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Equational Bases for Joins of Residuated-lattice Varieties

Abstract. Given a positive universal formula in the language of residuated lattices, we construct a recursive basis of equations for a variety, such that a subdirectly irreducible residuated lattice is in the variety exactly when it satisfies the positive universal formula. We use this correspondence to prove, among other things, that the join of two finitely based varieties of commutative residuated lattices is also finitely based. This implies that the intersection of two finitely axiomatized substructural logics over $\mathbf{FL}^+$ is also finitely axiomatized. Finally, we give examples of cases where the join of two varieties is their Cartesian product.

Keywords: residuated lattices, positive universal formulas, joins of varieties, basis of equations

1. Introduction

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice; $\langle L, \cdot, e\rangle$ is a monoid; and for all $a, b, c \in L$,

$$a \cdot b \leq c \;\Leftrightarrow\; a \leq c/b \;\Leftrightarrow\; b \leq a\backslash c.$$

It is not hard to see that $\mathcal{RL}$, the class of all residuated lattices, is equationally definable. Equivalently, it is a variety, i.e., a class of algebras closed under the formation of homomorphic images, subalgebras, and direct products.

Residuated lattices were first introduced by M. Ward and R. P. Dilworth, in [11], in an attempt to generalize properties of ideal lattices of rings. In their original definition they stipulated that multiplication is commutative and the multiplicative identity is the top element of the lattice reduct. The definition that we follow is due to K. Blount and C. Tsinakis (see [4]), who first studied the structure theory of residuated lattices in their generality. For a survey of residuated lattices, see [8].

Residuated lattices include many well-studied and diverse classes of algebras, such as lattice-ordered groups, Brouwerian algebras and generalized

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MV-algebras; see [7] for the latter. Moreover, they constitute algebraic semantics for the positive (without negation and 0) fragment $\mathbf{FL}^+$ of the full Lambek calculus $\mathbf{FL}$. Substructural propositional logics that contain $\mathbf{FL}^+$ are in a one-to-one correspondence with subvarieties of $\mathcal{RL}$; for connections between substructural logics and residuated lattices, see [10], [9] and [6].

A variety $\mathcal{V}$ is called a *discriminator variety*, if there exists a term $t(x, y, z)$ in the language of $\mathcal{V}$ such that, if an algebra $\mathbf{A}$ of $\mathcal{V}$ is subdirectly irreducible, then $t(a, a, c) = c$ and $t(a, b, c) = a$, for all $a, b, c \in \mathbf{A}$ with $a \neq b$. If $\mathcal{V}$ is a discriminator variety, to every first order formula corresponds a variety with the property that a subdirectly irreducible algebra is in the variety iff it satisfies the first order formula. In this case it is easy to construct an equational basis for the variety generated by the class of all models of a first order formula. Moreover, all subdirectly irreducible algebras are simple. Residuated lattices do not form a discriminator variety, since e.g. not all subdirectly irreducible residuated lattices are simple. Nevertheless, a similar correspondence can be developed for *positive universal* first order formulas.

We begin with some preliminary definitions about residuated lattices and a number of basic facts about their structure theory.

In section 3, we construct an equational basis for the variety generated by an arbitrary positive universal class of residuated lattices, in a recursive way. The main tool in the proof is the lattice isomorphism between the congruence relations of a residuated lattice and its convex normal subalgebras, developed in [4]. Even though the basis produced is infinite, it reduces to a finite one for certain classes of residuated lattices.

In section 4, we apply the above correspondence to obtain an equational basis for the join of two residuated lattice varieties. In particular, we provide varieties, including the variety of commutative residuated lattices, such that the join of any two of their finitely based subvarieties is also finitely based. This translates, on the logic side, to the fact that the intersection of two finitely axiomatized substructural propositional logics that include $\mathbf{FL_e}^+$ is also finitely axiomatized. See [6] for the precise connection.

Finally, in section 5, we give examples where the join of two varieties is their Cartesian product.

2. Preliminaries

We assume familiarity with basic definitions and results from universal algebra; a standard reference on the subject is [3]. We will use the same symbol, $\vee$, for the join operation in a residuated lattice, the disjunction of two first order formulas in the language of residuated lattices, and for the join of two residuated-lattice varieties in the subvariety lattice. This multiple usage will cause no confusion since it will be clear from the context which interpretation will be intended.

To reduce the number of parentheses in a residuated-lattice term, we assume that multiplication has priority over the other operations and that the division operations have priority over the lattice operations. So, for example, $x/yz \wedge u\backslash v$ means $[x/(yz)] \wedge (u\backslash v)$.

We start with some definitions and facts about the structure of residuated lattices, given in [4].

