D \to (x \to z))$ [(A2)]
- v. $(E \to (D \to (x \to z))) \to (((D \to (x \to z)) \to F) \to (E \to F))$ [(A2)]
- vi. $((D \to (x \to z)) \to F) \to (E \to F)$ [iv, v, (R1)]
- vii. $E \to F$ (as required) [iii, vi, (R1)]

**2.** The superficial discrepancy between our (A13), (A14) and Giambrone's A5, R3, R4 does not affect the relation of derivability of formulas from other formulas (that are not necessarily theorems).

**3.** The reader should be cautioned that in relevance logic (and elsewhere), fragments of deducibility relations are often disregarded and that by the 'positive fragment of **RW**', most relevance logicians would mean just the set of negation-free *theorems* of **RW**.

**4.** Relevance logic contains several conservative extension results of the general kind wanted here, most of which were proved by Meyer and appear in [22], [33], [2] or [34]. Of the published results that deal with the problem of adding negation, not all adopt the same signature: there is some variance over the inclusion of $\cdot$ and $t$. Some assume the contraction axiom $(x \to (x \to y)) \to (x \to y)$, others weaken (A12) to the form of a postulated rule: $x \to \sim y \rhd y \to \sim x$ (see Note 6 below), while others again focus only on the conservation of *theorems* (as opposed to derivable rules). Conservative extension results for theorems can be converted to ones for derivable rules also if they were proved by suitable embedding arguments or if the larger logic has a deduction-detachment theorem in the general sense of [6] or [11], which is formulated in the vocabulary of the smaller logic. **RW** has (demonstrably) no deduction-detachment theorem of this kind: see the conclusion of Section 8.

**5**. Meyer's result in [22], that **R** has the same negation-free theorems as the system axiomatized by the negation-free axioms of **R**, can be extended to derivable rules because **R** has a deduction-detachment theorem:

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{R}} \psi \text{ iff } \Gamma \vdash_{\mathbf{R}} (\varphi \wedge t) \to \psi \quad [23].$$

This last theorem relies on the presence in **R** of the contraction axiom. Making some use of this deduction-detachment theorem, A. Urquhart proved that **R**'s set of theorems — and even its set of negation-free theorems — is undecidable [37].

**6**. Giambrone and Meyer show in [16] that the $\{t\}$-free system got by deleting (A14) from **RW** can be extended conservatively by a strong ('classical') negation $\neg$ for which

$$(\neg\neg x) \leftrightarrow x,$$
$$x \to (y \to (z \vee \neg z)),$$
$$(x \wedge \neg x) \to y$$

are theorems, although only the 'rule' form $x \to \neg y \rhd y \to \neg x$ of contraposition for $\neg$ holds. 'Classical' negation cannot be used to derive Corollary 8.7 (iii) from Brady's Theorem. [16] shows also that **RW** itself (with $t$ and (A14)) *cannot* be extended conservatively by classical negation. According to [12, p. 214], the corresponding conservative extension of **R** by classical negation is not a strongly conservative extension of **R**.

**7**. In logical terms this shows, for example, that the positive fragment of **R**, formulated with $\cdot$ and $t$ in the usual way, can be extended in a strongly conservative manner by the axiom $t \to\, \sim t$ (along with de Morgan negation). This axiom strains the intuition that $t$ corresponds logically to the 'least' truth (i.e. the conjunction of all truths) and $f$ to the 'greatest' falsehood (i.e. the disjunction of all falsehoods). This point is not discussed in [22], where Meyer formulates **R** traditionally, i.e., without $t$, but it is taken up in his later paper [24, pp. 469–70].

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## References

[1] Abrusci, V. M., 'Phase semantics and sequent calculus for pure noncommutative classical linear logic', *J. Symbolic Logic* 56 (1991), 1403–1451.

[2] Anderson, A. R., and N. D. Belnap, Jnr., *Entailment: The Logic of Relevance and Necessity, Volume 1*, Princeton University Press, 1975.

[3] Anderson, A. R., N. D. Belnap, Jnr., and J. M. Dunn, *Entailment: The Logic of Relevance and Necessity, Volume 2*, Princeton University Press, 1992.

[4] AVRON, A., 'Relevance and paraconsistency — a new approach', *J. Symbolic Logic* 55 (1990), 707–732.

[5] BLOK W. J., and D. PIGOZZI, 'Algebraizable Logics', *Memoirs of the American Mathematical Society*, Number 396, Amer. Math. Soc., Providence, 1989.

[6] BLOK W. J., and D. PIGOZZI, 'Abstract algebraic logic and the deduction theorem', *The Bulletin of Symbolic Logic*, to appear.

[7] BLOK W. J., and J. G. RAFTERY, 'Fragments of R-mingle', *Studia Logica*, to appear.

[8] BLOUNT K., and C. TSINAKIS, 'The structure of residuated lattices', *International Journal of Algebra and Computation* 13 (2003), 437–461.

[9] BRADY R. T., 'The Gentzenization and decidability of RW', *J. Philosophical Logic* 19 (1990), 35–73.

[10] CHANG C. C., 'Algebraic analysis of many valued logic', *Trans. Amer. Math. Soc.* 88 (1958), 467–490.

[11] CZELAKOWSKI J., 'Algebraic aspects of deduction theorems', *Studia Logica* 44 (1985), 369–387.

