The Undecidability of the Word Problem for Distributive Residuated Lattices

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ABSTRACT. Let $\mathbf{A} = \langle X | R \rangle$ be a finitely presented algebra in a variety $\mathcal{V}$. The algebra $\mathbf{A}$ is said to have an undecidable word problem if there is no algorithm that decides whether or not any two given words in the absolutely free term algebra $T_{\mathcal{V}}(X)$ represent the same element of $\mathbf{A}$. If $\mathcal{V}$ contains such an algebra $\mathbf{A}$, we say that it has an undecidable word problem. (It is well known that the word problem for the varieties of semigroups, groups and l-groups is undecidable.)

The main result of this paper is the undecidability of the word problem for a range of varieties including the variety of distributive residuated lattices and the variety of commutative distributive ones. The result for a subrange, including the latter variety, is a consequence of a theorem by Urqhart [7]. The proof here is based on the undecidability of the word problem for the variety of semigroups and makes use of the concept of an n-frame, introduced by von Neumann. The methods in the proof extend ideas used by Lipshitz and Urquhart to establish undecidability results for the varieties of modular lattices and distributive lattice-ordered semigroups, respectively.

1 Introduction

Definition 1.1. A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, / \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice, $\langle L, \cdot, e \rangle$ is a monoid and multiplication is both left and right residuated, with $\backslash$ and $/$ as residuals, i.e., $a \cdot b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a \backslash c$, for all $a, b, c \in L$.

This definition of a residuated lattice is more general than the original given by Ward and Dilworth [1] in 1939. Here it is not stipulated that the monoid reduct is commutative nor that the lattice reduct has e as its top element. Residuated lattices have arisen also in logic, in connection to the Lambek calculus: see, e.g. the papers of J.M. Dunn and H. Ono in [9]. The structure theory of residuated lattices was first studied by K. Blount and C. Tsinakis in [2]. Although no further knowledge of what is included in this section is required for the comprehension of this paper, the reader is advised to refer to [2] for more information about residuated lattices.

It is not hard to see that $\mathcal{RL}$, the class of all residuated lattices, is a variety and

$$x \approx x \wedge (xy \vee z)/y, \quad x(y \vee z) \approx xy \vee xz, \quad (x/y)y \vee x \approx x$$
$$y \approx y \wedge x \setminus (yx \vee z), \quad (y \vee z)x \approx yx \vee zx, \quad y(y \setminus x) \vee x \approx x$$

together with the monoid and the lattice identities form an equational basis for it. Actually $\mathcal{RL}$ is an ideal variety, i.e., congruence relations are determined by their e-classes. Moreover, the latter are subalgebras with specific properties described in [2]: they have to be order convex and closed under all conjugation maps λ_a, ρ_a, where $\lambda_a(x) = (a \setminus (xa)) \wedge e$, $\rho_a(x) = ((ax)/a) \wedge e$, and a ranges over elements of the residuated lattice.

The following lemma of [2] contains all the necessary identities for algebraic manipulations in residuated lattices.

Lemma 1.2. *If x, y, z are elements of a residuated lattice, then the following properties hold.*

1) $x(y \vee z) = xy \vee xz$ *and* $(y \vee z)x = yx \vee zx$.

2) $x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z)$ *and* $(y \wedge z)/x = (y/x) \wedge (z/x)$.

3) $x/(y \vee z) = (x/y) \wedge (x/z)$ *and* $(y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x)$.

4) $(x/y)y \leq x$ *and* $y(y \setminus x) \leq x$.

5) $x(y/z) \leq (xy)/z$ *and* $(z \setminus y)x \leq z \setminus (yx)$.

6) $(x/y)/z = x/(zy)$ *and* $z \setminus (y \setminus x) = (yz) \setminus x$.

7) $x \setminus (y/z) = (x \setminus y)/z$.

8) $x/e = x = e \setminus x$.

9) $e \leq x/x$ *and* $e \leq x \setminus x$.

10) $x(x \setminus x) = x = (x/x)x$.

11) $(x \setminus x)^2 = (x \setminus x)$ *and* $(x/x)^2 = (x/x)$.

12) *If the residuated lattice has a bottom element 0, then it has a top element 1, as well, and for all x, $x0 = 0x = 0$, $x/0 = 0 \setminus x = 1$ and $1/x = x \setminus 1 = 1$.*

This paper is heavily influenced by work on similar problems. Lipshitz [5] established the undecidability of the word problem for modular lattices and Urquhart for DL-semigroups [8] and models of relevance logic [7]. Moreover, [3] contains undecidability results about relation algebras, while Freese [4] proved that the word problem for the free modular lattice on five generators is undecidable. The proofs of all the above make use of the notion of an n-*frame*, introduced by von Neumann in [10]. It is a geometric concept that was originally used in the definition of the von Staudt product of two points on a projective line. Taking advantage of the intrinsic connections between projective geometry and modular lattices, von Neumann defined this product in the latter. In other words, the notion of an n-frame can be used to define a semigroup structure in a modular lattice. Lipshitz used this fact to reduce the decidability of the word problem for modular lattices to the one for semigroups. Going one step further and using a modified version of an n-frame, Urquhart applied similar ideas to DL-semigroups. In this paper we give the definition of an n-frame and the results for modular lattices from [5], some of which will be used later on, before presenting the modified definition for residuated lattices together with the corresponding theorem.

2 Modular lattices.

We begin with a version of the original definition of von Neumann that is essentially equivalent to it.