LEMMA 2.1. *[4] Residuated lattices satisfy the following identities:*

(1) $x(y \vee z) \approx xy \vee xz$ *and* $(y \vee z)x \approx yx \vee zx$

(2) $x\backslash(y \wedge z) \approx (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x \approx (y/x) \wedge (z/x)$

(3) $x/(y \vee z) \approx (x/y) \wedge (x/z)$ *and* $(y \vee z)\backslash x \approx (y\backslash x) \wedge (z\backslash x)$

(4) $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$

(5) $x(y/z) \leq xy/z$ *and* $(z\backslash y)x \leq z\backslash yx$

(6) $(x/y)/z \approx x/zy$ *and* $z\backslash(y\backslash x) \approx yz\backslash x$

(7) $x\backslash(y/z) \approx (x\backslash y)/z$

(8) $x/e \approx x \approx e\backslash x$

(9) $e \leq x/x$ *and* $e \leq x\backslash x$

Let $\mathbf{L}$ be a residuated lattice and Y a set of variables. For $y \in Y$ and $x \in L \cup Y \cup \{e\}$, where e is the constant in the language of residuated lattices, we define the polynomials

$$\rho_x(y) = xy/x \wedge e \text{ and } \lambda_x(y) = x\backslash yx \wedge e,$$

the *right* and *left conjugate* of y with respect to x. An *iterated conjugate* is a composition of a number of left and right conjugates - we consider composition of conjugates with respect to their arguments. For X, A subsets of $L \cup Y \cup \{e\}$, and for $m \in \mathbb{N}$, we define the sets $\Gamma_X^0 = \{\lambda_e\}$,

$$\Gamma_X^m = \{\gamma_{x_1} \circ \gamma_{x_2} \circ \ldots \gamma_{x_m} \mid \gamma_{x_i} \in \{\lambda_{x_i}, \rho_{x_i}\},\ x_i \in X \cup \{e\},\ i \in \mathbb{N}\},$$

$$\Gamma_X^m(A) = \{\gamma(a) \mid \gamma \in \Gamma_X^m,\ a \in A\},$$

$$\Gamma_X = \bigcup\{\Gamma_X^n \mid n \in \mathbb{N}\},$$

$$\Gamma_X(A) = \bigcup\{\Gamma_X^n(A) \mid n \in \mathbb{N}\}.$$

Note that, if $\mathbf{L}$ is a residuated lattice, then $\lambda_e(x) = \rho_e(x) = x \wedge e$, $\gamma(x) \leq e$ and $\gamma(e) = e$, for all $x \in L$ and for every iterated conjugate $\gamma \in \Gamma_L$. In particular, if x is *negative*, i.e., $x \leq e$, then $\lambda_e(x) = \rho_e(x) = x$. If $\mathbf{L}$ is *commutative*, i.e., it has a commutative monoid reduct, then $x \wedge e \leq \gamma(x)$, for all $x \in L$ and $\gamma \in \Gamma_L$.

A subset N of L is called *normal* in $\mathbf{L}$, if it is closed under conjugation, i.e., $\gamma(N) \subseteq N$, for all $\gamma \in \Gamma_L$. A subset X of L is called *convex* in $\mathbf{L}$, if for every x, y in X and z in L, $x \leq z \leq y$ implies that z is in X.

The *negative cone* of a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is the algebra $\mathbf{L}^- = \langle L^-, \wedge, \vee, \cdot, \backslash_{\mathbf{L}^-}, /_{\mathbf{L}^-}, e\rangle$, where $L^- = \{x \in L \,|\, x \leq e\}$, $x\backslash_{\mathbf{L}^-} y = x\backslash y \wedge e$ and $x/_{\mathbf{L}^-} y = x/y \wedge e$. It is easy to see that $\mathbf{L}^-$ is a residuated lattice.

THEOREM 2.2. *[4], see also [6].*

(1) *The convex normal subalgebras of a residuated lattice* $\mathbf{L}$ *form a lattice* $\mathbf{CNS}(\mathbf{L})$, *which is isomorphic to the congruence lattice of* $\mathbf{L}$, *via* $S \mapsto \theta_S = \{(a, b) \in L^2 | \, (a/b \wedge e)(b/a \wedge e) \in H\}$ *and* $\theta \mapsto [e]_\theta$, *where* $[e]_\theta$ *is the* θ*-class of* e. *Moreover, the principal congruence generated by* (a, e) *corresponds to the convex normal subalgebra generated by* a.

(2) *The convex, normal in* $\mathbf{L}$ *submonoids of the negative cone of a residuated lattice* $\mathbf{L}$ *form a lattice isomorphic to* $\mathbf{CNS}(\mathbf{L})$, *via* $S \mapsto S^-$ *and* $M \mapsto S_M = \{x \,|\, m \leq x \leq e/m,\ m \in M\}$. *The convex normal submonoid generated by a negative element corresponds to the convex normal subalgebra generated by that element.*

(3) *If* $A \subseteq L^-$, *then the convex normal submonoid* $M(A)$ *of* $\mathbf{L}^-$ *generated by* A *is given by* $M(A) = \{x \mid g_1 g_2 \cdots g_n \leq x \leq e,\ g_i \in \Gamma_L(A), n \in \mathbb{N}\}$.

Let $\mathcal{RL}^C$ be the variety generated by the class of all totally ordered residuated lattices.

THEOREM 2.3. *[4] The equation* $\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx e$ *constitutes an equational basis for* $\mathcal{RL}^C$.