[12] DUNN J. M., 'Relevance logic and entailment', in D. Gabbay and F. Guenthner, (eds.), *Handbook of Philosophical Logic, Volume III*, Reidel, Dordrecht, 1986, pp. 117–226.

[13] DUNN J. M., 'Partial gaggles applied to logics with restricted structural rules', in P. Schroeder-Heister and K. Došen, (eds.), *Substructural Logics*, Clarendon Press, Oxford, 1993, pp. 63–108.

[14] GALATOS N., 'The undecidability of the word problem for distributive residuated lattices', in J. Martinez, (ed.), *Ordered Algebraic Structures*, Kluwer Academic Publishers, Dordrecht, 2002, pp. 231–243.

[15] GIAMBRONE S., 'TW+ and RW+ are decidable', *J. Philosophical Logic* 14 (1985), 235–254.

[16] GIAMBRONE S., and R. K. MEYER, 'Completeness and conservative extension results for some Boolean relevant logics', *Studia Logica* 48 (1989), 1–14.

[17] GRISHIN V. N., 'Impossibility of defining the class of $L_0$-algebras by means of identities', *Mathematical Notes* 38 (1985), 861–867. [Translation of *Matematicheskie Zametki* 38 (1985), 641–651.]

[18] JIPSEN P., and C. TSINAKIS, 'A survey of residuated lattices', in J. Martinez, (ed.), *Ordered Algebraic Structures*, Kluwer Academic Publishers, Dordrecht, 2002, pp. 19–56.

[19] LAMBEK J., 'Some lattice models of bilinear logic, *Algebra Universalis* 34 (1995), 541–550.

[20] LOŚ J., and R. SUSZKO, 'Remarks on sentential logics', *Proc. Kon. Nederl. Akad. van Wetenschappen, Series A* 61 (1958), 177–183.

[21] MEYER R. K., 'Conservative extension in relevant implication', *Studia Logica* 31 (1972), 39–46.

[22] MEYER R. K., 'On conserving positive logics', *Notre Dame J. Formal Logic* 14 (1973), 224–236.

[23] MEYER R. K., 'Intuitionism, entailment, negation', in H. Leblanc, (ed.), *Truth, Syntax and Modality*, North Holland, Amsterdam, 1973, pp. 168–198.

[24] MEYER R. K., 'A note on $R_{\rightarrow}$ matrices', *Notre Dame J. Formal Logic* 24 (1983), 450–472.

[25] MEYER R. K., and R.Z. PARKS, 'Independent axioms for the implicational fragment of Sobociński's three-valued logic', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 18 (1972), 291–295.

[26] MEYER R. K., and R. ROUTLEY, 'Algebraic analysis of entailment, I', *Logique et Analyse N.S.* 15 (1972), 407–428.

[27] ONO H., 'Semantics for substructural logics', in P. Schroeder-Heister and K. Došen, (eds.), *Substructural Logics*, Clarendon Press, Oxford, 1993, pp. 259–291.

[28] ONO H., 'Decidability and finite model property of substructural logics', in J. Ginzberg, Z. Khasidashvili, C. Vogel, J.-J. Levy, and E. Vallduví, (eds.), *The Tsibili Symposium on Logic, Language and Computation*, CSLI Publications, 1997, pp. 263–274.

[29] ONO H., 'Proof-theoretic methods in nonclassical logic — an introduction', in M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, (eds.), *Theories of Types and Proofs*, MSJ Memoirs 2, Mathematical Society of Japan, 1998, pp. 207–254.

[30] ONO H., and Y. KOMORI, 'Logics without the contraction rule', *J. Symbolic Logic* 50 (1985), 169–202.

[31] PAŁASINSKI M., 'An embedding theorem for BCK-algebras', *Math. Seminar Notes Kobe Univ.* 10 (1982), 749–751.

[32] RAFTERY J. G., 'On the variety generated by involutive pocrims', manuscript, 2003.

[33] ROUTLEY R., and R. K. MEYER, 'The semantics of entailment', in H. Leblanc, (ed.), *Truth, Syntax and Modality*, North Holland, Amsterdam, 1973, pp. 199–243.

[34] ROUTLEY R., R. K. MEYER, V. PLUMWOOD, and R. T. BRADY, *Relevant Logics and their Rivals 1*, Ridgeview Publishing Company, California, 1982.

[35] SOBOCIŃSKI B., 'Axiomatization of a partial system of three-valued calculus of propositions', *The Journal of Computing Systems* 1 (1952), 23–55.

[36] TROELSTRA A. S., *Lectures on Linear Logic*, CSLI Lecture Notes, No 29, 1992.

[37] URQUHART A., 'The undecidability of entailment and relevant implication', *J. Symbolic Logic* 49 (1984), 1059–1073.

[38] van Alten C. J., and J. G. Raftery, 'On the lattice of varieties of residuation algebras', *Algebra Universalis* 41 (1999), 283–315.

[39] van Alten C. J., and J. G. Raftery, 'Rule separation and embedding theorems for logics without weakening', *Studia Logica*, 76 (2004), 241–274.

[40] Wójcicki R., *Theory of Logical Calculi*, Kluwer, Dordrecht, 1988.

[41] Wroński A., 'Reflections and distensions of BCK-algebras', *Math. Japonica* 28 (1983), 215–225.

[42] Yetter D. N., 'Quantales and (noncommutative) linear logic', *J. Symbolic Logic* 55 (1990), 41–64.

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