Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 283 (2005) 254–291
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generalized MV-algebras

Nikolaos Galatos [a,*], Constantine Tsinakis [b]

[a] *School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Tatsunokuchi, Ishikawa, 923-1292, Japan*
[b] *Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA*

Received 9 April 2004
Available online 18 August 2004
Communicated by Efim Zelmanov

Abstract

We generalize the notion of an MV-algebra in the context of residuated lattices to include non-commutative and unbounded structures. We investigate a number of their properties and prove that they can be obtained from lattice-ordered groups via a truncation construction that generalizes the Chang–Mundici Γ functor. This correspondence extends to a categorical equivalence that generalizes the ones established by D. Mundici and A. Dvurečenskij. The decidability of the equational theory of the variety of generalized MV-algebras follows from our analysis.

Keywords: Residuated lattice; MV-algebra; Lattice-ordered group; Nucleus; Categorical equivalence

1. Introduction

A *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice; $\langle L, \cdot, e\rangle$ is a monoid; and for all $x, y, z \in \mathbf{L}$,

$$x \cdot y \leqslant z \quad \Leftrightarrow \quad x \leqslant z/y \quad \Leftrightarrow \quad y \leqslant x\backslash z.$$

* Corresponding author.
E-mail addresses: galatos@jaist.ac.jp (N. Galatos), constantine.tsinakis@vanderbilt.edu (C. Tsinakis).
URLs: http://www.jaist.ac.jp/~galatos (N. Galatos), http://sitemason.vanderbilt.edu/site/g4VM2c (C. Tsinakis).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.07.002

Residuated lattices form a finitely based equational class of algebras (see, for example, [4]), denoted by $\mathcal{RL}$.

It is important to remark that the elimination of the requirement that a residuated lattice have a bottom element has led to the development of a surprisingly rich theory that includes the study of various important varieties of cancellative residuated lattices, such as the variety of lattice-ordered groups. See, for example, [2,4,9,12–14,18,20].

A *lattice-ordered group* (*ℓ-group*) is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, e\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, \cdot, ^{-1}, e\rangle$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of ℓ-groups is term equivalent to the subvariety, $\mathcal{LG}$, of residuated lattices defined by the equations $(e/x)x \approx e \approx x(x\backslash e)$; the term equivalence is given by $x^{-1} = e/x$ and $x\backslash y = x^{-1}y,\ y/x = yx^{-1}$. See [1] for an accessible introduction to the theory of ℓ-groups.

A *residuated bounded-lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, 0\rangle$ such that $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice and $\mathbf{L}$ satisfies the equation $x \vee 0 \approx x$. Note that $\top = 0\backslash 0 = 0/0$ is the greatest element of such an algebra. A residuated (bounded-) lattice is called *commutative* if it satisfies the equation $xy \approx yx$ and *integral* if it satisfies $x \wedge e \approx x$.

Commutative, integral residuated bounded-lattices have been studied extensively in both algebraic and logical form, and include important classes of algebras, such as the variety of MV-algebras, which provides the algebraic setting for Łukasiewicz's infinite-valued propositional logic. Several term equivalent formulations of MV-algebras have been proposed (see, for example, [8]). Within the context of commutative, residuated bounded-lattices, MV-algebras are axiomatized by the identity $(x \to y) \to y \approx x \vee y$, which is a relativized version of the law $\neg\neg x \approx x$ of double negation; in commutative residuated lattices we write $x \to y$ for the common value of $x\backslash y$ and y/x, and $\neg x$ for $x \to 0$. The term equivalence with the standard signature is given by $x \odot y = x \cdot y$, $\neg x = x \to 0$, $x \oplus y = \neg(\neg x \cdot \neg y)$ and $x \to y = \neg x \oplus y$. The appropriate non-commutative generalization of an MV-algebra is a residuated bounded-lattice that satisfies the identities $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$. These algebras have recently been considered in [10,15,16] under the name pseudo-MV-algebras.

C.C. Chang proved in [7] that if $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, e\rangle$ is a totally ordered Abelian group and $u < e$, then the residuated-bounded lattice $\Gamma(\mathbf{G}, u) = \langle [u, e], \wedge, \vee, \circ, \backslash, /, e, u\rangle$—where $x \circ y = xy \vee u$, $x\backslash y = x^{-1}y \wedge e$ and $x/y = xy^{-1} \wedge e$—is an MV-algebra. Conversely, if $\mathbf{L}$ is a totally-ordered MV-algebra, then there exists a totally ordered Abelian group with a strong order unit $u < e$ such that $\mathbf{L} \cong \Gamma(\mathbf{G}, u)$. This result was subsequently generalized for arbitrary Abelian ℓ-groups by D. Mundici [24] and recently for arbitrary ℓ-groups by A. Dvurečenskij [10]. It should be noted that all three authors have expressed their results in terms of the positive, rather than the negative, cone. Mundici and Dvurečenskij have also shown that the object assignment Γ can be extended to an equivalence between the category of MV-algebras (respectively, pseudo-MV-algebras), and the category with objects Abelian (respectively, arbitrary) ℓ-groups with a strong order unit, and morphisms ℓ-group homomorphisms that preserve the unit.

We generalize the concept of an MV-algebra in the setting of residuated lattices—by dropping integrality ($x \wedge e \approx x$), commutativity ($xy \approx yx$) and the existence of bounds—to a class that includes ℓ-groups, their negative cones, generalized Boolean algebras and the 0-free reducts of MV-algebras. The aim of this paper is to extend the aforementioned results of Mundici and Dvurečenskij.

A *generalized MV-algebra* (*GMV-algebra*) is a residuated lattice that satisfies the identities $x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x$. It is shown in Section 2, see Lemma 2.9, that every GMV-algebra has a distributive lattice reduct.

The *negative cone* of a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is the algebra $\mathbf{L}^- = \langle L^-, \wedge, \vee, \cdot, \backslash_{\mathbf{L}^-}, /_{\mathbf{L}^-}, e\rangle$, where $L^- = \{x \in L \mid x \leqslant e\}$, $x\backslash_{\mathbf{L}^-} y = x\backslash y \wedge e$ and $x/_{\mathbf{L}^-} y = x/y \wedge e$. It is easy to verify that $\mathbf{L}^-$ is a residuated lattice. It will be shown that if $\mathbf{L}$ is a GMV-algebra, then $\mathbf{L}^-$ is a GMV-algebra, as well. As noted before a residuated lattice is called integral, if e is the greatest element of its lattice reduct. The negative cone of every residuated lattice is, obviously, integral.

By a *nucleus* on a residuated lattice $\mathbf{L}$ we understand a closure operator γ on $\mathbf{L}$ satisfying $\gamma(a)\gamma(b) \leqslant \gamma(ab)$, for all a, b in L.

We note that if $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice and γ is a nucleus on $\mathbf{L}$, then the image L_γ of γ can be endowed with a residuated lattice structure as follows (see Lemma 3.3):

$$\mathbf{L}_\gamma = \big\langle L_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(e)\big\rangle,$$

$$\gamma(a) \vee_\gamma \gamma(b) = \gamma(a \vee b), \qquad \gamma(a) \circ_\gamma \gamma(b) = \gamma(ab).$$

As an illustration, let u be a negative element of an ℓ-group $\mathbf{G}$, and let $\gamma_u : G^- \to G^-$ be defined by $\gamma_u(x) = x \vee u$, for all $x \in G^-$. Then, γ_u is a nucleus on $\mathbf{G}^-$ and $\mathbf{G}^-_{\gamma_u}$ is equal to the 0-free reduct of $\Gamma(\mathbf{G}, u)$.

We say that a residuated lattice $\mathbf{A}$ is the *direct sum* of two of its subalgebras $\mathbf{B}, \mathbf{C}$, in symbols $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$, if the map $f : B \times C \to A$ defined by $f(x, y) = xy$ is an isomorphism.

The primary purpose of this paper is to establish the following six results.

Theorem A (See Theorem 5.6). *A residuated lattice* $\mathbf{M}$ *is a GMV-algebra if and only if there are residuated lattices* $\mathbf{G}, \mathbf{L}$, *such that* $\mathbf{G}$ *is an* ℓ*-group,* $\mathbf{L}$ *is the negative cone of an* ℓ*-group,* γ *is a nucleus on* $\mathbf{L}$ *and* $\mathbf{M} = \mathbf{G} \oplus \mathbf{L}_\gamma$.

Theorem B (See Theorem 3.12). *A residuated lattice* $\mathbf{M}$ *is an integral GMV-algebra if and only if there exists an* ℓ*-group* $\mathbf{H}$ *and a nucleus* γ *on* $\mathbf{H}^-$, *such that* $\mathbf{M} \cong \mathbf{H}^-_\gamma$.

Let **IGMV** be the category with objects integral GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\mathbf{LG}^-_*$ be the category with objects algebras $\langle \mathbf{L}, \gamma\rangle$, such that $\mathbf{L}$ is the negative cone of an ℓ-group and γ is a nucleus on it such that its image generates L as a monoid. Let the morphisms of this category be homomorphisms between these algebras. The generalization of Mundici's and Dvurečenskij's results is provided by the following theorem.

Theorem C (See Theorem 4.12). *The categories* **IGMV** *and* $\mathbf{LG}^-_*$ *are equivalent.*

Theorem D (See Theorem 6.6). *A residuated lattice* $\mathbf{L}$ *is a GMV-algebra if and only if* $\mathbf{L} \cong \mathbf{G}_\beta$, *for some* ℓ*-group* $\mathbf{G}$ *and some core* β *on* $\mathbf{G}$. (*For the concept of a core, see page* 281 *and Lemma* 6.8.)

Let **GMV** be the category with objects GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\mathbf{LG}^*$ be the category with objects algebras $\langle \mathbf{G}, \beta \rangle$ such that $\mathbf{G}$ is an ℓ-group and β is core on $\mathbf{G}$ whose image generates $\mathbf{G}$; let the morphisms of this category be homomorphisms between these algebras.

Theorem E (See Theorem 6.9). *The categories* **GMV** *and* $\mathbf{LG}^*$ *are equivalent.*

Let $\mathcal{GMV}$ be the variety of GMV-algebras and let $\mathcal{IGMV}$ be the variety of integral GMV-algebras.

Theorem F (See Theorem 7.3). *The varieties* $\mathcal{GMV}$ *and* $\mathcal{IGMV}$ *have decidable equational theories.*

2. Definitions and basic properties

We refer the reader to [4] and [20] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

The operations $\backslash$ and $/$ may be viewed as generalized division operations, with x/y being read as "x over y" and $y\backslash x$ as "y under x". In either case, x is considered the *numerator* and y is the *denominator*. We refer to $\backslash$ as the *left division* operation and $/$ as the *right division* operation. Other commonly used terms for these operations are *left residuation* and *right residuation*, respectively.

As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parenthesis, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$. For example, $x/yz \wedge u\backslash v$ represents the expression $[x/(y \cdot z)] \wedge (u\backslash v)$. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has an *opposite* "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$). Examples of opposite equations can be seen in properties (i)–(vi) of Lemma 2.1 below.

The existence of the division operations has the following basic consequences.

Lemma 2.1 [4]. *Residuated lattices satisfy the following identities*:

(i) $x(y \vee z) \approx xy \vee xz$ *and* $(y \vee z)x \approx yx \vee zx$.
(ii) $x\backslash(y \wedge z) \approx (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x \approx (y/x) \wedge (z/x)$.
(iii) $x/(y \vee z) \approx (x/y) \wedge (x/z)$ *and* $(y \vee z)\backslash x \approx (y\backslash x) \wedge (z\backslash x)$.
(iv) $(x/y)y \leqslant x$ *and* $y(y\backslash x) \leqslant x$.
(v) $x(y/z) \leqslant (xy)/z$ *and* $(z\backslash y)x \leqslant z\backslash(yx)$.
(vi) $(x/y)/z \approx x/(zy)$ *and* $z\backslash(y\backslash x) \approx (yz)\backslash x$.
(vii) $x\backslash(y/z) \approx (x\backslash y)/z$.
(viii) $x/e \approx x \approx e\backslash x$.

(ix) $e \leqslant x/x$ *and* $e \leqslant x\backslash x$.
(x) $(x/x)^2 \approx x/x$ *and* $(x\backslash x)^2 \approx x\backslash x$.

A residuated lattice is called *commutative* (respectively, *cancellative*), if its monoid reduct is *commutative* (respectively, *cancellative*). It is shown in [2] that the class $Can\mathcal{RL}$ of all cancellative residuated lattices is a variety with defining equations $xy/y \approx x \approx y\backslash yx$. As mentioned before, a residuated lattice is called *integral* if it satisfies the identity $e \wedge x \approx x$. The variety of all integral residuated lattices will be denoted by $\mathcal{IRL}$. We will also have the occasion to refer to the subvariety of $\mathcal{RL}$ generated by all totally ordered residuated lattices. We denote this variety by $\mathcal{RL}^C$ and refer to its members as *representable* residuated lattices. It follows from Jónsson's Lemma on congruence-distributive varieties (see [21]) that all subdirectly irreducible algebras in $\mathcal{RL}^C$ are totally ordered and whence every representable residuated lattice is a subdirect product of totally ordered residuated lattices. The following result provides a concise equational basis for $\mathcal{RL}^C$.

Theorem 2.2 ([4,20], see also [18]). *A residuated lattice is representable, i.e., it is a member of the variety $\mathcal{RL}^C$, if and only if it satisfies the identity $(z\backslash(x/(x \vee y))z \wedge e) \vee (w(y/(x \vee y))/w \wedge e) \approx e$.*

Definition 2.3.

(i) A *generalized BL-algebra* (*GBL-algebra*) is a residuated lattice that satisfies the identities

$$\big((x \wedge y)/y\big)y \approx x \wedge y \approx y\big(y\backslash(x \wedge y)\big).$$

(ii) A *generalized MV-algebra* (*GMV-algebra*) is a residuated lattice that satisfies the identities

$$x/\big((x \vee y)\backslash x\big) \approx x \vee y \approx \big(x/(x \vee y)\big)\backslash x.$$

We denote the variety of all GBL-algebras by $\mathcal{GBL}$ and that of GMV-algebras by $\mathcal{GMV}$. GBL-algebras generalize BL-algebras, the algebraic counterparts of basic logic (see [17]). In particular, representable, commutative, bounded (integral) GBL-algebras are (term equivalent to) the 0-free reducts of BL-algebras.

Lemma 2.4 [2]. *The preceding sets of identities have, respectively, the following quasi-identity formulations*:

$$x \leqslant y \quad \Rightarrow \quad (x/y)y \approx x \approx y(y\backslash x)$$

and

$$x \leqslant y \quad \Rightarrow \quad x/(y\backslash x) \approx y \approx (x/y)\backslash x.$$

Moreover, the first set of identities is also equivalent to the property of *divisibility* in the setting of residuated lattices:

$$x \leqslant y \quad \Rightarrow \quad (\exists z, w)\ (zy \approx x \approx yw).$$

Lemma 2.5 [2]. *Every GMV-algebra is a GBL-algebra.*

Proof. Let x, y be elements of L such that $x \leqslant y$. Set $z = (x/y)y$ and note that, by Lemma 2.1, $z \leqslant x$ and $z/y \leqslant z/x$.

Using Lemma 2.1(vii), (vi) and the defining quasi-equation for GMV-algebras, we have the following:

$$\begin{aligned} z \leqslant x \quad &\Rightarrow \quad (z/x)\backslash z = x \\ &\Rightarrow \quad \big((z/x)\backslash z\big)/y = x/y \\ &\Rightarrow \quad (z/x)\backslash(z/y) = x/y \\ &\Rightarrow \quad (z/y)/\big((z/x)\backslash(z/y)\big) = (z/y)/(x/y) \\ &\Rightarrow \quad z/x = z/(x/y)y \\ &\Rightarrow \quad (z/x)\backslash z = \big(z/(x/y)y\big)\backslash z \\ &\Rightarrow \quad x = (x/y)y. \end{aligned}$$

Thus, $x \leqslant y$ implies $x = (x/y)y$. Likewise, $x \leqslant y$ implies $y(y\backslash x) = x$. □

Lattice-ordered groups and their negative cones are examples of cancellative GMV-algebras. Non-cancellative examples include generalized Boolean algebras.

Definition 2.6. An element a in a residuated lattice $\mathbf{L}$ is called *invertible*, if $a(a\backslash e) = e = (e/a)a$; a is called *integral*, if $e/a = a\backslash e = e$. We denote the set of invertible elements of $\mathbf{L}$ by $G(\mathbf{L})$ and the set of integral elements by $I(\mathbf{L})$.

Note that a is invertible if and only if there exists an element a^{-1} such that $aa^{-1} = e = a^{-1}a$. In this case $a^{-1} = e/a = a\backslash e$. It is easy to see that multiplication by an invertible element is an order automorphism.

Lemma 2.7. *Let* $\mathbf{L}$ *be a GBL-algebra.*

(i) *Every positive element of* $\mathbf{L}$ *is invertible.*
(ii) $\mathbf{L}$ *satisfies the identities* $x/x \approx x\backslash x \approx e$.
(iii) $\mathbf{L}$ *satisfies the identity* $e/x \approx x\backslash e$.