Let $Can\mathcal{RL}$ be the class of residuated lattices that have a cancellative monoid reduct. It is easy to see that $Can\mathcal{RL}$ is a variety and it is axiomatized, relative to $\mathcal{RL}$, by the equations $y\backslash yx \approx x \approx xy/y$. For a study of cancellative residuated lattices, see [2].

The variety of *lattice-ordered groups* (*ℓ-groups*) is term equivalent to the subvariety $\mathcal{LG}$ of residuated lattices defined, relative to $\mathcal{RL}$, by the equation $x(e/x) \approx e$. The term equivalence is given by $x^{-1} = e/x$ and $x/y = xy^{-1}$, $x\backslash y = x^{-1}y$. Obviously, every ℓ-group, as well as its negative cone, is a cancellative residuated lattice. For basic properties of ℓ-groups, see [1].

A residuated lattice is called *integral*, if it satisfies the equation $x \wedge e \approx x$, or, equivalently, if the multiplicative identity is the top element of the lattice reduct. We denote the variety of integral residuated lattices by $\mathcal{IRL}$.

3. Varieties of residuated lattices generated by positive universal classes

An *open positive universal formula* in a given language is an open first order formula that can be written as a disjunction of conjunctions of equations in the language. A *(closed) positive universal formula* is the universal closure of an open one. A *positive universal class* is the collection of all models of a set of positive universal formulas.

LEMMA 3.1. *Every open (closed) positive universal formula, ϕ, in the language of residuated lattices is equivalent to (the universal closure of) a disjunction ϕ' of equations of the form $e \approx r$, where the evaluation of the term r is negative in all residuated lattices.*

PROOF. Every equation $t \approx s$ in the language of residuated lattices, where t, s are terms, is equivalent to the conjunction of the two inequalities $t \leq s$ and $s \leq t$, which in turn is equivalent to the conjunction of the inequalities $e \leq s/t$ and $e \leq t/s$. Moreover, a conjunction of a finite number of inequalities of the form $e \leq t_i$, for $1 \leq i \leq n$ is equivalent to the inequality $e \leq t_1 \wedge \ldots \wedge t_n$. So, a conjunction of a finite number of equations is equivalent to a single inequality of the form $e \leq p$, which in turn is equivalent to the equation $e \approx r$, where $r = p \wedge e$. ∎

Recall the definition of the set of conjugate terms Γ_Y^m. For a positive universal formula $\phi(\bar{x})$ and a countable set of variables Y disjoint from $\bar{x}$, we define the sets of residuated-lattice equations

$$B_Y^m(\phi'(\bar{x})) = \{e \approx \gamma_1(r_1(\bar{x})) \vee \ldots \vee \gamma_n(r_n(\bar{x})) \mid \gamma_i \in \Gamma_Y^m\}$$

and

$$B_Y(\phi'(\bar{x})) = \bigcup \{B_Y^m(\phi'(\bar{x})) \mid m \in \mathbb{N}\},$$

where $m \in \mathbb{N}$ and

$$\phi'(\bar{x}) = (r_1(\bar{x}) \approx e) \vee \ldots \vee (r_n(\bar{x}) \approx e)$$

is the formula equivalent to $\phi(\bar{x})$, given in Lemma 3.1.

It is clear that $B_Y^m(\phi'(\bar{x}))$ is an infinite set for $m \geq 1$. Nevertheless, if we enumerate $Y = \{y_i \mid i \in I\}$ and insist that the indices of the conjugating elements of Y in $\gamma_1, \gamma_2, \ldots, \gamma_n$ appear in the natural order and they form an initial segment of the natural numbers, then we obtain a finite subset of $B_Y^m(\phi'(\bar{x}))$, which is equivalent to the latter. In that respect $B_Y^m(\phi'(\bar{x}))$ is essentially finite.

LEMMA 3.2. *Let* $\mathbf{L}$ *be a residuated lattice and* $A_1, \ldots, A_n$ *finite subsets of* L. *If* $a_1 \vee \ldots \vee a_n = e$, *for all* $a_i \in A_i$, $i \in \{1, \ldots, n\}$, *then for all* $i \in \{1, \ldots, n\}$, $n_i \in \mathbb{N}$, *and for all* $a_{i1}, a_{i2}, \ldots, a_{in_i} \in A_i$, *we have* $p_1 \vee \ldots \vee p_n = e$, *where* $p_i = a_{i1}a_{i2} \cdots a_{in_i}$.

PROOF. The proof is a simple induction argument. For the basis of induction and for the induction step we use Lemma 2.1(1). If $a \vee b = a \vee c = e$, then $e = (a \vee b)(a \vee c) = a^2 \vee ac \vee ba \vee bc \leq a \vee bc \leq a \vee b = e$. So, $a \vee bc = e$. ■

THEOREM 3.3. *Let* ϕ *be an open positive universal formula in the language of residuated lattices and* $\mathbf{L}$ *a residuated lattice.*

(1) *If* $\mathbf{L}$ *satisfies* $(\forall \bar{x})(\phi(\bar{x}))$, *then* $\mathbf{L}$ *satisfies* $(\forall \bar{x}, \bar{y})(\varepsilon(\bar{x}, \bar{y}))$, *for all* $\varepsilon(\bar{x}, \bar{y})$ *in* $B_Y(\phi'(\bar{x}))$ *and* $\bar{y} \in Y^l$, *for some appropriate* $l \in \mathbb{N}$.

(2) *If* $\mathbf{L}$ *is subdirectly irreducible, then* $\mathbf{L}$ *satisfies* $(\forall \bar{x})(\phi(\bar{x}))$ *iff* $\mathbf{L}$ *satisfies the equation* $(\forall \bar{x}, \bar{y})(\varepsilon(\bar{x}, \bar{y}))$, *for all* $\varepsilon(\bar{x}, \bar{y})$ *in* $B_Y(\phi'(\bar{x}))$ *and* $\bar{y} \in Y^l$.

PROOF. 1) Let $\mathbf{L}$ be a residuated lattice that satisfies $(\forall \bar{x})(\phi(\bar{x}))$. Moreover, let $\varepsilon(\bar{x}, \bar{y})$ be an equation in $B_Y(\phi'(\bar{x}))$, $\bar{c} \in L^k$ and $\bar{d} \in L^l$, for some appropriate $k, l \in \mathbb{N}$. We will show that $\varepsilon(\bar{c}, \bar{d})$ holds in $\mathbf{L}$. Since $\mathbf{L}$ satisfies $(\forall \bar{x})(\phi(\bar{x}))$, $\phi'(\bar{c})$ holds in $\mathbf{L}$. So, $r_i(\bar{c}) = e$, for some $i \in \{1, 2, \ldots, n\}$; hence $\gamma(r_i(\bar{c})) = e$, for all $\gamma \in \Gamma_Y$. Consequently, $\varepsilon(\bar{c}, \bar{d})$ holds.

2) Let $\mathbf{L}$ be a subdirectly irreducible residuated lattice that satisfies $B_Y(\phi'(\bar{x}))$, and let $\bar{c} \in L^k$ and $a_i = r_i(\bar{c})$. We will show that $a_i = e$ for some i.

Let $b \in M(a_1) \cap \cdots \cap M(a_n)$, where $M(x)$ symbolizes the convex normal submonoid of the negative cone generated by x. Using Theorem 2.2(3), we have that for all $i \in \{1, 2, \ldots, n\}$, $\prod_{j=1}^{s_i} g_{ij} \leq b \leq e$, for some $s_1, s_2, \ldots, s_n \in \mathbb{N}$ and $g_{i1}, g_{i2}, \ldots, g_{is_i} \in \Gamma_L(a_i)$. So,

$$\prod_{j=1}^{s_1} g_{1j} \vee \prod_{j=1}^{s_2} g_{2j} \vee \ldots \vee \prod_{j=1}^{s_n} g_{2j} \leq b \leq e.$$

On the other hand,

$$\gamma_1(a_1) \vee \gamma_2(a_2) \vee \ldots \vee \gamma_n(a_n) = e,$$

for all $\gamma_i \in \Gamma_L$, since every equation of $B_Y(\phi'(\bar{x}))$ holds in $\mathbf{L}$. So, for all $i \in \{1, 2, \ldots, n\}$ and $g_i \in \Gamma_L(a_i)$, we have $g_1 \vee g_2 \vee \ldots \vee g_n = e$ and, by Lemma 3.2,

$$\prod_{j=1}^{s_1} g_{1j} \vee \prod_{j=1}^{s_2} g_{2j} \vee \ldots \vee \prod_{j=1}^{s_n} g_{2j} = e.$$

Thus, $b = e$ and $M(a_1) \cap \cdots \cap M(a_n) = \{e\}$.

Using the lattice isomorphisms of Theorem 2.2, we obtain

$$\Theta(a_1, e) \cap \Theta(a_2, e) \cap \cdots \cap \Theta(a_n, e) = \Delta,$$

where $\Theta(a, e)$ denotes the principal congruence generated by (a, e) and Δ denotes the diagonal congruence. Since $\mathbf{L}$ is subdirectly irreducible, this implies that $\Theta(a_i, e) = \Delta$, i.e., $a_i = e$, for some i. Thus, $(\forall \bar{x})(\phi'(\bar{x}))$ holds in $\mathbf{L}$. ■

COROLLARY 3.4. *Let $\{\phi_i | i \in I\}$ be a collection of positive universal formulas. Then, $\bigcup\{B(\phi_i') \mid i \in I\}$ is an equational basis for the variety generated by the (subdirectly irreducible) residuated lattices that satisfy ϕ_i, for every $i \in I$.*