Proof. For the first property, let a be a positive element; by the defining identity for GBL-algebras, we get $a(a\backslash e) = e = (e/a)a$; that is, a is invertible. By (i) and Lemma 2.1(ix),

x/x and $x\backslash x$ are invertible for every x. Hence, by Lemma 2.1(x), $x/x = e = x\backslash x$. Finally, by (ii) and Lemma 2.1(v), $x(e/x) \leqslant x/x = e$, hence $e/x \leqslant x\backslash e$. Likewise, $x\backslash e \leqslant e/x$. $\square$

Lemma 2.8. *A residuated lattice is a GBL-algebra if and only if it satisfies the identities* $x(x\backslash y \wedge e) \approx x \wedge y \approx (y/x \wedge e)x$.

Proof. Assume that $\mathbf{L}$ is a GBL-algebra and $x, y \in L$. By Lemmas 2.7(ii) and 2.1(ii), we get

$$x \wedge y = x\big(x\backslash(x \wedge y)\big) = x(x\backslash x \wedge x\backslash y) = x(e \wedge x\backslash y).$$

Likewise, we get the opposite identity.

Conversely assume that the identities in the statement of the lemma hold. We first show that every positive element a is invertible. Indeed, $e = a(a\backslash e \wedge e) \leqslant a(a\backslash e) \leqslant e$. So, $a(a\backslash e) = e$ and likewise $(e/a)a = e$. Arguing as in the proof of (ii) of Lemma 2.7, we show that $x\backslash x = x/x = e$, for every $x \in L$. Using Lemma 2.1(ii), we get

$$x\big(x\backslash(x \wedge y)\big) = x(x\backslash x \wedge x\backslash y) = x(e \wedge x\backslash y) = x \wedge y.$$

Likewise, we obtain the opposite equation. $\square$

Lemma 2.9. *Every GBL-algebra has a distributive lattice reduct.*

Proof. Let $\mathbf{L}$ be a GBL-algebra and $x, y, z \in L$. Invoking Lemmas 2.1 and 2.8, we have

$$\begin{aligned} x \wedge (y \vee z) &= \big[x/(y \vee z) \wedge e\big](y \vee z) \\ &= \big[x/(y \vee z) \wedge e\big]y \vee \big[x/(y \vee z) \wedge e\big]z \\ &\leqslant (x/y \wedge e)y \vee (x/z \wedge e)z \\ &= (x \wedge y) \vee (x \wedge z), \end{aligned}$$

for all x, y, z. Thus, the lattice reduct of $\mathbf{L}$ is distributive. $\square$

Lemma 2.10. *If* x, y *are elements of a GBL-algebra and* $x \vee y = e$ (x, y *are orthogonal*), *then* $xy = x \wedge y$.

Proof. We have, $x = x/e = x/(x \vee y) = x/x \wedge x/y = e \wedge x/y = y/y \wedge x/y = (y \wedge x)/y$. So, $xy = ((x \wedge y)/y)y = x \wedge y$. $\square$

The variety of integral GBL-algebras is denoted by $\mathcal{IGBL}$ and that of integral GMV-algebras by $\mathcal{IGMV}$. Obviously, $\mathcal{IGBL} = \mathcal{IRL} \cap \mathcal{GBL}$ and $\mathcal{IGMV} = \mathcal{IRL} \cap \mathcal{GMV}$.

Lemma 2.11.

(i) *The variety $\mathcal{IGBL}$ is axiomatized, relative to $\mathcal{RL}$, by the equations* $(x/y)y \approx x \wedge y \approx y(y\backslash x)$.
(ii) *The variety $\mathcal{IGMV}$ is axiomatized, relative to $\mathcal{RL}$, by the equations* $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$.

Proof. In view of the alternative axiomatizations of $\mathcal{GBL}$ and $\mathcal{GMV}$ given in Lemma 2.8, the proposed equations hold in the corresponding varieties. For the reverse direction we verify that the proposed identities imply integrality. This is obvious for the first set of identities for $y = e$. For the second set, observe that for every x,

$$e \leqslant e \vee e/x = e/\big((e/x)\backslash e\big) = e/(e \vee x);$$

so $e \vee x \leqslant e$, i.e., $x \leqslant e$. □

Negative cones of ℓ-groups are examples of integral GMV-algebras, hence also of integral GBL-algebras. Moreover, these are cancellative residuated lattices, that is, members of $\mathit{Can}\mathcal{RL}$. It is shown in [2] that the class $\mathcal{LG}^-$ of negative cones of ℓ-groups is a variety and $\mathcal{LG}^- = \mathcal{IGMV} \cap \mathit{Can}\mathcal{RL} = \mathcal{IGBL} \cap \mathit{Can}\mathcal{RL}$. This result provides an equational basis for $\mathcal{LG}^-$.

Theorem 2.12 [2]. *The class, $\mathcal{LG}^-$, of negative cones of ℓ-groups is a variety and the equations $xy/y \approx x \approx y\backslash yx$, $x(x\backslash y) \approx x \wedge y \approx (y/x)x$ form an equational basis for it, relative to $\mathcal{RL}$.*

The variety of Brouwerian algebras is term equivalent to the subvariety $\mathcal{B}r$ of residuated lattices axiomatized by the identity $xy \approx x \wedge y$. It is clear that $\mathcal{B}r \subseteq \mathcal{IGBL}$. The variety $\mathcal{GBA}$ of generalized Boolean algebras is generated, in the setting of residuated lattices, by the two-element residuated lattice **2** and $\mathcal{GBA} = \mathcal{IGMV} \cap \mathcal{B}r$ (see [13]).

Lemma 2.13.

(i) *Every integral GBL-algebra satisfies the identity* $(y/x)\backslash(x/y) \approx x/y$ *and its opposite.*
(ii) *Every integral GMV-algebra satisfies the identity* $x/y \vee y/x \approx e$ *and its opposite.*
(iii) *Every integral GMV-algebra satisfies the identities* $x/(y \wedge z) \approx x/y \vee x/z$, $(x \vee y)/z \approx x/z \vee y/z$ *and the opposite ones.*
(iv) *Every commutative integral GMV-algebra is representable. Consequently, the subdirectly irreducible, commutative, integral GMV-algebras are totally ordered.*

Proof. (i) For every integral GBL-algebra, $y/x \leqslant e$, so $(y/x)\backslash(x/y) \geqslant x/y$. To show the reverse inequality, we need to check that

$$\big((y/x)\backslash(x/y)\big)y \leqslant x.$$

By Lemma 2.1(vii), it suffices to show that

$$\big(\big((y/x)\backslash x\big)/y\big)y \leqslant x.$$

Using Lemma 2.11(i), we see that the last equation is equivalent to

$$\big(y/\big((y/x)\backslash x\big)\big)\big((y/x)\backslash x\big) \leqslant x,$$

which in turn is equivalent to

$$y/\big((y/x)\backslash x\big) \leqslant x/\big((y/x)\backslash x\big).$$

To show that this holds note that

$$y/\big((y/x)\backslash x\big) \leqslant y/x,$$

since $y/x \leqslant e$, and that

$$y/x \leqslant x/\big((y/x)\backslash x\big),$$

since $u \leqslant v/(u\backslash v)$ is valid in every residuated lattice, by Lemma 2.1(iv).

(ii) Using Lemma 2.11(ii), we have $x/y \vee y/x = (x/y)/((y/x)\backslash(x/y))$, which simplifies to $(x/y)/(x/y)$, by invoking (i) and the fact that integral GMV-algebras are integral GBL-algebras. Finally, the last term is equal to e in integral residuated lattices.

(iii) Since every GMV-algebra has a distributive lattice reduct by Lemma 2.9, the equations in (iii) follow from (ii) and Proposition 6.10(ii) of [4].

(iv) This follows from (ii) and [18]. □

It will be shown in Section 5, refer to Corollary 5.5, that the assumption of integrality in condition (iv) is not needed.

3. A concrete realization of integral generalized MV-algebras

A *closure operator* on a poset $\mathbf{P}$ is a map $\gamma : P \to P$ with the usual properties of preserving the order, being extensive ($x \leqslant \gamma(x)$), and being idempotent. Such a map is completely determined by its image

$$C = \operatorname{im}\gamma \tag{3.1}$$

by virtue of the formula

$$\gamma(x) = \min\{c \in C\colon\ x \leqslant c\}. \tag{3.2}$$

A *closure retract* is any subset $C \subseteq P$ such that the minima (3.2) exist for all $x \in P$. Conditions (3.1) and (3.2) establish a bijective correspondence between all closure operators

γ and all closure retracts C of P. In what follows, we will use P_γ to denote the closure retract on P corresponding to the closure operator γ.

A *nucleus* on a residuated lattice $\mathbf{L}$ is a closure operator γ on the lattice reduct of $\mathbf{L}$ such that $\gamma(a)\gamma(b) \leqslant \gamma(ab)$, for all $a, b \in L$. It is clear that a closure operator γ on $\mathbf{L}$ is a nucleus if and only if $\gamma(\gamma(a)\gamma(b)) = \gamma(ab)$, for all $a, b \in L$. Note that the monotonicity condition in the definition of a nucleus can be replaced by the inequality $\gamma(x)\gamma(x\backslash y) \leqslant \gamma(y)$; so, the property that γ is a nucleus on a residuated lattice can be expressed equationally in the expansion of the language of residuated lattices by a unary operation. A closure retract C of a residuated lattice L is called a *subact* of L if $x/y,\ y\backslash x \in C$, for all $x \in C$ and $y \in L$.

As an example, note that if u is an element of an integral residuated lattice $\mathbf{L}$, then $\gamma_u : L \to L$—defined by $\gamma_u(x) = x \vee u$, for all $x \in L$—is a nucleus on $\mathbf{L}$. Its image L_{γ_u} is the principal filter $\uparrow u = \{x \in L \mid u \leqslant x\}$.

The next result describes the relationship between nuclei and subacts (see [25, p. 30]; and [26, Corollary 3.7], for an earlier result in the setting of Brouwerian meet-semilattices).

Lemma 3.1. *Let γ be a closure operator on a residuated lattice* $\mathbf{L}$, *and let L_γ be the closure retract associated with γ. The following statements are equivalent.*

(i) *γ is a nucleus.*
(ii) *$\gamma(a)/b, b\backslash\gamma(a) \in L_\gamma$, for all $a, b \in L$.*
(iii) *L_γ is a subact of L.*

Proof. (i) $\Rightarrow$ (ii). Let $a, b \in L$. We have,

$$\begin{aligned}
\gamma(\gamma(a)/b)b &\leqslant \gamma\big(\gamma(a)/b\big)\gamma(b) && (\gamma \text{ is extensive})\\
&\leqslant \gamma\big(\big(\gamma(a)/b\big)b\big) && \text{(i)}\\
&\leqslant \gamma\big(\gamma(a)\big) && \big(\gamma \text{ is monotone, Lemma 2.1(iv)}\big)\\
&= \gamma(a) && (\gamma \text{ is idempotent}).
\end{aligned}$$

So, $\gamma(\gamma(a)/b) \leqslant \gamma(a)/b$, by the defining property of residuated lattices. Since the reverse inequality follows from the fact that γ is extensive, we have $\gamma(a)/b = \gamma(\gamma(a)/b) \in L_\gamma$. Likewise, we obtain the result for the other division operation.

(ii) $\Rightarrow$ (i). Let $a, b \in L$. Since γ is extensive, $ab \leqslant \gamma(ab)$, so $a \leqslant \gamma(ab)/b$. By the monotonicity of γ and the hypothesis, $\gamma(a) \leqslant \gamma(ab)/b$. Using the defining property of residuated lattices, we get $b \leqslant \gamma(a)\backslash\gamma(ab)$. Invoking, once more, the monotonicity of γ and the hypothesis, we obtain $\gamma(b) \leqslant \gamma(a)\backslash\gamma(ab)$, namely $\gamma(a)\gamma(b) \leqslant \gamma(ab)$.

(ii) $\Leftrightarrow$ (iii). This is trivial by the definition of a subact. $\square$

Actually, it can be shown that an arbitrary map γ on a residuated lattice $\mathbf{L}$ is a nucleus if and only if $\gamma(a)/b = \gamma(a)/\gamma(b)$ and $b\backslash\gamma(a) = \gamma(b)\backslash\gamma(a)$, for all $a, b \in L$ (see [25, p. 30]).

Corollary 3.2. *Conditions* (3.1) *and* (3.2) *establish a bijective correspondence between nuclei on and subacts of a residuated lattice.*

Proof. Use Lemma 3.1. $\square$

The next result shows that every subact of a residuated lattice is a residuated lattice in its own right.

Lemma 3.3. *Let $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ be a residuated lattice, γ be a nucleus on $\mathbf{L}$ and L_γ be the subact associated with γ. Then the algebraic system $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(e)\rangle$—where $x \circ_\gamma y = \gamma(x \cdot y)$ and $x \vee_\gamma y = \gamma(x \vee y)$—is a residuated lattice.*

Proof. Obviously, $\gamma(e)$ is the multiplicative identity of $\mathbf{L}_\gamma$. Further, L_γ, being the image of a closure operator on $\mathbf{L}$, is a lattice with joins and meets defined by $x \vee_\gamma y = \gamma(x \vee y)$ and $x \wedge_\gamma y = x \wedge y$, for all $x, y \in L_\gamma$. One can easily check that multiplication is associative. Finally, to check that $\circ_\gamma$ is residuated, recall that L_γ is closed under the division operations by Lemma 3.1. Consider $x, y, z \in L_\gamma$. We have $x \circ_\gamma y \leqslant z \Leftrightarrow \gamma(xy) \leqslant z \Leftrightarrow xy \leqslant z$ (since $z = \gamma(z)$ and $xy \leqslant \gamma(xy)$) $\Leftrightarrow y \leqslant x\backslash z$. Likewise, we have $x \circ_\gamma y \leqslant z \Leftrightarrow x \leqslant z/y$. $\square$

Theorem 3.4. *If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a GMV-algebra, γ a nucleus on it and L_γ the associated subact, then*

(i) $\vee_\gamma = \vee$;
(ii) *γ preserves binary joins*;
(iii) $\gamma(e) = e$;
(iv) *$\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee, \circ_\gamma, \backslash, /, e\rangle$ is a GMV-algebra*; *and*
(v) *L_γ is a filter in $\mathbf{L}$.*

Proof. (i) Since $\mathbf{L}$ is a GMV-algebra, if $x \in L_\gamma$, then $x \vee y = x/((x \vee y)\backslash x) \in L_\gamma$, by Lemma 3.1(iv). Thus, $\vee_\gamma$ is the restriction of $\vee$ on $\mathbf{L}_\gamma$.

(ii) It is well known, and easy to prove, that if γ is a closure operator on a poset P and X is a subset of P such that ${}^{P}\bigvee X$ exists, then ${}^{P_\gamma}\bigvee \gamma(X)$ exists and ${}^{P_\gamma}\bigvee \gamma(X) = \gamma({}^{P}\bigvee X)$. Thus, (i) and (ii) are equivalent.

(iii) Since γ is extensive, $e \leqslant \gamma(e)$. Hence, $\gamma(e)$ is invertible, by Lemma 2.7(i). Since γ is a nucleus, $\gamma(e)\gamma(e) \leqslant \gamma(e)$, so $\gamma(e) \leqslant e$. Thus, $\gamma(e) = e$.

(iv) By Lemma 3.3, $\mathbf{L}_\gamma$ is a residuated lattice. It is a GMV-algebra because the join and division operations of $\mathbf{L}_\gamma$ are the restrictions of the corresponding operations in $\mathbf{L}$, and $\mathbf{L}$ is a GMV-algebra.

(v) If $x \in L_\gamma$, $y \in L$ and $x \leqslant y$, then by Lemma 3.1, $y = x \vee y = x/((x \vee y)\backslash x)$ is an element of L_γ. Since $\mathbf{L}_\gamma$ is also a sublattice, it is a lattice-filter. $\square$

Corollary 3.5. *If $\mathbf{L}$ is an integral GMV-algebra and γ is a nucleus on $\mathbf{L}$, then $\mathbf{L}_\gamma$ is an integral GMV-algebra.*

Lemma 3.6. *Let γ be a nucleus on the negative cone $\mathbf{L}$ of an ℓ-group. If $z \in L$ and $u = \gamma(z)$, then γ agrees with the nucleus γ_u on the principal filter $\uparrow z$.*

Proof. Let $x \geqslant z$. We will show that $\gamma(x) = u \vee x$. Note that $u \vee x = \gamma(z) \vee x \leqslant \gamma(x)$, since γ is monotone and extensive. On the other hand, $x \leqslant u \vee x$, so $\gamma(x) \leqslant \gamma(u \vee x) = u \vee x$, because L_γ is a filter, by Theorem 3.4(v). □

Corollary 3.7. *Every nucleus on a GMV-algebra is a lattice homomorphism.*

Proof. In view of Theorem 3.4(ii), we need only show that γ preserves binary meets. Let x, y be elements of a GMV-algebra and set $z = x \wedge y$ and $u = \gamma(z)$. Recall that a GMV-algebra has a distributive lattice reduct; refer to Lemmas 2.5 and 2.9. Whence by Lemma 3.6, $\gamma(x \wedge y) = \gamma_u(x \wedge y) = u \vee (x \wedge y) = (u \vee x) \wedge (u \vee y) = \gamma_u(x) \wedge \gamma_u(y) = \gamma(x) \wedge \gamma(y)$. □

By Corollary 3.5, the image of a nucleus on the negative cone of an ℓ-group is an integral GMV-algebra. In the remainder of this section we are concerned with the proof of the converse, namely that every integral GMV-algebra is the image of a nucleus on the negative cone of an ℓ-group. Our proof is based on Theorem 3.11, which is due to B. Bosbach, see [5] and [6].