PROOF. By the previous theorem a subdirectly irreducible residuated lattice satisfies ϕ_i iff it satisfies all the equations in $B(\phi_i')$, so

$$\begin{aligned}(\mathrm{Mod}(\textstyle\bigcup\{\phi_i \mid i \in I\}))_{SI} &= \textstyle\bigcap\{(\mathrm{Mod}(\phi_i))_{SI} \mid i \in I\} \\ &= \textstyle\bigcap\{(\mathrm{Mod}(B(\phi_i')))_{SI} \mid i \in I\} \\ &= (\mathrm{Mod}(\textstyle\bigcup\{B(\phi_i') \mid i \in I\}))_{SI},\end{aligned}$$

where for every variety $\mathcal{V}$ and every set of equations $\mathcal{E}$, $\mathcal{V}_{SI}$ denotes the class of all subdirectly irreducible algebras of $\mathcal{V}$ and $\mathrm{Mod}(\mathcal{E})$ denotes the variety of all models of $\mathcal{E}$. Consequently,

$$\begin{aligned}\mathbf{V}((\mathrm{Mod}(\textstyle\bigcup\{\phi_i \mid i \in I\}))_{SI}) &= \mathbf{V}((\mathrm{Mod}(\textstyle\bigcup\{B(\phi_i') \mid i \in I\}))_{SI}) \\ &= \mathrm{Mod}(\textstyle\bigcup\{B(\phi_i') \mid i \in I\}),\end{aligned}$$

where $\mathbf{V}(\mathcal{K})$ denotes the variety generated by a class of algebras $\mathcal{K}$. ■

Note that the equational basis for the variety generated by the models of a recursive positive universal class is recursive. In particular, the equational basis is recursive if the positive universal class is defined by a single formula.

The basis given in Theorem 3.3 is by no means of minimal cardinality. It is always infinite, while, as it can be easily seen, for commutative subvarieties it simplifies to the conjunction of commutativity and the equation of $B^0(\phi')$. So, for example, the variety generated by the commutative residuated lattices, whose underlying set is the union of its positive and negative cone, is axiomatized by $xy \approx yx$ and $e \approx (x \wedge e) \vee (e/x \wedge e)$.

4. Equational basis for joins of subvarieties

In what follows, we apply the correspondence established above to obtain an equational basis for the join of a finite number of residuated lattice varieties. Moreover, we provide sufficient conditions for a variety of residuated lattices, in order for the join of any two of its finitely based subvarieties to be finitely based, as well.

COROLLARY 4.1. *If the varieties $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ are axiomatized by the sets of equations $B_1, B_2, \ldots, B_n$, where the sets of all variables in each B_i are pairwise disjoint, then $\bigcup\{B(\phi_i') \mid i \in I\}$ is an equational basis for the join $\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n$, where ϕ_i ranges over all possible disjunctions of n equations, one from each of $B_1, B_2, \ldots, B_n$.*

PROOF. The variety $\mathcal{RL}$ is congruence distributive, because its members have lattice reducts. So, by Jónsson's Lemma, a subdirectly irreducible residuated lattice in the join of finitely many varieties is in one of the varieties. Moreover, by the definition of ϕ_i, it is clear that a subdirectly irreducible residuated lattice satisfies ϕ_i, for all $i \in I$, if and only if it is in one of the varieties $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$. So,

$$\begin{aligned}\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n &= \mathbf{V}((\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n)_{SI}) \\ &= \mathbf{V}((\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_n)_{SI}) \\ &= \mathbf{V}(\mathrm{Mod}(\bigcup\{\phi_i \mid i \in I\})_{SI}) \\ &= \mathrm{Mod}(\bigcup\{B(\phi_i') \mid i \in I\}).\end{aligned}$$ ■

COROLLARY 4.2. *The join of finitely many recursively based varieties of residuated lattices is recursively based.*

In the case of the join of finitely based varieties the situation is simpler.

COROLLARY 4.3. *If $B_1, B_2, \ldots, B_n$ are finite equational bases for the varieties $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, then $B(\phi')$ is an equational basis for the join $\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n$ of the varieties, where $\phi = (\bigwedge B_1 \vee \bigwedge B_2 \vee \ldots \vee \bigwedge B_n)$ and for every $i \in \{1, 2, \ldots, n\}$, $\bigwedge B_i$ denotes the conjunction of the equations in B_i.*

PROOF. Retaining the notation of Corollary 4.1, we see that $\bigcup\{\phi_i \mid i \in I\}$ is equivalent to ϕ and $\bigcup\{B(\phi_i') \mid i \in I\}$ is equivalent to $B(\phi')$. ■

We define the varieties

$$\mathcal{C}_k^-\mathcal{RL} = \mathrm{Mod}((x \wedge e)^k y \approx y(x \wedge e)^k),$$

for all $k \in \mathbb{N}$ and the variety

$$\mathcal{C}an\mathcal{C}_1^-\mathcal{RL} = \mathcal{C}an\mathcal{RL} \cap \mathcal{C}_1^-\mathcal{RL}.$$

For all natural numbers m, n, we set $B_n^m = B^m(\phi'_n)$, where $\phi'_n = (x_1 \approx e) \vee (x_2 \approx e) \vee \ldots \vee (x_n \approx e)$. We say that a residuated lattice $\mathbf{L}$ satisfies the implication $B_n^m \to B_n^k$ iff, for all $\bar{a} \in L^n$, if $e = r(\bar{a})$ for all $(e \approx r) \in B_n^m$, then $e = r(\bar{a})$ for all $(e \approx r) \in B_n^k$.

THEOREM 4.4.

(1) *If $\mathcal{V}_1$, $\mathcal{V}_2$ are finitely based varieties of residuated lattices that satisfy the implication $B_2^m \to B_2^{m+1}$, for some natural number m, then their join is also finitely based.*

(2) *The join of any pair of finitely based subvarieties of the varietey $\mathcal{LG} \vee \mathcal{C}an\mathcal{C}_1^-\mathcal{RL} \vee \mathcal{RL}^C \vee \mathcal{C}_k^-\mathcal{RL}$ is also finitely based, for every $k \geq 1$.*

PROOF. 1) First note that, since B_2^m is equivalent to a finite set of equations and since the conjunction of a finite set of equations is equivalent to a single equation, B_2^m is equivalent to a single equation. All subdirectly irreducible algebras in the join $\mathcal{V}_1 \vee \mathcal{V}_2$ coincide with the subdirectly irreducible algebras in the union $\mathcal{V}_1 \cup \mathcal{V}_2$, so they satisfy the implication $B_2^m \to B_2^{m+1}$. Since all residuated lattices in the join $\mathcal{V}_1 \vee \mathcal{V}_2$ are subdirect products of subdirectly irreducible algebras, and quasi-equations are preserved under products and subalgebras, the join satisfies the implication. Moreover, if B_1, B_2 are finite equational bases for $\mathcal{V}_1$, $\mathcal{V}_2$, respectively, then $B(\phi')$ is an equational basis for $\mathcal{V}_1 \vee \mathcal{V}_2$, where $\phi = \bigwedge B_1 \vee \bigwedge B_2$. So, the implication is a consequence of a finite subset B of $B(\phi')$, by compactness. It is clear then, that $B \cup B^m(\phi')$ is a finite equational basis for $\mathcal{V}_1 \vee \mathcal{V}_2$.

2) Note that $\mathcal{LG}$ satisfies $\lambda_z(\lambda_w(x)) \approx \lambda_{wz}(x)$ and $\rho_z(x) \approx \lambda_{z^{-1}}(x)$, where $z^{-1} = z\backslash e$ since

$$\begin{aligned}
\lambda_z(\lambda_w(x)) &= z\backslash(w\backslash xw \wedge e)z \wedge e \\
&= z^{-1}(w^{-1}xw \wedge e)z \wedge e \\
&= z^{-1}w^{-1}xwz \wedge z^{-1}z \wedge e \\
&= (wz)^{-1}xwz \wedge e \\
&= wz\backslash xwz \wedge e \\
&= \lambda_{wz}(x)
\end{aligned}$$

and

$$\rho_z(x) = zx/z \wedge e = zxz^{-1} \wedge e = z^{-1}\backslash xz^{-1} \wedge e = \lambda_{z^{-1}}(x).$$

So, $\lambda_z(\lambda_w(x \wedge e)) \approx \lambda_{wz}(x \wedge e)$ and $\rho_z(x \wedge e) \approx \lambda_{z^{-1}}(x \wedge e)$ hold in $\mathcal{LG}$. The same two equations hold in $\mathcal{C}an\mathcal{C}_1^-\mathcal{RL}$, since for any negative element a and any element b,

$$\lambda_b(a) = b\backslash ab \wedge e = b\backslash ba \wedge e = a \wedge e = a$$

and $\rho_b(a) = a \wedge e = a$. Thus, $\mathcal{LG}$ and $\mathcal{C}an\mathcal{C}_1^-\mathcal{RL}$ satisfy $B_2^1 \to B_2^2$.

On the other hand, the variety $\mathcal{RL}^C$ satisfies the implication

$$x \vee y = e \Rightarrow \lambda_z(x) \vee \rho_w(y) = e,$$

by Theorem 2.3. We will show that the same implication holds in $\mathcal{C}_k^-\mathcal{RL}$. If $x \vee y = e$, then, by Lemma 3.2, $x^k \vee y^k = e$. Since, $x \leq e$, we have $x^k \leq x \leq e$; so, for all z, $x^k z = z x^k$, hence $x^k \leq z\backslash x^k z$ and $x^k \leq z x^k/z$. Since $x^k \leq e$, this implies

$$x^k \leq z\backslash x^k z \wedge e \text{ and } x^k \leq z x^k/z \wedge e,$$

i.e., $x^k \leq \lambda_z(x^k)$ and $x^k \leq \rho_z(x^k)$, for all z. Thus, $\lambda_z(x^k) \vee \rho_w(y^k) = e$. Moreover, left and right conjugates are increasing in their arguments, so $\lambda_z(x) \vee \rho_w(y) = e$. So, $\mathcal{RL}^C$ and $\mathcal{C}_k^-\mathcal{RL}$ satisfy the implication $B_2^0 \to B_2^1$, hence also the implication $B_2^1 \to B_2^2$.