Definition 3.8. A *cone algebra* is an algebra $\mathbf{C} = \langle C, \backslash, /, e\rangle$ that satisfies:

$$(x\backslash y)\backslash(x\backslash z) \approx (y\backslash x)\backslash(y\backslash z)(z/y)/(x/y) \approx (z/x)/(y/x),$$

$$e\backslash y \approx y, \qquad y/e \approx y,$$

$$x\backslash(y/z) \approx (x\backslash y)/z, \qquad x/(y\backslash x) \approx (y/x)\backslash y,$$

$$x\backslash x \approx e, \qquad x/x \approx e.$$

Lemma 3.9 [5,6]. *If* $\mathbf{C} = \langle C, \backslash, /, e\rangle$ *is a cone algebra, then*

(i) *for all* $a, b \in C$, $a\backslash b = e \Leftrightarrow b/a = e$;
(ii) *the relation* $\leqslant$ *on* C, *defined by* $a \leqslant b \Leftrightarrow a\backslash b = e$, *is a semilattice order with* $a \vee b = a/(b\backslash a)$; *in particular* $a \leqslant e$, *for all* a;
(iii) *if* $a \leqslant b$, *then* $c\backslash a \leqslant c\backslash b$ *and* $a/c \leqslant b/c$.

It is easy to see that if $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is an integral GMV-algebra—for example, $\mathbf{L} \in \mathcal{LG}^-$—then $\langle L, \backslash, /, e\rangle$ is a cone algebra, called *the cone algebra of* $\mathbf{L}$.

It will be shown that every cone algebra is a subalgebra of the cone algebra of a residuated lattice in $\mathcal{LG}^-$. In the following construction, the algebra in $\mathcal{LG}^-$ is defined as the union of an ascending chain $\langle \mathbf{C}_n \mid n \in \mathbb{N}\rangle$ of cone algebras, each of which is a subalgebra of its successor. In the process of constructing the algebras $\mathbf{C}_n$, we also define in $\mathbf{C}_{n+1}$ binary products of elements of $\mathbf{C}_n$. Each such product is identified with the congruence class of the corresponding ordered pair. The definition below of the division operations becomes less opaque if we note that the negative cones of any ℓ-group satisfies the law $ab\backslash cd = (b\backslash(a\backslash c)) \cdot (((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d))$ and its opposite.

Let $\mathbf{C}$ be a cone algebra. Define the operations $\backslash$ and $/$ and the relations Θ and Θ' on $C \times C$, by

$$(a,b)\backslash(c,d) = \big(b\backslash(a\backslash c), \big((a\backslash c)\backslash b\big)\backslash\big((c\backslash a)\backslash d\big)\big),$$
$$(d,c)/(b,a) = \big(\big(d/(a/c)\big)/\big(b/(c/a)\big), (c/a)/b\big),$$
$$(a,b)\ \Theta\ (c,d) \quad\Leftrightarrow\quad (a,b)\backslash(c,d) = (e,e) \quad\text{and}\quad (c,d)\backslash(a,b) = (e,e),$$
$$(a,b)\ \Theta'\ (c,d) \quad\Leftrightarrow\quad (a,b)/(c,d) = (e,e) \quad\text{and}\quad (c,d)/(a,b) = (e,e).$$

Lemma 3.10 [6]. *Let $\mathbf{C} = \langle C, \backslash, /, e\rangle$ be a cone algebra. Then*:

(i) $\Theta = \Theta'$.
(ii) *Θ is a congruence relation of $\mathbf{C} \times \mathbf{C}$.*
(iii) *$s(\mathbf{C}) = \langle C \times C, \backslash, /, e\rangle/\Theta$ is a cone algebra.*
(iv) *For each $x \in C$, let $[(x,e)]_\Theta$ denote the Θ-congruence class of (x,e). Then the map $x \mapsto [(x,e)]_\Theta$ is an embedding of $\mathbf{C}$ into $s(\mathbf{C})$.*

Let $\mathbf{C}_0 = \mathbf{C}$, $\mathbf{C}_{n+1} = s(\mathbf{C}_n)$, for every natural number n, and $\overline{\mathbf{C}} = \bigcup \mathbf{C}_n$, the directed union of the $\mathbf{C}_n$'s.

We can now establish the main result of [6].

Theorem 3.11 [6]. *Every cone algebra $\mathbf{C}$ is a subalgebra of the cone algebra of some $\widehat{\mathbf{C}} \in \mathcal{LG}^-$. Moreover, every element of $\widehat{C}$ can be written as a product of elements of C.*

Proof. We will show that the algebra $\overline{\mathbf{C}}$ defined above is the cone algebra, i.e., the $\{\backslash, /, e\}$-reduct, of a $\widehat{\mathbf{C}} \in \mathcal{LG}^-$.

For two elements of $\overline{C}$, we define their product, ab, to be the element $[(a,b)]_\Theta$. This is well defined, because of the embedding of $\mathbf{C}_n$ into $\mathbf{C}_{n+1}$, for every n. Let $\widehat{\mathbf{C}} = \langle \overline{C}, \wedge, \vee, \cdot, \backslash, /, e\rangle$, where $\backslash = \backslash_{\overline{\mathbf{C}}}$, $/ = /_{\overline{\mathbf{C}}}$, $x \vee y = x/(y\backslash x)$ and $x \wedge y = (x/y)y$. We will show that $\widehat{\mathbf{C}} \in \mathcal{LG}^-$.

By the definition of the operations in $\widehat{\mathbf{C}}$ and Lemma 3.9(ii), $\widehat{\mathbf{C}}$ is a join semilattice. Note that $ab\backslash cd = (b\backslash(a\backslash c)) \cdot (((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d))$. In particular, $ab\backslash c = b\backslash(a\backslash c)$ and $a\backslash ab = b$. The opposite equations hold, as well. Finally, note that $e/a = e = a\backslash e$.

Multiplication is order preserving

Let $a \leqslant c$; then $e = a\backslash c$, by the definition of $\leqslant$. To show that $ab \leqslant cb$, we note that

$$ab\backslash cb = b\backslash\big[(c\backslash a)\backslash b\big] = \big[(c\backslash a)b\big]\backslash b.$$

On the other hand,

$$b/\big[(c\backslash a)b\big] = (b/b)/(c\backslash d) = e/(c\backslash d) = e.$$

This successively yields, $(c\backslash a)b \leqslant b$, $[(c\backslash a)b]\backslash b = e$, $ab\backslash cb = e$ and $ab \leqslant cb$. Likewise $a \leqslant c$ implies $ba \leqslant bc$.

Multiplication is residuated

Note that $a(a\backslash c) \leqslant c$, since $[a(a\backslash c)]\backslash c = (a\backslash c)\backslash(a\backslash c) = e$. If $ab \leqslant c$, then $a\backslash ab \leqslant a\backslash c$, so $b \leqslant a\backslash c$. Conversely, if $b \leqslant a\backslash c$, then $ab \leqslant a(a\backslash c) \leqslant c$. The other equivalence is obtained similarly.

Multiplication is associative

We have the following sequence of equivalences:

$$\begin{aligned}(ab)c \leqslant d \quad &\Leftrightarrow \quad ab \leqslant d/c \quad \Leftrightarrow \quad b \leqslant a\backslash(d/c) \quad \Leftrightarrow \quad b \leqslant (a\backslash d)/c \\ &\Leftrightarrow \quad bc \leqslant (a\backslash d) \quad \Leftrightarrow \quad a(bc) \leqslant d.\end{aligned}$$

$\wedge$ is the meet operation

We have $a(a\backslash b) \leqslant b$ and $a(a\backslash b) \leqslant ae = a$. Additionally, if $c \leqslant a$ and $c \leqslant b$, then $e = c\backslash a = c\backslash b$. We have, $c\backslash a(a\backslash b) = (c\backslash a)\backslash(c\backslash b) = e$, so $c \leqslant a(a\backslash b)$. Interchanging the roles of a and b we get that $c \leqslant a, b \Leftrightarrow c \leqslant b(b\backslash a)$. The opposite properties are obtained similarly.

Thus, $\widehat{\mathbf{C}}$ is a residuated lattice. Since it satisfies the identities $x\backslash xy \approx y \approx yx/x$ and $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$, it is in $\mathcal{LG}^-$, by Theorem 2.12. Finally, by construction, every element of $\widehat{C}$ is the product of elements of C. $\square$

The algebra $\widehat{\mathbf{C}}$ is called the *product extension* of $\mathbf{C}$.

We can now prove the main result of this section.

Theorem 3.12. *The residuated lattice $\mathbf{M}$ is an integral GMV-algebra if and only if $\mathbf{M} \cong \mathbf{L}_\gamma$, for some $\mathbf{L} \in \mathcal{LG}^-$ and some nucleus γ on $\mathbf{L}$.*

Proof. One direction follows from Corollary 3.5. For the opposite implication, let $\mathbf{M} = \langle M, \wedge, \vee, \circ, \backslash, /, e\rangle$ be an integral GMV-algebra. Using Lemmas 2.5, 2.11(ii), 2.1(vi), 2.7(ii), 2.1(viii), 2.1(vii) and 2.11(i), we see that $\langle M, \backslash, /, e\rangle$ is a cone algebra. So, by Theorem 3.11, it is a subreduct of a residuated lattice $\mathbf{L} = \widehat{\mathbf{M}} \in \mathcal{LG}^-$ such that M generates $\mathbf{L}$ as a monoid.

Since the division operations of $\mathbf{M}$ are the restrictions of the division operations of $\mathbf{L}$, we use the symbols $\backslash$ and $/$ for the latter, as well. Moreover, the same holds for the join and the constant e, because in integral GMV-algebras they are term definable by the division operations: $x \vee y \approx x/(y\backslash x)$ and $e \approx x/x$. We use "$\cdot$" to denote the multiplication of $\mathbf{L}$.

Since M generates L as a monoid, for every $x \in L$ there exists a sequence $(x_1, \ldots, x_n)$ of elements of M such that $x = x_1 \cdots x_n$.

Claim 1. *If $z \in M$, $x \in L$ and $(x_1, \ldots, x_n)$ is a sequence of elements of M such that $x = x_1 \cdots x_n$, then $z \vee x = z \vee x_1 \circ \cdots \circ x_n$.*

Indeed,

$$z \vee x = z/(x\backslash z) \quad \text{(axiom of IGMV-algebras)}$$

$$
\begin{aligned}
&= z/\big((x_1 \cdots x_n)\backslash z\big) \\
&= z/\big[x_n\backslash\big(\cdots\big(x_2\backslash(x_1\backslash z)\big)\cdots\big)\big] \quad \big(\text{Lemma 2.1(vi)}\big) \\
&= z/\big((x_1 \circ \cdots \circ x_n)\backslash z\big) \quad \big(\text{Lemma 2.1(vi)}\big) \\
&= z \vee x_1 \circ \cdots \circ x_n \quad (\text{axiom of IGMV-algebras}).
\end{aligned}
$$

Claim 2. *Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ be sequences of elements of M such that $x_1 \cdots x_n = y_1 \cdots y_m$. Then, $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_m$.*

Indeed, $x_1 \circ \cdots \circ x_n \vee y_1 \circ \cdots \circ y_m = x_1 \circ \cdots \circ x_n \vee x_1 \circ \cdots \circ x_n$, by the preceding claim. It follows that $y_1 \circ \cdots \circ y_m \leqslant x_1 \circ \cdots \circ x_n$, and likewise, $x_1 \circ \cdots \circ x_n \leqslant y_1 \circ \cdots \circ y_m$. Hence, $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_m$.

We now define a map γ on L as follows: if $x \in L$ and $(x_1, \ldots, x_n)$ is a sequence of elements of M such that $x = x_1 \cdots x_n$, we let $\gamma(x) = x_1 \circ \cdots \circ x_n$. By Claim 2, γ is well defined. We will show that it is a nucleus on $\mathbf{L}$, $L_\gamma = M$ and $\mathbf{L}_\gamma \cong \mathbf{M}$.

Note that $\gamma(x) \in M$, for all $x \in L$, so by setting $z = \gamma(x)$ in the statement of Claim 1, we get $\gamma(x) \vee x = \gamma(x)$. So, $x \leqslant \gamma(x)$, for all $x \in L$. If $x \leqslant y$, then

$$
\begin{aligned}
\gamma(x) &\leqslant \gamma(y) \vee \gamma(x) \\
&= \gamma(y) \vee x \quad \big(\text{Claim 1 for } z = \gamma(y)\big) \\
&\leqslant \gamma(y) \vee y \quad (x \leqslant y) \\
&\leqslant \gamma(y) \quad (\text{since } \gamma \text{ is extensive}).
\end{aligned}
$$

This shows that γ is monotone. The following computation shows that γ is idempotent, and hence a closure operator.

$$\gamma(\gamma(x)) = \gamma(x_1 \circ \cdots \circ x_n) = x_1 \circ \cdots \circ x_n = \gamma(x).$$

Finally, if $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$, are two representations of x and y in terms of elements of M, then

$$
\begin{aligned}
\gamma(x)\gamma(y) &\leqslant \gamma\big(\gamma(x)\gamma(y)\big) \quad (\text{since } \gamma \text{ is extensive}) \\
&= \gamma(x) \circ \gamma(y) \quad (\text{definition of } \gamma) \\
&= (x_1 \circ \cdots \circ x_n) \circ (y_1 \circ \cdots \circ y_m) \quad (\text{definition of } \gamma) \\
&= \gamma(xy) \quad (\text{definition of } \gamma).
\end{aligned}
$$

Thus, γ is a nucleus.

It is clear that $L_\gamma = M$, by the definition of $\gamma(x)$. Further, we have already observed that the division operations, join and e agree on $\mathbf{L}_\gamma$ and $\mathbf{M}$. Also, for $x, y \in M$, $x \circ_\gamma y = \gamma(xy) = x \circ y$. Finally, the meet operation on the two structures is the same, since integral GMV-algebras satisfy the identity $x \wedge y \approx (x/y)y$. Thus, the two structures $\mathbf{M}$ and $\mathbf{L}_\gamma$ are identical. □

As an example, we note that the collection of all co-finite subsets of ω is the universe of a generalized Boolean algebra $\mathbf{A}$, hence an integral GMV-algebra. It is easy to see that $\mathbf{A} \cong ((\mathbb{Z}^-)^\omega)_\gamma$, where $\mathbb{Z}$ is the ℓ-group of the integers under addition and the natural order, and $\gamma((x_n)_{n\in\omega}) = (x_n \vee (-1))_{n\in\omega}$.

4. A categorical equivalence for integral GMV-algebras

In this section we extend the representation of integral GMV-algebras, discussed in the previous section, to a categorical equivalence.

Let **IGMV** be the category with objects integral GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\mathbf{LG}_*^-$ be the category with objects algebras $\langle L, \wedge, \vee, \cdot, \backslash, /, e, \gamma\rangle$, where $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle \in \mathcal{LG}^-$ and γ is a nucleus on $\mathbf{L}$ such that its image generates L as a monoid. (In what follows, we will use the notation $\langle \mathbf{L}, \gamma\rangle$ for the objects of $\mathbf{LG}_*^-$.) Let the morphisms of this category be homomorphisms between these algebras. The main result of this section, Theorem 4.12, asserts that the two categories defined above are equivalent.

Definition 4.2 and Lemma 4.3 below have been influenced by results in [24]. Lemmas 4.5 and 4.7 are non-commutative, unbounded versions of results in the same paper.

Lemma 4.1. *Let a, b, c be elements of a residuated lattice $\mathbf{L} \in \mathcal{LG}^-$. Then, $ab = c$ iff ($a = c/b$ and $c \leqslant b$) iff ($b = a\backslash c$ and $c \leqslant a$).*

Proof. We prove only the first equivalence. If $ab = c$, then $ab/b = c/b$, so, by Theorem 2.12, $a = c/b$. Moreover, $c = ab \leqslant eb \leqslant b$, by integrality. Conversely, if $a = c/b$ and $c \leqslant b$, then $ab = (c/b)b$. So, $ab = c \wedge b$, because $\mathbf{L} \in \mathcal{IGBL}$, by Theorem 2.12. Since $c \leqslant b$, we get $ab = c$. □

Recall that if γ is a nucleus on some $\mathbf{L} \in \mathcal{LG}^-$, the monoid multiplication $\circ_\gamma$ of $\mathbf{L}_\gamma$ is defined by $x \circ_\gamma y = \gamma(xy)$, for all elements $x, y \in L$ (see Lemma 3.3).

Definition 4.2. Let γ be a nucleus on $\mathbf{L} \in \mathcal{LG}^-$ and let x be an element of $\mathbf{L}$. A sequence $(x_1, \ldots, x_n)$ of elements of L_γ is called a *decomposition* of x with respect to γ if $x = x_1 \cdots x_n$. A decomposition is called *canonical* if, in addition, $x_i \circ_\gamma x_{i+1} = x_i$, for all $i \in \{1, \ldots, n-1\}$.