Using the same argumentation as in the proof of (1), it is easy to see that the join $\mathcal{LG} \vee \mathcal{C}an\mathcal{C}_1^-\mathcal{RL} \vee \mathcal{RL}^C \vee \mathcal{C}_k^-\mathcal{RL}$ of the four varieties satisfies the implication $B_2^1 \to B_2^2$. Consequently, every subvariety of the join satisfies the implication, as well. Statement (2) then follows from (1). ∎

COROLLARY 4.5. *The join of two finitely based commutative varieties of residuated lattices is finitely based.*

It is an open problem whether the join of two finitely based varieties of residuated lattices is finitely based.

COROLLARY 4.6. *The intersection of two finitely axiomatized substructural logics over* $\mathbf{FL_e}^+$ *or* $\mathbf{FL_e}$ *is finitely axiomatized, as well.*

5. Direct product decompositions

Certain pairs of subvarieties of $\mathcal{RL}$ are so different that their join decomposes into their Cartesian product - the class of all Cartesian products of algebras of the two varieties up to isomorphism. Such a pair is the variety of ℓ-groups and the variety of their negative cones. The following proposition is in the folklore of the subject and allows us to obtain such decompositions given two projection terms.

PROPOSITION 5.1. *Let $\mathcal{V}_1, \mathcal{V}_2$ be subvarieties of $\mathcal{RL}$ with equational bases B_1 and B_2, respectively, and let $\pi_1(x), \pi_2(x)$ be unary terms, such that $\mathcal{V}_1$ satisfies $\pi_1(x) \approx x$ and $\pi_2(x) \approx e$ and $\mathcal{V}_2$ satisfies $\pi_1(x) \approx e$ and $\pi_2(x) \approx x$. Then $\mathcal{V}_1 \vee \mathcal{V}_2 = \mathcal{V}_1 \times \mathcal{V}_2$ and the following list, $B_1 * B_2$, of equations is an equational basis for the variety $\mathcal{V}_1 \vee \mathcal{V}_2$.*

(1) $\pi_1(x) \cdot \pi_2(x) \approx x$

(2) $\pi_i(\pi_i(x)) \approx \pi_i(x)$ *and* $\pi_i(\pi_j(x)) \approx e$, *for* $i \neq j$; $i, j \in \{1, 2\}$

(3) $\pi_i(x \star y) \approx \pi_i(x) \star \pi_i(y)$, *where* $\star \in \{\wedge, \vee, \cdot, /, \backslash\}$ *and* $i \in \{1, 2\}$

(4) $\varepsilon(\pi_1(x_1), \ldots, \pi_1(x_n))$, *for all equations* $\varepsilon(x_1, \ldots, x_n)$ *of* B_1

(5) $\varepsilon(\pi_2(x_1), \ldots, \pi_2(x_n))$, *for all equations* $\varepsilon(x_1, \ldots, x_n)$ *of* B_2

The same decomposition holds for the join of any pair of subvarieties of $\mathcal{V}_1, \mathcal{V}_2$. Note that if B_1, B_2 are finite, then so is $B_1 * B_2$.

PROOF. It is easy to see that the equations in $B_1 * B_2$ hold both in $\mathcal{V}_1$ and $\mathcal{V}_2$, hence they hold in $\mathcal{V}_1 \vee \mathcal{V}_2$, also. Moreover, $\mathcal{V}_1 \times \mathcal{V}_2 \subseteq \mathcal{V}_1 \vee \mathcal{V}_2$. Finally, suppose that the residuated lattice A satisfies the equations $B_1 * B_2$; we will show that A is in $\mathcal{V}_1 \times \mathcal{V}_2$.

Define $A_1 = \{x \in A | \pi_2(x) = e\}$ and $A_2 = \{x \in A | \pi_1(x) = e\}$. Using (3) and (1), it is easy to see that A_1 and A_2 are subalgebras of A. Define the map $f : A \to A_1 \times A_2$, by $f(x) = (\pi_1(x), \pi_2(x))$. It is easy to check that f is well defined, using (2); that it is a homomorphism, using (3); one-to-one, using (1); and onto, using (3) and (1). Thus, A is isomorphic to $A_1 \times A_2 \in \mathcal{V}_1 \times \mathcal{V}_2$. ■

The first example of a pair of two varieties whose join is their Cartesian product is the variety of ℓ-groups and the variety of integral residuated lattices.

COROLLARY 5.2. *The join of $\mathcal{LG}$ and $\mathcal{IRL}$ is equal to their product. Moreover, if $B_1 = \{(e/x)x \approx e\}$ and $B_2 = \{e \wedge x \approx x\}$, then $B_1 * B_2$ is an equational basis for $\mathcal{LG} \vee \mathcal{IRL}$.*

PROOF. Let $\pi_1(x) = e/(e/x)$ and $\pi_2(x) = (e/x)x$. It is easy to see that $\mathcal{LG}$ satisfies $e/(e/x) \approx e(ex^{-1})^{-1} \approx x$ and $(e/x)x \approx x^{-1}x \approx e$. Moreover, $\mathcal{IRL}$ satisfies $e/x \approx e$, so it also satisfies $(e/x)x \approx ex \approx x$ and $e/(e/x) \approx e$. ■

It is shown in [2] that the class $\mathcal{LG}^-$, consisting of the negative cones of ℓ-groups, is a variety. As an application of the previous example we have $\mathcal{LG} \vee \mathcal{LG}^- = \mathcal{LG} \times \mathcal{LG}^-$.

Moreover, the class $\mathcal{IGMV} = \mathrm{Mod}(x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x)$ of integral generalized MV-algebras is easily shown to be a subvariety of $\mathcal{IRL}$. So, Corollary 5.2 provides an equational basis for $\mathcal{LG} \vee \mathcal{IGMV} = \mathcal{LG} \times \mathcal{IGMV}$. In [7], generalized MV-algebras are studied and an alternative, simpler equational basis is given for $\mathcal{LG} \vee \mathcal{IGMV}$.

The second example involves certain finitely generated varieties, that actually are in the variety $\mathcal{RL}^C$.

For every natural number n, set $T_n = \{T, e\} \cup \{u_k \mid k \in \mathbb{N}_n\}$, where $\mathbb{N}_n = \{1, 2, \ldots n\}$. Define an order relation on T_n by

$$u_k \leq u_l \text{ iff } k \geq l, \text{ and } u_k < e < T, \text{ for all natural numbers } k, l \leq n.$$

Also, define multiplication by

$$xT = Tx = x, \text{ for all } x \neq e \text{ and } u_k u_l = u_{min\{n,k+l\}}, \text{ for all } k, l \in \mathbb{N}_n;$$

and the two division operations by

$$x/y = \bigvee\{z \in T_n \mid zy \leq x\} \text{ and } y\backslash x = \bigvee\{z \in T_n \mid yz \leq x\}.$$

It is easy to verify that $\mathbf{T}_n = \langle T_n, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice.

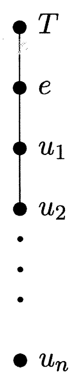

Figure 1. The residuated lattice $\mathbf{T}_n$.

The finitely generated varieties $\mathbf{V}(\mathbf{T}_n)$ are distinct atoms in the subvariety lattice of residuated lattices. For a discussion on minimal residuated-lattice varieties we refer the reader to [5].

COROLLARY 5.3. *Let $\mathcal{V}$ be a variety of residuated lattices that satisfies the identities $(e/(e/x))^n \leq x$ and $(x \wedge e)^n \approx (x \wedge e)^{n+1}$, for some $n \in \mathbb{N}$. Then, $\mathcal{C}an\mathcal{IRL} \vee \mathcal{V} = \mathcal{C}an\mathcal{IRL} \times \mathcal{V}$.*

PROOF. Let $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e$ and $\pi_2(x) = (e/(e/x))^n \vee x$. Note that $Can\mathcal{IRL}$ satisfies

$$\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e \approx (e \wedge x) \wedge e \approx x$$

and

$$\pi_2(x) = (e/(e/x))^n \vee x \approx e^n \vee x \approx e.$$

On the other hand, $\mathcal{V}$ satisfies

$$\pi_2(x) = (e/(e/x))^n \vee x \approx x$$

and

$$\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e \approx ((e \wedge x)^n/(e \wedge x)^n) \wedge e \approx e. \qquad \blacksquare$$

COROLLARY 5.4. *The join of the varieties* $\mathcal{V} = \mathbf{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, \ldots, \mathbf{T}_{i_k})$ *and* $Can\mathcal{IRL}$ *is equal to their Cartesian product, for all* $k, i_1, \ldots, i_k \in \mathbb{N}$.

PROOF. In view of the last corollary, we need only verify that $\mathcal{V}$ satisfies the identities $(x \wedge e)^n \approx (x \wedge e)^{n+1}$ and $(e/(e/x))^n \leq x$, for some $n \in \mathbb{N}$.

If $n \geq m$, then $(e \wedge x)^{n+1} = (e \wedge x)^m = (e \wedge x)^n$, for $x \in \mathbf{T}_m$. Moreover, $(e/(e/T))^n \leq T$, $(e/(e/e))^n = e$ and $(e/(e/x))^n = (e/T)^n = u^m \leq x$, for $x < e$. If $n \geq \max\{i_1, \ldots, i_k\}$, then $\mathcal{V} = \mathbf{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, \ldots, \mathbf{T}_{i_k})$ satisfies both identities. ■

Note that $\mathcal{IRL} \vee \mathbf{V}(\mathbf{T}_1) \neq \mathcal{IRL} \times \mathbf{V}(\mathbf{T}_1)$, since $\mathbf{A}$ is in $\mathbf{S}(\mathbf{2} \times \mathbf{T}_1)$ but not in $\mathcal{IRL} \times \mathbf{V}(\mathbf{T}_1)$, where $A = \{(1,T), (1,e), (1,u_1), (0,u_1)\}$ and $2 = \{0,1\}$.

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