Lemma 4.3. *Let $\mathbf{L} \in \mathcal{LG}^-$ and let γ be a nucleus on $\mathbf{L}$ such that L_γ generates L as a monoid. Then every element of L has a canonical decomposition with respect to γ. Moreover, if $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_m)$ are canonical decompositions of the same element with respect to γ and $m \geqslant n$, then $x_i = x'_i$, for all $i \in \{1, \ldots, n\}$ and $x'_i = e$ for all $i \in \{n+1, \ldots, m\}$.*

Proof. We first construct a canonical decomposition of an arbitrary element x of L. Let $x_1 = \gamma(x)$ and $x_{i+1} = \gamma((x_1 \cdots x_i)\backslash x)$, for all $i \geqslant 1$. We will prove that there exists a natural number n such that $x = x_1 \cdots x_n$ and $x_i \circ_\gamma x_{i+1} = x_i$ for all $i \in \{1, \ldots, n-1\}$.

We show, by induction, that $x \leqslant x_1 \cdots x_k$, for every integer $k \geqslant 1$. For $k = 1$ we have $x \leqslant \gamma(x) = x_1$. If $x \leqslant x_1 \cdots x_k$, then Lemma 2.11(i) yields

$$\begin{aligned} x &= x_1 \cdots x_k \wedge x \\ &= x_1 \cdots x_k \cdot \bigl[(x_1 \cdots x_k)\backslash x\bigr] \\ &\leqslant x_1 \cdots x_k \cdot \gamma\bigl((x_1 \cdots x_k)\backslash x\bigr) \\ &= x_1 \cdots x_k \cdot x_{k+1}. \end{aligned}$$

Next, let z be any element of L such that $z \leqslant x$ and set $u = \gamma(z)$. By Lemma 3.6, the maps γ and γ_u agree on $\uparrow x$. The arguments of γ in the definition of the elements x_i, as well as in the equality $\gamma(x_i \cdot x_{i+1}) = x_i$, are in $\uparrow x$, so we can replace γ by γ_u. In particular, a decomposition of an element x is canonical with respect to γ if and only if it is canonical with respect to some/every γ_u such that $u = \gamma(z)$ and $z \leqslant x$.

Applying Lemma 4.1, for $a = x_i$, $b = x_i\backslash((x_1 \cdots x_{i-1})\backslash x)$ and $c = (x_1 \cdots x_{i-1})\backslash x$, we obtain for all $i \geqslant 1$,

$$x_i\bigl[x_i\backslash\bigl((x_1 \cdots x_{i-1})\backslash x\bigr)\bigr] = (x_1 \cdots x_{i-1})\backslash x,$$

where $x_1 \cdots x_{i-1} = e$ for $i = 1$. It follows that, for all $i \geqslant 1$,

$$\begin{aligned} x_i \circ_\gamma x_{i+1} &= \gamma(x_i x_{i+1}) = \gamma_u(x_i x_{i+1}) \\ &= u \vee (x_i x_{i+1}) = u \vee \bigl(x_i \gamma_u\bigl((x_1 \cdots x_i)\backslash x\bigr)\bigr) \\ &= u \vee x_i\bigl(u \vee \bigl[(x_1 \cdots x_i)\backslash x\bigr]\bigr) \\ &= u \vee x_i u \vee x_i\bigl[(x_1 \cdots x_i)\backslash x\bigr] \\ &= u \vee x_i u \vee x_i\bigl[x_i\backslash\bigl((x_1 \cdots x_{i-1})\backslash x\bigr)\bigr] \\ &= u \vee x_i u \vee \bigl[(x_1 \cdots x_{i-1})\backslash x\bigr] \\ &= u \vee \bigl[(x_1 \cdots x_{i-1})\backslash x\bigr] \\ &= \gamma_u\bigl((x_1 \cdots x_{i-1})\backslash x\bigr) \\ &= \gamma\bigl((x_1 \cdots x_{i-1})\backslash x\bigr) = x_i. \end{aligned}$$

We next show that $(x_1 \cdots x_k)\backslash x = u^k\backslash x$, for all $k \geqslant 1$, using induction on k. For $k = 1$, we have

$$x_1\backslash x = \gamma(x)\backslash x = \gamma_u(x)\backslash x = (x \vee u)\backslash x = x\backslash x \wedge u\backslash x = e \wedge u\backslash x = u\backslash x.$$

Assume that the statement is true for k. To show that it is true for $k + 1$, note that, using properties (iii) and (vi) of Lemma 2.1, we get

$$\begin{aligned}(x_1\cdots x_{k+1})\backslash x &= x_{k+1}\backslash\big[(x_1\cdots x_k)\backslash x\big]\\ &= \gamma\big((x_1\cdots x_k)\backslash x\big)\backslash\big[(x_1\cdots x_k)\backslash x\big]\\ &= \big[u\vee\big((x_1\cdots x_k)\backslash x\big)\big]\backslash\big[(x_1\cdots x_k)\backslash x\big]\\ &= u\backslash\big[(x_1\cdots x_k)\backslash x\big]\wedge\big[(x_1\cdots x_k)\backslash x\big]\backslash\big[(x_1\cdots x_k)\backslash x\big]\\ &= u\backslash\big[(x_1\cdots x_k)\backslash x\big]\wedge e\\ &= u\backslash\big(u^k\backslash x\big) = u^{k+1}\backslash x.\end{aligned}$$

We have shown that $(x_1\cdots x_k)\backslash x = u^k\backslash x$, for all $k\geqslant 1$.

Since L is the monoid generated by L_γ, there exist a natural number n and elements $a_1,\dots,a_n\in L_\gamma$ such that $x=a_1\cdots a_n$. Thus, $u\leqslant\gamma(x)=\gamma(a_1\cdots a_n)=a_1\circ_\gamma\cdots\circ_\gamma a_n\leqslant a_i$, for all i. It follows that $u^n\leqslant a_1\cdots a_n = x$. Consequently, $e\leqslant u^n\backslash x=(x_1\cdots x_n)\backslash x$, that is, $x_1\cdots x_n\leqslant x$. Since the reverse inequality was established above, we have $x=x_1\cdots x_n$.

To establish uniqueness, let $(x_1,\dots,x_n)$ and $(x'_1,\dots,x'_m)$ be canonical decompositions of an element x with respect to γ and $m\geqslant n$. Then, $x_i\circ_\gamma x_{i+1}=x_i$, $x'_i\circ_\gamma x'_{i+1}=x'_i$, for all appropriate values of i, and $x_1\cdots x_n = x'_1\cdots x'_m$. So, $\gamma(x_1\cdots x_n)=\gamma(x'_1\cdots x'_m)$, i.e.,

$$x_1\circ_\gamma\cdots\circ_\gamma x_n = x'_1\circ_\gamma\cdots\circ_\gamma x'_m.$$

Hence $x_1=x'_1$, by the defining property of canonical decompositions. Consequently, $x_1\backslash x_1x_2\cdots x_n = x'_1\backslash x'_1x'_2\cdots x'_m$, so $x_2\cdots x_n = x'_2\cdots x'_m$, by cancellativity. Proceeding inductively, we get $x_i=x'_i$, for all $i\in\{1,\dots,n\}$. Another application of cancellativity yields $e=x'_{n+1}\cdots x'_m$, hence $x'_i=e$ for all $i\in\{n+1,\dots,m\}$, by integrality. □

It follows from the preceding lemma that each element has a canonical decomposition unique up to the addition of extra terms, equal to e, at the end of the sequence. Thus, when we consider canonical decompositions of a finite set of elements, we may assume that all have the same length.

Corollary 4.4. *Let* $\mathbf{L}\in\mathcal{LG}^-$ *and let* γ *be a nucleus on* $\mathbf{L}$*. If* $(x_1,\dots,x_n)$ *and* $(y_1,\dots,y_n)$ *are canonical decompositions of the elements* x *and* y*, respectively, with respect to* γ *and* $x\leqslant y$*, then* $x_i\leqslant y_i$*, for all* $i\leqslant n$*.*

Proof. In view of the preceding lemma, we may assume that x_i and y_i are given by the formulas at the beginning of its proof. Let z be an element of L such that $z\leqslant x\wedge y$ and let $u=\gamma(z)$. From the proof of the previous theorem we have that $(x_1\cdots x_k)\backslash x=u^k\backslash x$, and $(y_1\cdots y_k)\backslash y = u^k\backslash y$, for all $k\in\{1,\dots,n\}$. Thus, for all $i\in\{1,\dots,n\}$,

$$\begin{aligned}x_i &= \gamma\big((x_1\cdots x_{i-1})\backslash x\big)=\gamma\big(u^{i-1}\backslash x\big)\\ &\leqslant \gamma\big(u^{i-1}\backslash y\big)=\gamma\big((y_1\cdots y_{i-1})\backslash y\big)=y_i,\end{aligned}$$

where $x_1\cdots x_{i-1}=y_1\cdots y_{i-1}=e$, if $i=1$. □

Lemma 4.5. *Let $\mathbf{L} \in \mathcal{LG}^-$ and let γ be a nucleus on $\mathbf{L}$ such that L_γ generates L as a monoid. Also, let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be canonical decompositions for the elements x and y, respectively. Then,*

$$x \wedge y = \prod_{i=1}^{n} (x_i \wedge y_i) \quad \textit{and} \quad x \vee y = \prod_{i=1}^{n} (x_i \vee y_i).$$

Proof. Let $(z_1, \ldots, z_n)$ be a canonical decomposition of $z = x \wedge y$. Without loss of generality we assume that the length of the decomposition of z is n. We can do that by extending the decompositions of x and y or of z with extra terms each equal to e. Obviously,

$$\prod_{i=1}^{n} (x_i \wedge y_i) \leqslant \prod_{i=1}^{n} x_i \wedge \prod_{i=1}^{n} y_i = x \wedge y = z.$$

Moreover, $z \leqslant x, y$, so $z_i \leqslant x_i, y_i$, for all i, by Corollary 4.4; hence $z_i \leqslant x_i \wedge y_i$. Consequently,

$$z = \prod_{i=1}^{n} z_i \leqslant \prod_{i=1}^{n} (x_i \wedge y_i).$$

Thus,

$$z = \prod_{i=1}^{n} (x_i \wedge y_i).$$

The proof for joins is analogous. □

The following refinement lemma can be found in [11]. Its importance in the proof of the categorical equivalence was suggested to us by the considerations in [10]. For completeness, we give the proof in the language of negative cones of ℓ-groups.

Lemma 4.6 [11, Theorem 1, p. 68]. *Let $\mathbf{L} \in \mathcal{LG}^-$ and let $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ be elements of L. The following statements are equivalent.*

(1) *The equality $a_1 \cdots a_n = b_1 \cdots b_m$ holds.*
(2) *There exist elements c_{ij} of L, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, such that for all i, j,*

$$a_j = \prod_{i=1}^{m} c_{ij}, \quad b_i = \prod_{j=1}^{n} c_{ij} \quad \textit{and} \quad \prod_{l=j+1}^{m} c_{il} \vee \prod_{k=i+1}^{n} c_{kj} = e.$$

Notation. We denote the fact that condition (2) holds by the following configuration:

$$\begin{array}{c c} & \begin{array}{ccc} a_1 & \cdots & a_n \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\!\!\left[\begin{array}{ccc} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mn} \end{array}\right]\!\!\right] \end{array}$$

Thus, with respect to this description, condition (2) states that for all i and j, a_j is the product of the elements of the jth column, b_i is the product of the elements of the ith row and that the product of the elements to the right of c_{ij} is orthogonal to the product of elements below it.

Proof. First, we show that (2) implies (1). Recall that if $x \vee y = e$, then $xy = yx$, by Lemma 2.10. For $m = n = 2$, we have $a_1a_2 = c_{11}c_{21}c_{12}c_{22} = c_{11}c_{12}c_{21}c_{22} = b_1b_2$. We proceed by induction on the pair (m, n). Let $m \geqslant 2$, $n > 2$ and assume that the lemma is true for all pairs (m, k), where $k < n$. We will show it is true for the pair (m, n).

Suppose that condition (2) holds. It is easy to see that

$$\begin{array}{c c} & \begin{array}{ccc} a_2 & \cdots & a_n \end{array} \\ \begin{array}{c} c_1 \\ \vdots \\ c_m \end{array} & \left[\!\!\left[\begin{array}{ccc} c_{12} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{m2} & \cdots & c_{mn} \end{array}\right]\!\!\right] \end{array} \quad \text{and} \quad \begin{array}{c c} & \begin{array}{cc} a_1 & c \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\!\!\left[\begin{array}{cc} c_{11} & c_1 \\ \vdots & \vdots \\ c_{m1} & c_m \end{array}\right]\!\!\right] \end{array}$$

where $c = c_1 \cdots c_m$. So, $a_1a_2 \cdots a_n = a_1(c_1 \cdots c_m) = a_1c = b_1b_2 \cdots b_m$. Note that the lemma holds for the pair (m, n) if and only if it holds for the pair (n, m), a fact that completes the induction proof.

For the converse we use induction, as well. We first prove it for $m = n = 2$. Assume that $a_1a_2 = b_1b_2 = c$ and set

$$c_{11} = a_1 \vee b_1, \qquad c_{12} = a_2/c_{22},$$
$$c_{21} = c_{11}\backslash a_1, \qquad c_{22} = a_2 \vee b_2.$$

Using Lemmas 2.13(iii), 4.1 and 2.1 we get

$$\begin{aligned} c_{12} &= a_2/c_{22} = a_2/(a_2 \vee b_2) \\ &= (a_1\backslash c)/(a_1\backslash c \vee b_1\backslash c) = (a_1\backslash c)/\big((a_1 \wedge b_1)\backslash c\big) \\ &= a_1\backslash\big[c/\big((a_1 \wedge b_1)\backslash c\big)\big] = a_1\backslash\big[(a_1 \wedge b_1) \vee c\big] \\ &= a_1\backslash(a_1 \wedge b_1) = a_1\backslash a_1 \wedge a_1\backslash b_1 \\ &= e \wedge a_1\backslash b_1 = a_1\backslash b_1 \wedge b_1\backslash b_1 \\ &= (a_1 \vee b_1)\backslash b_1 = c_{11}\backslash b_1. \end{aligned}$$

Similarly, we show that $c_{21} = b_2/c_{22}$. So, we have

$$c_{11}c_{21} = c_{11}(c_{11}\backslash a_1) = c_{11} \wedge a_1 = (a_1 \vee b_1) \wedge a_1 = a_1,$$
$$c_{12}c_{22} = (a_2/c_{22})c_{22} = a_2 \wedge c_{22} = a_2,$$
$$c_{11}c_{12} = c_{11}(c_{11}\backslash b_1) = c_{11} \wedge b_1 = b_1,$$
$$c_{21}c_{22} = (b_2/c_{22})c_{22} = b_2 \wedge c_{22} = b_2.$$

Finally, $c_{21} \vee c_{12} = c_{11}\backslash a_1 \vee c_{11}\backslash b_1 = c_{11}\backslash(a_1 \vee b_1) = c_{11}\backslash c_{11} = e$.

For the general case, we proceed by induction on the pair (m,n). Let $m \geqslant 2$, $n > 2$ and assume that the lemma is true for all pairs (m,k), where $k < n$. We will show it is true for the pair (m,n). Assume that $a_1a_2\cdots a_n = b_1b_2\cdots b_m$ and set $a = a_2a_3\cdots a_n$. So, $a_1a = b_1b_2\cdots b_m$. By the induction hypothesis, we get

$$\begin{array}{c c} & \begin{array}{cc} a_1 & a \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\left[\begin{array}{cc} c_{11} & c_{12} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{array}\right]\right] \end{array} \quad \text{and} \quad \begin{array}{c c} & \begin{array}{ccc} a_2 & \cdots & a_n \end{array} \\ \begin{array}{c} c_{12} \\ \vdots \\ c_{m2} \end{array} & \left[\left[\begin{array}{ccc} d_{12} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{m2} & \cdots & d_{mn} \end{array}\right]\right] \end{array}$$

for some c_{ij}, d_{kl}, with appropriate indices. So, we have

$$\begin{array}{c c} & \begin{array}{cccc} a_1 & a_2 & \ldots & a_n \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\left[\begin{array}{cccc} c_{11} & d_{12} & \ldots & d_{1n} \\ \vdots & & & \vdots \\ c_{m1} & d_{m2} & \ldots & d_{mn} \end{array}\right]\right] \end{array}. \quad \square$$

Lemma 4.7. *Let* $\mathbf{L} \in \mathcal{LG}^-$, γ *be a nucleus on it and* $a, a_1, \ldots, a_n \in L_\gamma$. *Then* $a = a_1 \cdot a_2 \cdots a_n$ *if and only if* $a = a_1 \circ_\gamma a_2 \circ_\gamma \cdots \circ_\gamma a_n$ *and* $a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n)$, *for all* $1 \leqslant k < n$.

Proof. We use induction on n. For $n = 2$, if $a = a_1a_2$, then $\gamma(a) = \gamma(a_1a_2)$, so $a = a_1 \circ_\gamma a_2$. Moreover, by Lemma 4.1, $a_1 = a/a_2$, so $a_1 = (a_1 \circ_\gamma a_2)/a_2$. Conversely, if $a = a_1 \circ_\gamma a_2$ and $(a_1 \circ_\gamma a_2)/a_2 = a_1$, then $a = \gamma(a_1a_2) \leqslant \gamma(a_2) = a_2$. Since $a_1 = a/a_2$, we get $a = a_1a_2$, by Lemma 4.1.

Assume now that the statement is true for all numbers less than n. Note that if $a_1a_2\cdots a_n \in L_\gamma$, then $a_2\cdots a_n \in L_\gamma$, since $a_1a_2\cdots a_n \leqslant a_2\cdots a_n$ and L_γ is a filter, by Theorems 3.4 and 2.12,

$$\begin{aligned} & a = a_1(a_2\cdots a_n) \\ & \quad \Leftrightarrow \quad a = a_1b, \quad b = a_2\cdots a_n \quad \text{and} \quad b \in L_\gamma \\ & \quad \Leftrightarrow \quad a = a_1 \circ_\gamma b, \quad a_1 = a/b, \quad b = a_2 \circ_\gamma \cdots \circ_\gamma a_n \quad \text{and} \end{aligned}$$

$$a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n) \quad \text{for all } 2 \leqslant k < n$$

$$\Leftrightarrow \quad a = a_1 \circ_\gamma a_2 \circ_\gamma \cdots \circ_\gamma a_n \quad \text{and}$$

$$a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n)$$

$$\text{for all } 1 \leqslant k < n. \qquad \square$$

Lemma 4.8. *Assume that* $\mathbf{K}, \mathbf{L} \in \mathcal{LG}^-$, γ_1, γ_2 *are nuclei on* $\mathbf{K}, \mathbf{L}$ *and* K_{γ_1}, L_{γ_2} *generate* $\mathbf{K}$ *and* $\mathbf{L}$ *as monoids, respectively. Let* $f : \mathbf{K}_{\gamma_1} \to \mathbf{L}_{\gamma_2}$ *be a residuated lattice homomorphism and let* $a_1, \ldots, a_n, b_1, \ldots, b_m$ *be elements of* K_{γ_1}, *such that* $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m$, *where multiplication is in* $\mathbf{K}$. *Then,* $f(a_1) f(a_2) \cdots f(a_n) = f(b_1) f(b_2) \cdots f(b_m)$, *where multiplication takes place in* $\mathbf{L}$.

Proof. First note that, for all $c_1, c_2, \ldots, c_n \in K_{\gamma_1}$, if $c_1 c_2 \cdots c_n \in K_{\gamma_1}$, then

$$f(c_1 c_2 \cdots c_n) = f(c_1) f(c_2) \cdots f(c_n).$$

Indeed, by Lemma 4.7, the statement $c = c_1 c_2 \cdots c_n$, for an element $c \in K$, is equivalent to a system of IGMV-algebra equations in $\mathbf{K}_{\gamma_1}$. Since f is a homomorphism, the same equations hold for the images of the elements under f. Applying Lemma 4.7 again, we get $f(c) = f(c_1) f(c_2) \cdots f(c_n)$.

Next, the equality $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m$ implies, by Lemma 4.6, that there exist $c_{ij} \in K_{\gamma_1}$, such that if for all i, j,

$$a_j = \prod_{i=1}^{m} c_{ij}, \quad b_i = \prod_{j=1}^{n} c_{ij} \quad \text{and} \quad \prod_{l=j+1}^{m} c_{il} \vee \prod_{k=i+1}^{n} c_{kj} = e.$$

Note that all of the products above are in K_{γ_1}. Using the observation above and the fact that f preserves joins (recall that the join operation in $\mathbf{K}_{\gamma_1}$ is the restriction of the join operation in $\mathbf{K}$, by Theorems 3.4 and 2.12), we get that, for all i, j,

$$f(a_j) = \prod_{i=1}^{m} f(c_{ij}), \quad f(b_i) = \prod_{j=1}^{n} f(c_{ij}) \quad \text{and} \quad \prod_{l=j+1}^{m} f(c_{il}) \vee \prod_{k=i+1}^{n} f(c_{kj}) = e.$$

Finally, we obtain

$$f(a_1) f(a_2) \cdots f(a_n) = f(b_1) f(b_2) \cdots f(b_m)$$

by applying Lemma 4.6 once more. $\square$

The following result is an immediate consequence of Theorem 1.4.5 of [3].

Lemma 4.9. *Any multiplicative meet-homomorphism between two members of* $\mathcal{LG}^-$ *is a residuated lattice homomorphism.*

Lemma 4.10. *Let* $\mathbf{K}, \mathbf{L} \in \mathcal{LG}^-$, *and let* γ_1, γ_2 *be nuclei on* $\mathbf{K}, \mathbf{L}$, *respectively, such that* $K_{\gamma_1}, L_{\gamma_2}$ *generate* $\mathbf{K}, \mathbf{L}$ *as monoids. If* $f : \mathbf{K}_{\gamma_1} \to \mathbf{L}_{\gamma_2}$ *is a residuated lattice homomorphism, then there exists a unique homomorphism* $\bar{f} : \mathbf{K} \to \mathbf{L}$, *such that* $f \circ \gamma_1 = \gamma_2 \circ \bar{f}$.

Proof. By assumption every element of K is a product of elements of K_{γ_1}. By Lemma 4.8, the map $\bar{f} : K \to L$, defined by $\bar{f}(x_1 x_2 \cdots x_n) = f(x_1) f(x_2) \cdots f(x_n)$, for $x_1, x_2, \ldots, x_n \in K_{\gamma_1}$, is well defined and obviously preserves multiplication.

If $x \in K$, then there exist $x_1, \ldots, x_n \in K_{\gamma_1}$ such that $x = x_1 \cdots x_n$. Hence,

$$\begin{aligned} \bar{f}\big(\gamma_1(x)\big) &= f\big(\gamma_1(x)\big) = f\big(\gamma_1(x_1 \cdots x_n)\big) = f(x_1 \circ_{\gamma_1} \cdots \circ_{\gamma_1} x_n) \\ &= f(x_1) \circ_{\gamma_2} \cdots \circ_{\gamma_2} f(x_n) = \gamma_2\big(f(x_1) \cdots f(x_n)\big) = \gamma_2\big(\bar{f}(x)\big). \end{aligned}$$

Thus, $\bar{f} \circ \gamma_1 = \gamma_2 \circ \bar{f}$.

Moreover, if $(x_1, \ldots, x_n)$ is a canonical decomposition for x with respect to γ_1, then $x = x_1 \cdots x_n$ and $x_i \circ_{\gamma_1} x_{i+1} = x_i$. So, $\bar{f}(x) = f(x_1) \cdots f(x_n)$ and $f(x_i) \circ_{\gamma_2} f(x_{i+1}) = f(x_i)$, i.e., $(f(x_1), \ldots, f(x_n))$ is a canonical decomposition for $\bar{f}(x)$ with respect to γ_2.

We can now show that $\bar{f}$ preserves meets. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be canonical decompositions for x, y. Then, by Lemma 4.5,

$$\bar{f}(x \wedge y) = \bar{f}\left(\prod_{i=1}^{n} (x_i \wedge y_i)\right) = \prod_{i=1}^{n} f(x_i \wedge y_i) = \prod_{i=1}^{n} \big(f(x_i) \wedge f(y_i)\big) = \bar{f}(x) \wedge \bar{f}(y),$$

where the last equality is given by Lemma 4.5, since f preserves canonical decompositions. Thus $\bar{f}$ preserves multiplication and meet, and hence it is a residuated lattice homomorphism, by Lemma 4.9. □

Corollary 4.11. *Under the hypothesis of the previous lemma, if* f *is an injection, a surjection or an isomorphism, then so is* $\bar{f}$.

Proof. Assume that f is onto and let $y \in L$. There exist $y_1, \ldots, y_n \in L_{\gamma_2}$, such that $y = y_1 \cdots y_n$. Moreover, there exist $x_1, \ldots, x_n \in K_{\gamma_1}$, such that $f(x_i) = y_i$ for all i. Then, $\bar{f}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n) = y_1 \cdots y_n = y$.

Assume that f is injective. If $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ are canonical decompositions for x, y and $\bar{f}(x) = \bar{f}(y)$, namely $f(x_1) \cdots f(x_n) = f(y_1) \cdots f(y_n)$ then, by the preservation of the canonicity of the decomposition under $\bar{f}$, established in the proof of the previous lemma, we get $f(x_i) = f(y_i)$ for all i. By the injectivity of f we get $x_i = y_i$, for all i, so $x = y$. □

Theorem 4.12. *The categories* **IGMV** *and* $\mathbf{LG}_*^-$ *are equivalent.*

Proof. For an object $\langle \mathbf{K}, \gamma \rangle$ of $\mathbf{LG}_*^-$, let $\Gamma(\langle \mathbf{K}, \gamma \rangle) = \mathbf{K}_\gamma$; for a homomorphism $f : \langle \mathbf{K}, \gamma_1 \rangle \to \langle \mathbf{L}, \gamma_2 \rangle$, let $\Gamma(f)$ be the restriction of f to K_{γ_1}.

By Corollary 3.5, $\Gamma(\langle \mathbf{K}, \gamma \rangle)$ is an object in **IGMV**. Using the fact that f commutes with the nuclei γ_1 and γ_2, it is easy to see that $\Gamma(f)$ is a morphism of **IGMV**. To

Table 1
Categorical equivalences

$\mathbf{K} =$		**IGMV**	$\mathbf{LG}_*^-$	$\mathbf{LG}_*$
K	Obj	$\mathcal{IGMV}$	$\langle \mathbf{L}, \gamma\rangle \in \mathcal{LG}_*^-$ $L = \langle \gamma(L)\rangle$	$(\mathbf{G}, \gamma)$, $\mathbf{G} \in \mathcal{LG}$ $G^- = \langle \gamma(G^-)\rangle$ γ is a nucleus on $\mathbf{G}^-$
	Mor	$H(\mathcal{IGMV})$	$H(\mathcal{LG}_*^-)$	$f \in H(\mathcal{LG})$, $f : \mathbf{G} \to \mathbf{H}$ $f\|_{G^-} \circ \gamma = \gamma \circ f\|_{G^-}$
b**K**	Obj	$b\mathcal{IGMV}$	$\langle \mathbf{L}, \gamma_u\rangle \in \mathcal{LG}_*^-$ $u \in L = \langle \gamma(L)\rangle$	$(\mathbf{G}, \gamma_u)$, $\mathbf{G} \in \mathcal{LG}$ $u \in G^- = \langle \gamma_u(G^-)\rangle$ γ_u is a nucleus on $\mathbf{G}^-$
	Mor	$H(\mathcal{IGMV})$	$H(\mathcal{LG}_*^-)$	$f \in H(\mathcal{LG})$, $f : \mathbf{G} \to \mathbf{H}$ $f\|_{G^-} \circ \gamma = \gamma \circ f\|_{G^-}$
Kb	Obj	$\mathcal{IGMV}$	$\langle \mathbf{L}, \gamma\rangle \in \mathcal{LG}_*^-$ $L = \langle \gamma(L)\rangle$	$(\mathbf{G}, \gamma)$, $\mathbf{G} \in \mathcal{LG}$ $G^- = \langle \gamma(G^-)\rangle$ γ is a nucleus on $\mathbf{G}^-$
	Mor	$f \in H(\mathcal{IGMV})$ $f : \mathbf{M} \to \mathbf{N}$, $\uparrow f[M] = N$	$f \in H(\mathcal{LG}_*^-)$ $f : \mathbf{K} \to \mathbf{L}$, $\uparrow f[K] = L$	$f \in H(\mathcal{LG})$, $f : \mathbf{G} \to \mathbf{H}$ $f\|_{G^-} \circ \gamma = \gamma \circ f\|_{G^-}$ $\uparrow f[G] = H$
b**K**b	Obj	$b\mathcal{IGMV}$	$\langle \mathbf{L}, \gamma_u\rangle \in \mathcal{LG}_*^-$ $u \in L = \langle \gamma(L)\rangle$	$(\mathbf{G}, \gamma_u)$, $\mathbf{G} \in \mathcal{LG}$ $u \in G^- = \langle \gamma_u(G^-)\rangle$ γ_u is a nucleus on $\mathbf{G}^-$
	Mor	$f \in H(\mathcal{IGMV})$ $f : \mathbf{M} \to \mathbf{N}$, $\uparrow f[M] = N$	$f \in H(\mathcal{LG}_*^-)$ $f : \mathbf{K} \to \mathbf{L}$, $\uparrow f[K] = L$	$f \in H(\mathcal{LG})$, $f : \mathbf{G} \to \mathbf{H}$ $f\|_{G^-} \circ \gamma = \gamma \circ f\|_{G^-}$ $\uparrow f[G] = H$

check, for example, that it preserves multiplication, note that $\Gamma(f)(x \circ_{\gamma_1} y) = f(\gamma_1(xy)) = \gamma_2(f(xy)) = \gamma_2(f(x)f(y)) = f(x) \circ_{\gamma_2} f(y)$.

Moreover, it is obvious that $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$ and that $\Gamma(\mathrm{id}_{\mathbf{K}_{\gamma_1}}) = \mathrm{id}_{\mathbf{K}_{\gamma_2}}$. Thus, Γ is a functor between the two categories.

By Theorem 3.12, Γ is onto the objects of **IGMV** and by Lemma 4.10, Γ is full. Finally, Γ is faithful, because if two morphisms agree on a generating set, they agree on the whole negative cone of the ℓ-group. Thus, Γ is a categorical equivalence between the two categories, by Theorem 1, page 93, of [23]. □

In addition to **IGMV** and $\mathbf{LG}_*^-$, we also consider the following categories, the definitions of which we organize in Table 1.

We first explain the notation that is used. A *bounded* GMV-algebra is a residuated bounded-lattice whose 0-free reduct is a GMV-algebra; bounded GMV-algebras are called *pseudo-MV-algebras* in [15]. It is easy to see that every bounded GMV-algebra is integral. Bounded GMV-algebras form a variety, which we denote by $b\mathcal{IGMV}$. We denote the class of integral GMV-algebras by $\mathcal{IGMV}$, and the class of objects of the category $\mathbf{LG}_*^-$ by $\mathcal{LG}_*^-$. If $\mathcal{K}$ is a class of algebras, we denote by $H(\mathcal{K})$ the class of all homomorphisms between the algebras of $\mathcal{K}$; we denote the submonoid of a residuated lattice generated by a set X by $\langle X\rangle$. The category $\mathbf{K}$ in the first column takes as values the categories in the top row. For example, the last entry of the table describes the category $b\mathbf{LG}_*b$.

Note that the functor defining the equivalence of Theorem 4.12 specializes to pairs of domain and range as described in (the first two columns of) the last three rows of the table. Moreover, since the category of ℓ-groups and the category of their negative cones are equivalent, by [2], the categories $\mathbf{LG}_*^-$ and $\mathbf{LG}_*$ are equivalent. Consequently, all three categories in the first row of the table are equivalent. The same arguments apply to the last two columns of the remaining three rows, so each of the four rows consists of a triple of equivalent categories. The categorical equivalence of the last row is the one established by A. Dvurečenskij in [10]. If we restrict further to the commutative case, we obtain D. Mundici's result in [24].

5. Decomposition of GBL-algebras

The primary objective of this section is to establish Theorem A (see Theorem 5.6 below). Its proof is based on the decomposition result of Theorem 5.2. We refer the reader to [22] for a comprehensive discussion of products of residuated structures.

Lemma 5.1. *GBL-algebras satisfy the identity* $x \approx (x\vee e)(x\wedge e)$.

Proof. By Lemma 2.8, $(e/x\wedge e)x = x\wedge e$. Moreover, by Lemma 2.7(i), $x\vee e$ is invertible and $(x\vee e)^{-1} = e/(x\vee e) = e/x\wedge e$. Thus, $(x\vee e)^{-1}x = x\wedge e$, or $x = (x\vee e)(x\wedge e)$. □

The following theorem shows that if $\mathbf{L}$ is a GBL-algebra then the sets $G(\mathbf{L})$ and $I(\mathbf{L})$, given in Definition 2.6, are subuniverses of $\mathbf{L}$. We denote the corresponding subalgebras by $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$.

Theorem 5.2. *Every GBL-algebra* $\mathbf{L}$ *decomposes into the direct sum* $\mathbf{G}(\mathbf{L})\oplus\mathbf{I}(\mathbf{L})$.

Proof. We begin with a series of claims.

Claim 1. $G(\mathbf{L})$ *is a subuniverse of* $\mathbf{L}$.

Let x, y be invertible elements. It is clear that xy is invertible. Additionally, for all $x, y\in G(\mathbf{L})$ and $z\in L$, $z\leqslant x^{-1}y \Leftrightarrow xz\leqslant y \Leftrightarrow z\leqslant x\backslash y$. It follows that $x\backslash y = x^{-1}y$, hence $x\backslash y$ is invertible. Likewise, $y/x = yx^{-1}$ is invertible.

Moreover, $x\vee y = (xy^{-1}\vee e)y$. So, $x\vee y$ is invertible, since every positive element is invertible, by Lemma 2.7(i), and the product of two invertible elements is invertible. By Lemma 2.1(iii), $x\wedge y = e/(x^{-1}\vee y^{-1})$, which is invertible, since we have already shown that $G(\mathbf{L})$ is closed under joins and the division operations.

Claim 2. $I(\mathbf{L})$ *is a subuniverse of* $\mathbf{L}$.

Note that every integral element a is negative, since $e = e/a$ implies $e\leqslant e/a$ and $a\leqslant e$. For $x, y\in I(\mathbf{L})$, using Lemma 2.1 repeatedly, we get:

$$e/xy = (e/y)/x = e/x = e, \quad \text{so} \quad xy \in I(\mathbf{L}),$$

$$e/(x \vee y) = e/x \wedge e/y = e, \quad \text{so} \quad x \vee y \in I(\mathbf{L}),$$

$$e \leqslant e/x \leqslant e/(x \wedge y) \leqslant e/xy = e, \quad \text{so} \quad x \wedge y \in I(\mathbf{L}),$$

$$e = e/(e/y) \leqslant e/(x/y) \leqslant e/(x/e) = e/x = e, \quad \text{so} \quad x/y \in I(\mathbf{L}).$$

Claim 3. *For every $g \in (G(\mathbf{L}))^-$ and every $h \in I(\mathbf{L})$, $g \vee h = e$.*

Let $g \in (G(\mathbf{L}))^-$ and $h \in I(\mathbf{L})$. We have $e/(g \vee h) = e/g \wedge e/h = e/g \wedge e = e$, since $e \leqslant e/g$. Moreover, $g \leqslant g \vee h$, so $e \leqslant g^{-1}(g \vee h)$. Thus, by the GBL-algebra identities and Lemma 2.1

$$\begin{aligned} e &= \bigl(e/\bigl[g^{-1}(g \vee h)\bigr]\bigr)\bigl[g^{-1}(g \vee h)\bigr] = \bigl(\bigl[e/(g \vee h)\bigr]/g^{-1}\bigr)g^{-1}(g \vee h) \\ &= \bigl(e/g^{-1}\bigr)g^{-1}(g \vee h) = gg^{-1}(g \vee h) = g \vee h. \end{aligned}$$

Claim 4. *For every $g \in (G(\mathbf{L}))^-$ and every $h \in I(\mathbf{L})$, $gh = g \wedge h$.*

In light of Lemma 5.1, $g^{-1}h = (g^{-1}h \vee e)(g^{-1}h \wedge e)$. Multiplication by g yields $h = (h \vee g)(g^{-1}h \wedge e)$. Using Claim 3, we have $gh = g(g^{-1}h \wedge e) = h \wedge g$, since multiplication by an invertible element is an order automorphism.

Claim 5. *For every $g \in G(\mathbf{L})$ and every $h \in I(\mathbf{L})$, $gh = hg$.*

The statement is true if $g \leqslant e$, by Claim 4. If $g \geqslant e$ then $g^{-1} \leqslant e$, thus $g^{-1}h = hg^{-1}$, hence $hg = gh$. For arbitrary g, note that both $g \vee e$ and $g \wedge e$ commute with h. Using Lemma 5.1, we get $gh = (g \vee e)(g \wedge e)h = (g \vee e)h(g \wedge e) = h(g \vee e)(g \wedge e) = hg$.

Claim 6. *For every $x \in L$, there exist $g_x \in G(\mathbf{L})$ and $h_x \in I(\mathbf{L})$, such that $x = g_x h_x$.*

By Lemma 5.1, $x = (x \vee e)(x \wedge e)$. Since $e \leqslant x \vee e$ and $e \leqslant e/(x \wedge e)$, by Lemma 2.7(i), these elements are invertible. Set $g_x = (x \vee e)(e/(x \wedge e))^{-1}$ and $h_x = (e/(x \wedge e))(x \wedge e)$. It is clear that $x = g_x h_x$, g_x is invertible and h_x is integral.

Claim 7. *For every $g_1, g_2 \in G(\mathbf{L})$ and $h_1, h_2 \in I(\mathbf{L})$, $g_1h_1 \leqslant g_2h_2$ if and only if $g_1 \leqslant g_2$ and $h_1 \leqslant h_2$.*

For the non-trivial direction we have

$$g_1h_1 \leqslant g_2h_2 \quad \Rightarrow \quad g_2^{-1}g_1h_1 \leqslant h_2 \quad \Rightarrow \quad g_2^{-1}g_1 \leqslant h_2/h_1 \leqslant e \quad \Rightarrow \quad g_1 \leqslant g_2.$$

Moreover,

$$\begin{aligned} g_2^{-1}g_1 \leqslant h_2/h_1 \quad &\Rightarrow \quad e \leqslant g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow \quad e = \big[e/g_1^{-1}g_2(h_2/h_1)\big]g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow \quad e = \big[\big(e/(h_2/h_1)\big)/g_1^{-1}g_2\big]g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow \quad e = g_2^{-1}g_1g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow \quad e = h_2/h_1 \\ &\Rightarrow \quad h_1 \leqslant h_2. \end{aligned}$$

By Claims 1 and 2, $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$ are subalgebras of $\mathbf{L}$. Define $f : \mathbf{G}(\mathbf{L}) \times \mathbf{I}(\mathbf{L}) \to \mathbf{L}$ by $f(g,h) = gh$. We will show that f is an isomorphism. It is onto by Claim 6 and an order isomorphism by Claim 7. So, it is a lattice isomorphism, as well. To verify that f preserves the other operations note that $gg'hh' = ghg'h'$, for all $g, g' \in G(\mathbf{L})$ and $h, h' \in I(\mathbf{L})$, by Claim 5. Moreover, for all $g, g', \bar{g} \in G(\mathbf{L})$ and $h, h', \bar{h} \in I(\mathbf{L})$, $\bar{g}\bar{h} \leqslant gh/g'h'$ if and only if $\bar{g}\bar{h}g'h' \leqslant gh$. By Claim 5, this is equivalent to $\bar{g}g'\bar{h}h' \leqslant gh$, and, by Claim 7, to $\bar{g}g' \leqslant g$ and $\bar{h}h' \leqslant h$. This is in turn equivalent to $\bar{g} \leqslant g/g'$ and $\bar{h} \leqslant h/h'$, which is equivalent to $\bar{g}\bar{h} \leqslant (g/g')(h/h')$ by Claim 7. Thus, for all $g, g' \in G(\mathbf{L})$ and $h, h' \in I(\mathbf{L})$, $gh/g'h' = (g/g')(h/h')$ and, likewise, $g'h'\backslash gh = (g'\backslash g)(h'\backslash h)$. $\square$

Corollary 5.3. *The varieties $\mathcal{GBL}$ and $\mathcal{GMV}$ decompose as follows*:

$$\mathcal{GBL} = \mathcal{LG} \times \mathcal{IGBL} = \mathcal{LG} \vee \mathcal{IGBL} \quad \textit{and} \quad \mathcal{GMV} = \mathcal{LG} \times \mathcal{IGMV} = \mathcal{LG} \vee \mathcal{IGMV}.$$

Taking intersections with $\mathcal{C}an\mathcal{RL}$ and recalling Theorem 2.12, we get:

Corollary 5.4. $\mathit{Can}\mathcal{GMV} = \mathit{Can}\mathcal{GBL} = \mathcal{LG} \times \mathcal{LG}^-$.

Here we have set $\mathit{Can}\mathcal{GMV} = \mathit{Can}\mathcal{RL} \cap \mathcal{GMV}$ and $\mathit{Can}\mathcal{GBL} = \mathit{Can}\mathcal{RL} \cap \mathcal{GBL}$. Moreover, in conjunction with Lemma 2.13(iv) and Theorem 2.2, Corollary 5.3 yields:

Corollary 5.5. *Every commutative GMV-algebra is representable.*

By combining Theorems 5.2 and 3.12, we obtain the main result of this section.

Theorem 5.6. *A residuated lattice $\mathbf{M}$ is a GMV-algebra if and only if there exist residuated lattices $\mathbf{G}, \mathbf{L}$, such that $\mathbf{G}$ is an ℓ-group, $\mathbf{L} \in \mathcal{LG}^-$, γ is a nucleus on $\mathbf{L}$ and $\mathbf{M} = \mathbf{G} \oplus \mathbf{L}_\gamma$. Equivalently, $\mathbf{M}$ is a GMV-algebra if and only if it has a direct product decomposition $\mathbf{M} \cong \mathbf{G} \times \mathbf{H}_\gamma^-$, where $\mathbf{G}, \mathbf{H}$ are ℓ-groups and γ is a nucleus on $\mathbf{H}^-$.*

6. A categorical equivalence for GMV-algebras

The goal of this section is to establish Theorems D and E (see Theorems 6.6 and 6.9 below).

If $\mathbf{G}, \mathbf{H}$ are ℓ-groups and γ is a nucleus on $\mathbf{H}^-$, define $\delta(g,h) = (g, h \wedge e)$ and $\gamma'(g,h') = (g, \gamma(h'))$, for all $g \in G$, $h \in H$ and $h' \in H^-$. It follows from Theorem 5.6 that the underlying set of every GMV-algebra $\mathbf{M}$ is of the form $\gamma'(\delta(G \times H))$, where $\mathbf{G}, \mathbf{H}$ are ℓ-groups and γ is a nucleus on $\mathbf{H}^-$.

Note that δ is an interior operator on $\mathbf{L} = \mathbf{G} \times \mathbf{H}$, i.e., it is *contracting* ($\delta(x) \leqslant x$, for all $x \in L$), *monotone* (if $x \leqslant y$, then $\delta(x) \leqslant \delta(y)$, for all $x, y \in L$) and *idempotent* ($\delta(\delta(x)) = \delta(x)$, for all $x \in L$). Moreover, its image $L_\delta = \delta(L)$ is a submonoid and a lattice ideal of $\mathbf{L}$. More explicitly, we have $\delta(\delta(x)\delta(y)) = \delta(x)\delta(y)$, $\delta(e) = e$, $\delta(x) \wedge y = \delta(\delta(x) \wedge y)$ and $\delta(x) \vee \delta(y) = \delta(\delta(x) \vee \delta(y))$, for all x, y in L. We call an interior operator on a residuated lattice that satisfies the above properties a *kernel* operator; note that the last equality follows from the fact that δ is an interior operator and is not needed in the definition of a kernel. A *core* operator on a residuated lattice $\mathbf{L}$ is the composition $\gamma \circ \delta$ of a kernel operator δ on $\mathbf{L}$ and a nucleus γ on the image $\mathbf{L}_\delta$ of δ; see Lemma 6.1.

6.1. The object level: representations of GMV-algebras

The main result of this subsection is Theorem D (see Theorem 6.6 below). En route, we show that any core on a GMV-algebra has a unique representation as the composition of a nucleus and a kernel operator.

Lemma 6.1. *If $\mathbf{L}$ is a residuated lattice and δ a kernel on it, then the algebra $\mathbf{L}_\delta = \langle \delta(L), \wedge, \vee, \cdot, \backslash_\delta, /_\delta, e\rangle$, where $x /_\delta y = \delta(x/y)$ and $x \backslash_\delta y = \delta(x \backslash y)$, is a residuated lattice. Moreover, $\mathbf{L}_\delta$ is a lattice ideal of $\mathbf{L}$. If $\mathbf{L}$ is a GMV-algebra or a GBL-algebra, then so is $\mathbf{L}_\delta$.*

Proof. L_δ is closed under join, since δ is an interior operator, and under multiplication, by the first property of a kernel. Moreover, it contains e and it is obviously closed under $\backslash_\delta$ and $/_\delta$. By the third property of a kernel and the fact that it is closed under joins, L_δ is an ideal of $\mathbf{L}$. So, $\mathbf{L}_\delta$ is a submonoid and a subsemilattice of $\mathbf{L}$. Moreover, $\mathbf{L}_\delta$ is residuated. For all $x, y, z \in L_\delta$, $x \leqslant z /_\delta y$ is equivalent to $x \leqslant \delta(x/y)$, which in turn is equivalent to $x \leqslant z/y$, since δ is contracting and $x = \delta(x)$.

If $\mathbf{L}$ is a GMV-algebra, then

$$(x \vee y)\backslash x = x\backslash x \wedge y\backslash x = e \wedge y\backslash x \leqslant e.$$

Since L_δ is an ideal that contains e, we have $\delta((x \vee y)\backslash x) = (x \vee y)\backslash x$, for $x, y \in L_\delta$. So,

$$x /_\delta \big[(x \vee y)\backslash_\delta x\big] = \delta\big(x/\delta\big((x \vee y)\backslash x\big)\big) = \delta\big(x/\big((x \vee y)\backslash x\big)\big) = \delta(x \vee y) = x \vee y.$$

Similarly, if $\mathbf{L}$ is a GBL-algebra, we have

$$\big((x \wedge y)/_\delta y\big) y = \delta\big((x \wedge y)/y\big) y = \big((x \wedge y)/y\big) y = x \wedge y,$$

since $(x \wedge y)/y \leqslant e$. $\square$

Note that the map δ on a residuated lattice $\mathbf{L}$, defined by $\delta(x) = x \wedge e$, is a kernel on $\mathbf{L}$ and $\mathbf{L}_\delta = \mathbf{L}^-$.

For a class of algebras $\mathcal{K}$ we denote by $\mathbf{n}(\mathcal{K})$ and $\mathbf{k}(\mathcal{K})$ the class of all images of nuclei and kernels, respectively, of members of $\mathcal{K}$. We already know that $\mathbf{n}(\mathcal{LG}^-) = \mathcal{IGMV}$, from Theorem 3.12, and $\mathcal{GMV} \subseteq \mathbf{n}(\mathbf{k}(\mathcal{LG}))$. We will show that $\mathbf{k}(\mathcal{LG}) = \mathit{Can}\mathcal{GMV}$ and $\mathbf{n}(\mathit{Can}\mathcal{GMV}) = \mathcal{GMV}$. Moreover, we will give an alternative characterization of core operators. It follows from the lemma below that $\mathbf{n}(\mathcal{LG}) = \mathcal{LG}$ and $\mathbf{k}(\mathcal{IGMV}) = \mathcal{IGMV}$.

Lemma 6.2.

(i) *The identity map is the only nucleus on an ℓ-group.*
(ii) *The identity is the only kernel on an integral GMV-algebra.*

Proof. (i) Assume γ is a nucleus on the ℓ-group $\mathbf{G}$. Since $\mathbf{G}$ is a GMV-algebra, we have $e = \gamma(e) \in G_\gamma$, by Theorem 3.4; hence $G^+ \subseteq G_\gamma$. Moreover, by Lemma 3.1, for every $x \in G$, $e/x \in G_\gamma$, that is, $x^{-1} \in G_\gamma$. Thus, $G_\gamma = G$. Since a closure operator is uniquely defined by its image, γ is the identity on G.

(ii) Assume that δ is a kernel on an integral GMV-algebra $\mathbf{M}$. By Lemma 6.1, M_δ is an ideal of M. Moreover, $e = \delta(e) \in M_\delta$. So, $M_\delta = M$ and δ is the identity map on M. $\square$

The following corollary describes the action of a kernel on a GMV-algebra and shows that $\mathbf{k}(\mathcal{LG}) \subseteq \mathit{Can}\mathcal{GMV}$. In what follows, we will use the term ℓ-subgroup for a subalgebra of a residuated lattice that happens to be an ℓ-group.

Corollary 6.3. *If δ is a kernel on a GMV-algebra* $\mathbf{M}$*, then there exist a GMV-subalgebra* $\mathbf{N}$ *and an ℓ-subgroup* $\mathbf{H}$ *of* $\mathbf{M}$*, such that* $\mathbf{M} = \mathbf{N} \oplus \mathbf{H}$ *and $\delta(nh) = n(h \wedge e)$, for all $n \in N$ and $h \in H$. Thus,* $\mathbf{M}_\delta = \mathbf{N} \oplus \mathbf{H}^-$. *If* $\mathbf{M}$ *is an ℓ-group, then so is* $\mathbf{N}$.

Proof. By Theorem 5.6, there exist ℓ-groups $\mathbf{G}, \mathbf{L}$, and a nucleus γ on $\mathbf{L}^-$, such that $\mathbf{M} = \mathbf{G} \oplus \mathbf{L}_\gamma^-$. The restrictions of δ on $\mathbf{G}$ and $\mathbf{L}_\gamma^-$, also denoted by δ, are kernels, because of the equational definition of a kernel.

First, note that $\delta(L_\gamma^-) \subseteq L_\gamma^-$ and $\delta(G) \subseteq G$. To verify this, observe that the image of $\mathbf{M}$ under δ is an ideal of $\mathbf{M}$, that contains the identity e, by Lemma 6.1; hence the negative cone of $\mathbf{M}$ is fixed by δ. In particular, L_γ^- and G^- are fixed by δ. Consider an element x in G. We will show that $\delta(x)$ is also in G. Let $\delta(x) = yk$, where $y \in G$ and $k \in L_\gamma^-$. Since $yk = \delta(x) \leqslant x = xe$, we have $y \leqslant x$. Both yk and e are fixed by δ, so the same holds for their join $(y \vee e)(k \vee e) = y \vee e$, since the image of δ is a lattice ideal. Likewise, y is fixed by δ since $y \leqslant y \vee e$. The element $\delta(x)$ is the maximum element below x fixed by δ; so $y \leqslant \delta(x)$, since $y \leqslant x$. On the other hand, $\delta(x) = yk \leqslant y$; hence $\delta(x) = y \in G$.

We will show that there exist ℓ-subgroups $\mathbf{K}, \mathbf{H}$ of $\mathbf{G}$, such that $\mathbf{G} = \mathbf{K} \oplus \mathbf{H}$ and $\delta(kh) = k(h \wedge e)$, for all $k \in K$ and $h \in H$. Observe that $\mathbf{G}_\delta$ is a GMV-algebra, by Lemma 6.1, so there are ℓ-groups $\mathbf{K}, \mathbf{H}$ and a nucleus γ on $\mathbf{H}^-$, such that

$$\mathbf{G}_\delta = \mathbf{K} \oplus \mathbf{H}_\gamma^-,$$

by Theorem 5.6. Since $\mathbf{K} \times \mathbf{H}_\gamma^-$ is isomorphic to $\mathbf{G}_\delta$, the negative cones $\mathbf{K}^- \times \mathbf{H}_\gamma^-$ and $\mathbf{G}_\delta^-$ are isomorphic. Moreover, we have $(G_\delta)^- = G^-$, because G_δ is an ideal of $\mathbf{G}$ that contains e. The operations on $(\mathbf{G}_\delta)^-$ and $\mathbf{G}^-$ agree, since the lattice and monoid operations on both algebras are the restrictions to $(G_\delta)^- = G^-$ of the operations on $\mathbf{G}$. Additionally, for all $z \in G$, $z \wedge e$ is the greatest element fixed under δ that is below z; so, $z \wedge e = \delta(z) = \delta(z) \wedge e$, and for all $x, y \in G^-$, $x \backslash_{(\mathbf{G}_\delta)^-} y = x \backslash_\delta y \wedge e = \delta(x \backslash y) \wedge e = x \backslash y \wedge e = x \backslash_{\mathbf{G}^-} y$ and likewise for right division. Consequently, $\mathbf{K}^- \times \mathbf{H}_\gamma^-$ is isomorphic to $\mathbf{G}^-$ via the map $(k, h) \mapsto kh$; i.e.,

$$\mathbf{G}^- = \mathbf{K}^- \oplus \mathbf{H}_\gamma^-.$$

Since $\mathbf{H}_\gamma^-$ is a subalgebra of $\mathbf{G}^- \in \mathcal{LG}^-$, we have $\mathbf{H}_\gamma^- \in \mathcal{LG}^-$. For simplicity of the presentation, and without loss of generality, we assume that $\mathbf{H}$ is such that γ is the identity on $\mathbf{H}^-$. So,

$$\mathbf{G}^- = \mathbf{K}^- \oplus \mathbf{H}^- = (\mathbf{K} \oplus \mathbf{H})^-$$

and $\mathbf{G}$ is isomorphic to $\mathbf{K} \oplus \mathbf{H}$. We simplify notation by identifying isomorphic algebras, so $\mathbf{G} = \mathbf{K} \oplus \mathbf{H}$.

We have shown that $(K \oplus H)_\delta = K \oplus H^-$. Thus, $\delta(K \oplus H) = \delta'(K \oplus H)$, where $\delta'(gh) = g(h \wedge e)$ is a interior operator. Since an interior operator is defined by its image, we get $\delta(gh) = g(h \wedge e)$. So $\mathbf{M} = \mathbf{K} \oplus \mathbf{H} \oplus \mathbf{L}_\gamma^-$. Moreover, δ is the identity on $\mathbf{L}_\gamma^-$. If we set $\mathbf{N} = \mathbf{K} \oplus \mathbf{L}_\gamma^-$, we get $\mathbf{M} = \mathbf{N} \oplus \mathbf{H}$ and $\delta(nh) = n(h \wedge e)$, for all $n \in N$ and $h \in H$. □

Definition 6.4.

(i) If δ is a map on a residuated lattice $\mathbf{L}$ and γ a map on $\delta(L)$, define the map $\beta_{(\gamma,\delta)}$ on L by $\beta_{(\gamma,\delta)}(x) = \gamma(\delta(x))$.
(ii) If β is a map on a residuated lattice $\mathbf{L}$, define the maps δ_β on L and γ_β on $\delta_\beta(L)$ by $\delta_\beta(x) = \beta(x) \wedge x$ and $\gamma_\beta(x) = \beta(x)$.

Lemma 6.5. *Let $\mathbf{L}$ be a GMV-algebra. If δ is a kernel on $\mathbf{L}$ and γ a nucleus on $\mathbf{L}_\delta$, then $\delta_{\beta_{(\gamma,\delta)}} = \delta$, $\gamma_{\beta_{(\gamma,\delta)}} = \gamma$.*

Proof. We have $\delta_{\beta_{(\gamma,\delta)}}(x) = \beta_{(\gamma,\delta)}(x) \wedge x = \gamma(\delta(x)) \wedge x$. In view of Corollary 6.3, to show that $\delta_{\beta_{(\gamma,\delta)}} = \delta$, it will suffice to verify that $\gamma(\delta(x)) \wedge x = \delta(x)$, only for the cases $\delta(x) = x$ and $\delta(x) = x \wedge e$. In the first case, the equation holds, because γ is extensive. In the second case, the equation reduces to $\gamma(x \wedge e) \wedge x = x \wedge e$. Since γ is extensive, we have $x \wedge e = x \wedge e \wedge x \leqslant \gamma(x \wedge e) \wedge x$. Invoking the monotonicity of γ we get $\gamma(x \wedge e) \wedge x \leqslant \gamma(e) \wedge x = e \wedge x$, by Theorem 3.4(iii).

For every x in the range of $\delta_{\beta_{(\gamma,\delta)}} = \delta$, namely for $x = \delta(x)$, we have $\gamma_{\beta_{(\gamma,\delta)}}(x) = \beta_{(\gamma,\delta)}(x) = \gamma(\delta(x)) = \gamma(x)$. □

Therefore cores on GMV-algebras decompose uniquely as compositions of kernels and nuclei. For a GMV-algebra $\mathbf{L}$ and a core β on it, define $\mathbf{L}_\beta = (\mathbf{L}_{\delta_\beta})_{\gamma_\beta}$.

Theorem 6.6. *A residuated lattice* $\mathbf{L}$ *is a GMV-algebra if and only if* $\mathbf{L} \cong \mathbf{G}_\beta$, *for some* ℓ*-group* $\mathbf{G}$ *and some core* β *on* $\mathbf{G}$.

Proof. By Lemma 6.1, if $\mathbf{G}$ is an ℓ-group and δ a kernel on it, then $\mathbf{G}_\delta$ is a GMV-algebra. Moreover, by Theorem 3.4, $(\mathbf{G}_\delta)_\gamma$ is a GMV-algebra, as well.

Conversely, let $\mathbf{L}$ be a GMV-algebra. By Corollary 5.6, $\mathbf{L} \cong \mathbf{K} \times \mathbf{H}^-_\gamma$, for some ℓ-groups $\mathbf{K}$ and $\mathbf{H}$, and a nucleus γ on $\mathbf{H}^-$. Define a map δ on $K \times H$, by $\delta(k,h) = (k, h \wedge e)$. We will show that δ is a kernel. It is obviously an interior operator and $\delta(e,e) = (e,e)$. Note that

$$\begin{aligned}\delta(k,h)\delta(k',h') &= (k, h\wedge e)(k', h' \wedge e)\\ &= \big(kk', (h\wedge e)(h'\wedge e)\big)\\ &= \big(kk', hh' \wedge h \wedge h' \wedge e\big)\end{aligned}$$

and $\delta(kk', hh' \wedge h \wedge h' \wedge e) = (kk', hh' \wedge h \wedge h' \wedge e)$. Similarly

$$\delta(k,h) \wedge (k',h') = (k, h\wedge e) \wedge (k', h) = (k \wedge k', h \wedge e \wedge h')$$

and $\delta(k \wedge k', h \wedge e \wedge h') = (k \wedge k', h \wedge e \wedge h')$.

Note that the underlying set of $(\mathbf{K} \times \mathbf{H})_\delta$ is $K \times H^-$. Define $\bar\gamma$ on $K \times H^-$, by $\bar\gamma(k,h) = (k, \gamma(h))$. We will show that $\bar\gamma$ is a nucleus on $(\mathbf{K} \times \mathbf{H})_\delta$. It is obviously a closure operator. Moreover,

$$\begin{aligned}\bar\gamma(k,h)\bar\gamma(k',h') &= \big(k,\gamma(h)\big)\big(k',\gamma(h')\big)\\ &= \big(kk', \gamma(h)\gamma(h')\big)\\ &\leqslant \big(kk', \gamma(hh')\big)\\ &= \bar\gamma(kk', hh')\\ &= \bar\gamma\big((k,h)(k',h')\big).\end{aligned}$$

We have $\bar\gamma((K \times H)_\delta) = \bar\gamma(K \times H^-) = K \times H^-_\gamma$. So $\mathbf{K} \times \mathbf{H}^-_\gamma$ and $((\mathbf{K} \times \mathbf{H})_\delta)_{\bar\gamma}$ have the same underlying set. Recalling the definitions of the image of a residuated lattice under a kernel and under a nucleus, we see that the lattice operations on the two algebras coincide. To show that the other operations coincide, note that for all $(k,h), (k'h') \in K \times H^-_\gamma$,

$$\begin{aligned}(k,h) \circ_{((\mathbf{K}\times\mathbf{H})_\delta)_{\bar\gamma}} (k',h') &= (k,h) \circ_{\bar\gamma} (k',h')\\ &= \bar\gamma\big((k,h)\cdot(k',h')\big)\\ &= \bar\gamma\big(kk', hh'\big)\\ &= \big(kk', \gamma(hh')\big)\\ &= \big(kk', h \circ_\gamma h'\big)\\ &= (k,h) \circ_{\mathbf{K}\times\mathbf{H}^-_\gamma} (k',h'),\end{aligned}$$

$$
\begin{aligned}
(k,h)\backslash_{((\mathbf{K}\times\mathbf{H})_\delta)_{\bar\gamma}}(k',h') &= \delta\big((k,h)\backslash_{\mathbf{K}\times\mathbf{H}}(k',h')\big)\\
&= \delta\big(\big(k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}}h'\big)\big)\\
&= \big(k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}}h' \wedge e\big)\\
&= \big(k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}^-}h'\big)\\
&= \big(k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}^-_\gamma}h'\big)\\
&= (k,h)\backslash_{\mathbf{K}\times\mathbf{H}^-_\gamma}\big(k',h'\big).
\end{aligned}
$$

The proof for the other division operation is analogous. □

It follows from the preceding theorem that $\mathbf{k}(\mathbf{n}(\mathcal{LG})) = \mathcal{GMV}$. We show below that $\mathbf{k}(\mathcal{LG}) = \mathit{Can}\mathcal{GMV}$ and $\mathbf{n}(\mathit{Can}\mathcal{GMV}) = \mathcal{GMV}$. Further, we provide an equational description for a core operator.

Corollary 6.7.

(i) *A residuated lattice* $\mathbf{L}$ *is a cancellative GMV-algebra if and only if* $\mathbf{L} \cong \mathbf{G}_\delta$, *for some* ℓ*-group* $\mathbf{G}$ *and some kernel* δ *on* $\mathbf{G}$.
(ii) *A residuated lattice* $\mathbf{L}$ *is a GMV-algebra if and only if* $\mathbf{L} \cong \mathbf{K}_\gamma$, *for some cancellative GMV-algebra* $\mathbf{K}$ *and some nucleus* γ *on* $\mathbf{K}$.

Proof. (i) One direction follows from Corollaries 6.3 and 5.4. For the other direction, assume that $\mathbf{L}$ is a cancellative GMV-algebra. By Corollary 5.4, $\mathbf{L} = \mathbf{K} \times \mathbf{H}^-$, for some ℓ-groups $\mathbf{K}, \mathbf{H}$. We have already remarked that the map δ on $\mathbf{K} \times \mathbf{H}$, defined by $\delta(k,h) = (k, h \wedge e)$, is a kernel and that $(\mathbf{K} \times \mathbf{H})_\delta = \mathbf{K} \times \mathbf{H}^- = \mathbf{L}$.

(ii) One direction follows from Theorem 3.4. Conversely, if $\mathbf{L}$ is a GMV-algebra, then, by Theorem 5.6, there exist ℓ-groups $\mathbf{G}, \mathbf{H}$ and a nucleus γ on $\mathbf{H}^-$, such that $\mathbf{L} = \mathbf{G} \times \mathbf{H}^-_\gamma$. It is easy to check that the map $\bar\gamma$ on $\mathbf{G} \times \mathbf{H}^-$, defined by $\bar\gamma(g,h) = (g, \gamma(h))$, is a nucleus and that $(\mathbf{G} \times \mathbf{H}^-)_{\bar\gamma} = \mathbf{G} \times \mathbf{H}^-_\gamma = \mathbf{L}$. Finally, $\mathbf{K} = \mathbf{G} \times \mathbf{H}^-$ is a cancellative GMV-algebra, by Corollary 5.4. □

Lemma 6.8. *A map* β *on a GMV-algebra* $\mathbf{L}$ *is a core if and only if it is monotone, idempotent and satisfies the following properties*:

(i) $\beta(x)\beta(y) \leqslant \beta(xy)$,
(ii) $\beta(e) = e$,
(iii) $(\beta(x) \wedge x)(\beta(y) \wedge y) \leqslant \beta((\beta(x) \wedge x)(\beta(y) \wedge y))$,
(iv) $\beta(x) \wedge x \wedge y \leqslant \beta(\beta(x) \wedge x \wedge y)$,
(v) $\beta(\beta(x) \wedge x) = \beta(x)$.

Proof. The result is a consequence of the following two claims and Lemma 6.5.

Claim 1. *If β is a monotone, idempotent map on* $\mathbf{L}$ *that satisfies the properties above, then* δ_β *is a kernel on* $\mathbf{L}$, γ_β *is a nucleus on* $\mathbf{L}_{\delta_\beta}$ *and* $\beta_{(\gamma_\beta,\delta_\beta)} = \beta$.

Since γ_β is the restriction of β, we have $\gamma_\beta(x)\gamma_\beta(y) \leqslant \gamma_\beta(xy)$, by the first property. Moreover, γ_β is monotone and idempotent, being a restriction of β. It is also extensive on $\mathbf{L}_{\delta_\beta}$ because $\delta_\beta(x) = \beta(x) \wedge x \leqslant \beta(\beta(x) \wedge x) = \gamma_\beta(\delta_\beta(x))$, by (iv). Thus, γ_β is a nucleus.

Obviously, $\delta_\beta(e) = \beta(e) \wedge e = e$, by the second property. The remaining two properties of a kernel state that $\delta_\beta(x)\delta_\beta(y)$ and $\delta_\beta(x) \wedge y$ are elements fixed by δ_β. It is easy to see that for every x, $\delta_\beta(x) = x$ if and only if $x \leqslant \beta(x)$. So, the remaining properties are equivalent to properties (iii) and (iv) of the lemma. Additionally, δ_β is an interior operator, since $\delta_\beta(x) = \beta(x) \wedge x \leqslant x$; $\delta_\beta(\delta_\beta(x)) = \beta(\beta(x) \wedge x) \wedge x = \beta(x) \wedge x = \delta_\beta(x)$, by (v); and if $x \leqslant y$, then $\delta_\beta(x) = \beta(x) \wedge x \leqslant \beta(y) \wedge y = \delta_\beta(y)$. Thus, δ_β is a kernel.

Finally, $\beta_{(\gamma_\beta,\delta_\beta)}(x) = \gamma_\beta(\delta_\beta(x)) = \beta(\beta(x) \wedge x) = \beta(x)$.

Claim 2. *If δ is a kernel on* $\mathbf{L}$ *and γ a nucleus on* $\mathbf{L}_\delta$, *then the map $\beta_{(\gamma,\delta)}$ is monotone, idempotent and it satisfies the properties in the statement of the lemma.*

For the first property we have

$$\begin{aligned}\beta(x)\beta(y) &= \gamma\bigl(\delta(x)\bigr)\gamma\bigl(\delta(y)\bigr) \leqslant \gamma\bigl(\delta(x)\delta(y)\bigr)\\ &= \gamma\bigl(\delta\bigl(\delta(x)\delta(y)\bigr)\bigr) \leqslant \gamma\bigl(\delta(xy)\bigr)\\ &= \beta(xy).\end{aligned}$$

Also, $\beta(e) = \gamma(\delta(e)) = \gamma(e) = e$, by Theorem 3.4(iii).

Since for every x, $x \leqslant \beta_{(\gamma,\delta)}(x)$ if and only if $\delta_{\beta_{(\gamma,\delta)}}(x) = x$, properties (iii) and (iv) hold for $\beta_{(\gamma,\delta)}$ if and only if the corresponding properties of a kernel hold for $\delta_{\beta_{(\gamma,\delta)}}$. This is actually the case, since $\delta_{\beta_{(\gamma,\delta)}} = \delta$, by Lemma 6.5.

The last property for $\beta_{(\gamma,\delta)}$ is equivalent to $\beta_{(\gamma,\delta)}(\delta_{\beta_{(\gamma,\delta)}}(x)) = \beta_{(\gamma,\delta)}(x)$, that is, $\beta_{(\gamma,\delta)}(\delta(x)) = \beta_{(\gamma,\delta)}(x)$, which follows from the idempotency of δ. $\square$

6.2. The morphism level

Let **GMV** be the category with objects GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\mathbf{LG}^*$ be the category with objects algebras $\langle \mathbf{G}, \beta\rangle$ such that $\mathbf{G}$ is an ℓ-group and β is a core on $\mathbf{G}$ whose image generates $\mathbf{G}$; let the morphisms of this category be homomorphisms between these algebras.

Theorem 6.9. *The categories* **GMV** *and* $\mathbf{LG}^*$ *are equivalent.*

Proof. For an object $\langle \mathbf{G}, \beta\rangle$ of $\mathbf{LG}^*$, define $\Gamma(\langle \mathbf{G}, \beta\rangle) = \mathbf{G}_\beta$. For a morphism f of $\mathbf{LG}^*$ with domain $\langle \mathbf{G}, \beta\rangle$, define $\Gamma(f)$ to be the restriction of f to G_β.

Let $\delta = \delta_\beta$ and $\gamma = \gamma_\beta$. By Lemma 6.1 and Theorem 3.4, the algebra $\Gamma(\langle \mathbf{G}, \beta\rangle)$ is an object of **GMV**. Actually, it can be easily seen that $\mathbf{G}_\beta = \langle (G_\delta)_\gamma, \wedge, \vee, \circ_\gamma, \backslash_\delta, /_\delta, e\rangle$. To show that $\Gamma(f)$ is a morphism of **GMV**, we use the fact that f commutes with β—we use the same symbol for the cores in the domain and in the codomain.

First note that f commutes with δ on $\mathbf{L}$ and γ on $\delta(\mathbf{L})$. Indeed, by Lemma 6.5,

$$\begin{aligned}\delta\big(f(x)\big) &= \beta\big(f(x)\big) \wedge f(x) = f\big(\beta(x)\big) \wedge f(x)\\ &= f\big(\beta(x) \wedge x\big) = f\big(\delta(x)\big).\end{aligned}$$

Moreover, $\gamma(f(x)) = \gamma(\delta(f(x))) = f(\gamma(\delta(x))) = f(\gamma(x))$. In particular, if $x = \beta(x)$, then $x = \gamma(x) = \delta(x)$ and $f(x) = \delta(f(x)) = \gamma(f(x))$.

We can now show that f preserves multiplication. For $x, y \in \beta(G)$, $x = \delta(x) = \gamma(x)$ and $y = \delta(y) = \gamma(y)$, so

$$\delta(xy) = \delta\big(\delta(x)\delta(y)\big) = \delta(x)\delta(y) = xy.$$

Thus,

$$f(x \circ_\gamma y) = f\big(\gamma(xy)\big) = \gamma\big(f(xy)\big) = \gamma\big(f(x)f(y)\big) = f(x) \circ_\gamma f(y).$$

Additionally,

$$f(x/_\delta y) = f\big(\delta(x/y)\big) = \delta\big(f(x/y)\big) = \delta\big(f(x)/f(y)\big) = f(x)/_\delta f(y).$$

The proof for the other division is analogous. $\Gamma(f)$ preserves the lattice operations, because they are restrictions of the lattice operations of the ℓ-group, so $\Gamma(f)$ is a homomorphism.

By Theorem 6.6, Γ is onto the objects of **GMV**. Moreover, Γ is faithful, because if two morphisms agree on a generating set, they agree on the whole ℓ-group.

To see that Γ is full, let $g : \mathbf{M} \to \mathbf{N}$, be a morphism in **GMV**. By Theorem 5.6, there exist ℓ-groups $\mathbf{K}, \mathbf{H}, \overline{\mathbf{K}}, \overline{\mathbf{H}}$ and nuclei γ on $\mathbf{H}^-$ and $\overline{\gamma}$ on $\overline{\mathbf{H}}^-$, such that

$$\mathbf{M} = \mathbf{K} \times \mathbf{H}^-_\gamma \quad \text{and} \quad \mathbf{N} = \overline{\mathbf{K}} \times \overline{\mathbf{H}}^-_{\overline{\gamma}}.$$

Moreover, by the proof of Theorem 6.6, there exist kernels δ on $\mathbf{K} \times \mathbf{H}$, $\bar{\delta}$ on $\overline{\mathbf{K}} \times \overline{\mathbf{H}}$, and nuclei γ' on $(\mathbf{K} \times \mathbf{H})_\delta$ and $\overline{\gamma}'$ on $(\overline{\mathbf{K}} \times \overline{\mathbf{H}})_{\bar{\delta}}$, such that $\delta(k, h) = (k, h \wedge e)$, $\bar{\delta}(\bar{k}, \bar{h}) = (\bar{k}, \bar{h} \wedge e)$, $\gamma'(k, h) = (k, \gamma(h))$ and $\overline{\gamma}'(\bar{k}, \bar{h}) = (\bar{k}, \overline{\gamma}(\bar{h}))$, for $h \in H$, $\bar{h} \in \overline{H}$, $k \in K$ and $\bar{k} \in \overline{K}$. For the cores $\beta = \gamma' \circ \delta$ and $\overline{\beta} = \overline{\gamma}' \circ \bar{\delta}$, there exist homomorphisms $g_1 : \mathbf{K} \to \overline{\mathbf{K}}$ and $g_2 : \mathbf{H}^-_{\gamma_1} \to \overline{\mathbf{H}}^-_{\gamma_2}$ such that $g = (g_1, g_2)$; the reason for this is that invertible and integral elements are preserved under homomorphisms. By Theorem 4.10, there exists a homomorphism $f_2^- : \mathbf{H}^- \to \overline{\mathbf{H}}^-$ that extends g_2 and commutes with the γ's. By the results in [2], there exists a homomorphism $f_2 : \mathbf{H} \to \overline{\mathbf{H}}$ that extends f_2^-. Let $f : \langle \mathbf{K} \times \mathbf{H}, \beta\rangle \to \langle \overline{\mathbf{K}} \times \overline{\mathbf{H}}, \overline{\beta}\rangle$ be defined by $f = (g_1, f_2)$. It is clear that $\Gamma(f) = g$. We will show that $g(\beta(x)) = \overline{\beta}(f(x))$. Let $(k, h) \in K \times H$.

$$\begin{aligned}
g\big(\beta(k,h)\big) &= g\big(\overline{\gamma}\big(\delta(k,h)\big)\big) = g\big(k,\gamma(h\wedge e)\big)\\
&= \big(g_1(k), g_2\big(\gamma(h\wedge e)\big)\big) = \big(g_1(k), \overline{\gamma}\big(f_2^-(h\wedge e)\big)\big)\\
&= \big(g_1(k), \overline{\gamma}\big(f_2(h)\wedge e\big)\big) = \overline{\gamma}'\big(g_1(k), f_2(h)\wedge e\big)\\
&= \overline{\gamma}\big(\bar{\delta}\big(\big(g_1(k), f_2(h)\big)\big)\big) = \overline{\beta}\big(f(k,h)\big).
\end{aligned}$$

Thus, by [23, Theorem 1, p. 93], Γ is an equivalence between the two categories. □

7. Decidability of the equational theories

In this section, we obtain the decidability of the equational theories of the varieties $\mathcal{IGMV}$ and $\mathcal{GMV}$ as an easy application of the representation theorems established in the previous sections.

For a residuated lattice term t and a variable $z \notin \mathrm{Var}(t)$, we define the term t_z inductively on the complexity of a term, by

$$x_z = x \vee z, \qquad e_z = e,$$
$$(s\vee r)_z = s_z \vee r_z (s\wedge r)_z = s_z \wedge r_z,$$
$$(s/r)_z = s_z/r_z, \qquad (s\backslash r)_z = s_z\backslash r_z, \qquad (sr)_z = s_z r_z \vee z,$$

for every variable x and every pair of terms s, r.

For a term t and an algebra $\mathbf{L}$, we write $t^{\mathbf{L}}$ for the term operation on $\mathbf{L}$ induced by t.

For a residuated lattice term t, a residuated lattice $\mathbf{L}$ and a map γ on $\mathbf{L}$, we define the operation t_γ on L, of arity equal to that of t, by

$$x_\gamma = \gamma\big(x^{\mathbf{L}}\big), \qquad e_\gamma = e^{\mathbf{L}},$$
$$(s\vee r)_\gamma = s_\gamma \vee r_\gamma (s\wedge r)_\gamma = s_\gamma \wedge r_\gamma,$$
$$(s/r)_\gamma = s_\gamma/r_\gamma, \qquad (s\backslash r)_\gamma = s_\gamma\backslash r_\gamma, \qquad (sr)_\gamma = \gamma(s_\gamma r_\gamma),$$

for every variable x and every pair of terms s, r.

Note that t_γ is obtained from $t^{\mathbf{L}}$ by replacing every product sr by $\gamma(sr)$ and every variable x by $\gamma(x)$; t_z is obtained from t by replacing every product sr by $sr \vee z$ and every variable x by $x \vee z$. We extend the above definitions to every residuated lattice identity $\varepsilon = (t \approx s)$ by $\varepsilon_z = (t_z \approx s_z)$, for a variable z that does not occur in ε. Moreover, we define $\varepsilon_\gamma(\bar{a}) = (t_\gamma(\bar{a}) = s_\gamma(\bar{a}))$, where $\bar{a}$ is an element of an appropriate power of L.

Proposition 7.1. *An identity ε holds in $\mathcal{IGMV}$ if and only if the identity ε_z holds in $\mathcal{LG}^-$, where $z \notin \mathrm{Var}(\varepsilon)$.*

Proof. We prove the contrapositive of the lemma. Let ε be an identity that fails in $\mathcal{IGMV}$. Then there exists an integral GMV-algebra $\mathbf{M}$, and an element $\bar{a}$ in an appropriate power, n, of M, such that $\varepsilon(\bar{a})$ is false. By Theorem 3.12, there exists an $\mathbf{L} \in \mathcal{LG}^-$ and a nucleus γ on $\mathbf{L}$ such that $\mathbf{M} = \mathbf{L}_\gamma$. By the definition of $\mathbf{L}_\gamma$, it follows that $\varepsilon_\gamma(\bar{a})$ does not hold in $\mathbf{L}$. Let p be the meet of all products $t_\gamma(\bar{a})s_\gamma(\bar{a})$, where t, s range over all subterms of ε and $u = \gamma(p)$. By Lemma 3.6, γ and γ_u agree on the principal filter of p. Since the arguments of all occurrences of γ in $\varepsilon_\gamma(\bar{a})$ are of the form $t_\gamma(\bar{a})s_\gamma(\bar{a})$, where t, s are subterms of ε, and $t_\gamma(\bar{a})s_\gamma(\bar{a})$ are in the principal filter of p, we can replace, working inductively inwards, all occurrences of γ in $\varepsilon_\gamma(\bar{a})$ by γ_u. Hence $\varepsilon_{\gamma_u}(\bar{a}) = \varepsilon_\gamma(\bar{a})$ and $\varepsilon_{\gamma_u}(\bar{a})$ fails in $\mathbf{L}$. Moreover, $\varepsilon_{\gamma_u}(\bar{a}) = (\varepsilon_z)^{\mathbf{L}}(\bar{a}, u)$. Thus ε_z fails in $\mathbf{L}$ and ε_z is not a valid identity of $\mathcal{LG}^-$.

Conversely, if ε_z, fails in $\mathcal{LG}^-$, there exist an $\mathbf{L} \in \mathcal{LG}^-$, $\bar{a}$ in an appropriate power, n, of L and $u \in L$ such that $(\varepsilon_z)^{\mathbf{L}}(\bar{a}, u)$ is false. Obviously, γ_u is a nucleus on $\mathbf{L}$, so $\mathbf{L}_{\gamma_u}$ is an integral GMV-algebra. Let $\bar{b}$ be the element of L^n, defined by $\bar{b}(i) = \bar{a}(i) \vee u$, for all $i \in \{1, \ldots, n\}$. Note that $(\varepsilon_z)^{\mathbf{L}}(\bar{a}, u) = \varepsilon_{\gamma_u}(\bar{a}) = \varepsilon_{\gamma_u}(\bar{b}) = \varepsilon^{\mathbf{L}_{\gamma_u}}(\bar{b})$ and $u, \bar{b}(i) \in \mathbf{L}_{\gamma_u}$, for all $i \in \{1, \ldots, n\}$. So ε fails in $\mathbf{L}_{\gamma_u}$ and hence in $\mathcal{IGMV}$. $\square$

In view of Theorem 5.6 we have the following corollary.

Corollary 7.2. *An identity ε holds in $\mathcal{GMV}$ if and only if ε holds in $\mathcal{LG}$ and ε_z holds in $\mathcal{LG}^-$, where $z \notin \mathrm{Var}(\varepsilon)$.*

The variety of ℓ-groups has a decidable equational theory by [19]. Based on this fact, it is shown in [2] that the same holds for $\mathcal{LG}^-$. So, we obtain the following result.

Theorem 7.3. *The varieties $\mathcal{IGMV}$ and $\mathcal{GMV}$ have decidable equational theories.*

Recall that a bounded GMV-algebra (also called a pseudo MV-algebra) is an expansion of a GMV-algebra by a constant 0 that satisfies the identity $x \wedge 0 \approx 0$. We denote the variety of all bounded GMV-algebras by $b\mathcal{GMV}$. Note that every bounded GMV-algebra is integral, as a consequence of Theorem 5.6.

For a term t in the language of residuated bounded-lattices and a variable $z \notin \mathrm{Var}(t)$, we define the term t_z inductively on the complexity of a term, by

$$x_z = x \vee z, \qquad e_z = e, \qquad 0_z = z,$$
$$(s \vee r)_z = s_z \vee r_z (s \wedge r)_z = s_z \wedge r_z,$$
$$(s/r)_z = s_z/r_z, \qquad (s\backslash r)_z = s_z\backslash r_z, \qquad (sr)_z = s_z r_z \vee z,$$

for every variable x and every pair of terms s, r. We use the same notation ε_z as before, since the two definitions agree if the equation ε does not contain any occurrences of the constant 0.

Minor modifications in the proof of Proposition 7.1 yield the following result.

Proposition 7.4. *An identity ε holds in $b\mathcal{GMV}$ if and only if the identity ε_z holds in $\mathcal{LG}^-$, where $z \notin \mathrm{Var}(\varepsilon)$.*

A careful analysis of the construction of an algebra in $\mathcal{LG}^-$ from an integral GMV-algebra shows that if the latter is commutative then so is the former. The same result is shown in [24]. So, the proof of Proposition 7.1 also shows the following.

Proposition 7.5. *An identity ε holds in $\mathcal{MV}$ if and only if the identity ε_z holds in $\mathcal{LG}^-$, where $z \notin \mathrm{Var}(\varepsilon)$.*

Consequently, we have the following result.

Theorem 7.6. *The varieties of MV-algebras and bounded GMV-algebras have decidable equational theories.*

References

[1] M. Anderson, T. Feil, Lattice-Ordered Groups: An Introduction, Reidel Publishing Company, 1988.
[2] P. Bahls, J. Cole, N. Galatos, P. Jipsen, C. Tsinakis, Cancellative residuated lattices, Algebra Universalis 50 (1) (2003) 83–106.
[3] A. Bigard, K. Keimel, S. Wolfenstein, Groupes at Anneaux Réticulés, Lecture Notes in Math., vol. 608, Springer-Verlag, Berlin, 1977.
[4] K. Blount, C. Tsinakis, The structure of residuated lattices, Internat. J. Algebra Comput. 13 (4) (2003) 437–461.
[5] B. Bosbach, Residuation groupoids, Result. Math. 5 (1982) 107–122.
[6] B. Bosbach, Concerning cone algebras, Algebra Universalis 15 (1982) 58–66.
[7] C.C. Chang, Algebraic analysis of many valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.
[8] R. Cignoli, I. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-Valued Reasoning, Trends in Logic—Studia Logica Library, vol. 7, Kluwer Academic Publishers, Dordrecht, 2000.
[9] J. Cole, Non-distributive cancellative residuated lattices, in: J. Martinez (Ed.), Ordered Algebraic Structures, Kluwer Academic Publishers, Dordrecht, 2002, pp. 205–212.
[10] A. Dvurečenskij, Pseudo MV-algebras are intervals in l-groups, J. Austr. Math. Soc. 72 (3) (2002) 427–445.
[11] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, Oxford, 1963.
[12] N. Galatos, The undecidability of the word problem for distributive residuated lattices, in: J. Martinez (Ed.), Ordered Algebraic Structures, Kluwer Academic Publishers, Dordrecht, 2002, pp. 231–243.
[13] N. Galatos, Minimal varieties of residuated lattices, Algebra Universalis, in press.
[14] N. Galatos, Equational bases for joins of residuated-lattice varieties, Studia Logica 76 (2) (2004) 227–240.
[15] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras: a noncommutative extension of MV algebras, in: Information Technology, Bucharest, 1999, Inforec, Bucharest, 1999, pp. 961–968.
[16] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras, in: G.C. Moisil memorial issue, Mult.-Valued Log. 6 (1–2) (2001) 95–135.
[17] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic—Studia Logica Library, vol. 4, Kluwer Academic Publishers, Dordrecht, 1998.
[18] J. Hart, L. Rafter, C. Tsinakis, The structure of commutative residuated lattices, Internat. J. Algebra Comput. 12 (4) (2002) 509–524.
[19] W.C. Holland, S.H. McCleary, Solvability of the word problem in free lattice-ordered groups, Houston J. Math. 5 (1) (1979) 99–105.
[20] P. Jipsen, C. Tsinakis, A survey of residuated lattices, in: J. Martinez (Ed.), Ordered Algebraic Structures, Kluwer Academic Publishers, Dordrecht, 2002, pp. 19–56.
[21] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand. 21 (1967) 110–121.
[22] B. Jónsson, C. Tsinakis, Products of classes of residuated structures, Studia Logica 77 (2) (2004) 267–292.
[23] S. Mac Lane, Categories for the Working Mathematician, Grad. Texts in Math., 2nd ed., Springer, 1997.

[24] D. Mundici, Interpretation of AF C*-algebras in Łukasiewicz sentential calculus, J. Funct. Anal. 65 (1) (1986) 15–63.
[25] K.I. Rosenthal, Quantales and Their Applications, Pitman Res. Notes Math. Ser., Longman, 1990.
[26] J. Schmidt, C. Tsinakis, Relative pseudo-complements, join-extensions and meet-retractions, Math. Z. 157 (1977) 271–284.