Definition 2.1. A *modular n-frame* in a lattice $\mathbf{L}$ is an $n \times n$ matrix, $C = [c_{ij}]$, $c_{ij} \in \mathbf{L}$, (set $a_i = c_{ii}$ and $e = \bigwedge\{a_i \mid i \in \mathbb{N}_n\}$; $\mathbb{N}_n = \{1, ..., n\}$), such that:

i) $\bigvee A_1 \wedge \bigvee A_2 = \bigvee(A_1 \cap A_2)$, for all $A_1, A_2 \subseteq \{a_1, a_2, ..., a_n\}$, where $\bigvee \emptyset = e$;

ii) $c_{ij} = c_{ji}$, for all $i, j \in \mathbb{N}_n$;

iii) $a_i \vee a_j = a_i \vee c_{ij}$, for all $i, j \in \mathbb{N}_n$;

iv) $a_i \wedge c_{ij} = e$, for all distinct $i, j \in \mathbb{N}_n$;

v) $(c_{ij} \vee c_{jk}) \wedge (a_i \vee a_k) = c_{ik}$, for all distinct triples $i, j, k \in \mathbb{N}_n$.

The following examples, taken from [3], give some idea of the motivation for the definition.

Example 2.2. Consider the real projective plane P. The lattice $\mathbf{L}$ of subspaces of P contains points, projective lines, P and $\emptyset$, ordered under inclusion. Meet is intersection of subsets of P, while the join of two projective subspaces is the least subspace containing both of them. Modularity of $\mathbf{L}$ is well known and easy to establish. A modular 3-frame, see Figure 1, will consist of essentially six points: $a_1, a_2, a_3, c_{12}, c_{13}, c_{23}$, because of (ii). The points a_1, a_2, a_3 are not collinear, by condition (i); c_{ij} has to be on

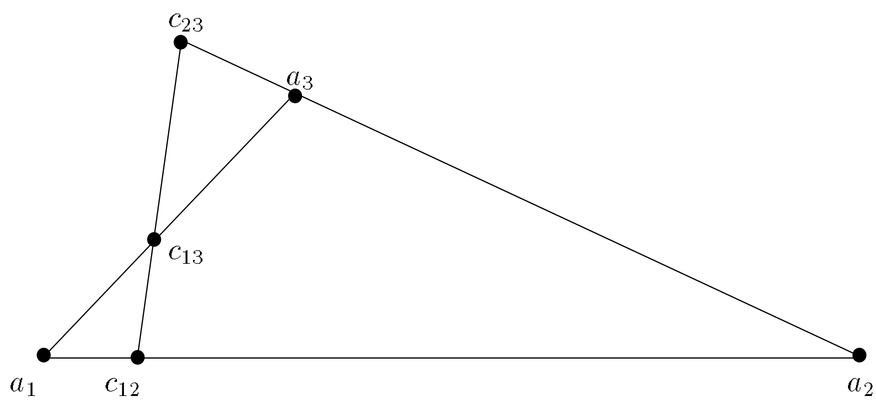

Figure 1: The geometric meaning of a modular 3-frame.

the line $a_i \vee a_j$, by (iii), while c_{12}, c_{13}, c_{23} are collinear, by condition (v); actually, c_{ik} is the point of intersection of the lines $a_i \vee a_j$ and $c_{ij} \vee c_{jk}$.

Example 2.3. Let $\mathbf{V}$ be an n-dimensional real inner product space, $\{\mathbf{e}_i \mid i \in \mathbb{N}_n\}$ an orthonormal base of $\mathbf{V}$, $a_i = \langle \mathbf{e}_i \rangle$, the subspace generated by $\mathbf{e}_i$, and $c_{ij} = \langle \mathbf{e}_i - \mathbf{e}_j \rangle$. Then $[c_{ij}]$, $i, j \in \mathbb{N}_n$, is a modular n-frame in the lattice $\mathbf{L}$ of subspaces of $\mathbf{V}$.

Given a modular n-frame one can define operations of multiplication and addition on certain elements of the lattice.

Definition 2.4. Let $[c_{ij}]$ be a modular n-frame in a modular lattice $\mathbf{L}$. Define

i) $L_{ij} = \{x \in \mathbf{L} \mid x \vee a_j = a_i \vee a_j \text{ and } x \wedge a_j = e\}$, for all distinct $i, j \in \mathbb{N}_n$;

ii) $b \otimes_{ijk} d = (b \vee d) \wedge (a_i \vee a_k)$, for all $b \in L_{ij}, d \in L_{jk}$;

iii) $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, for all $b, d \in L_{ij}$;

iv) $b \oplus_{ij} d = [((b \vee c_{ik}) \wedge (a_j \vee a_k)) \vee ((d \vee a_k) \wedge (a_j \vee c_{ik}))] \wedge (a_i \vee a_j)$, for all $b, d \in L_{ij}$.

In [5] it is shown that the definitions of $\odot_{ij}$ and $\oplus_{ij}$ are independent of the choice of $k \in \mathbb{N}_n$, for $k \neq i, k \neq j$.

Remark 2.5. The definitions of $b \odot_{ij} d$ and $b \oplus_{ij} d$ differ from [10] and [5]. There multiplication and addition are not defined for elements of L_{ij}, but for L-numbers. An L-number α in a modular n-frame C is a set of lattice elements indexed by $\{(i,j) \mid i, j \in \mathbb{N}_n,\ i \neq j\}$, such that $(\alpha)_{kh} = [P(i,j,k,h)]((\alpha)_{ij})$, where $(\alpha)_{ij}$ symbolizes

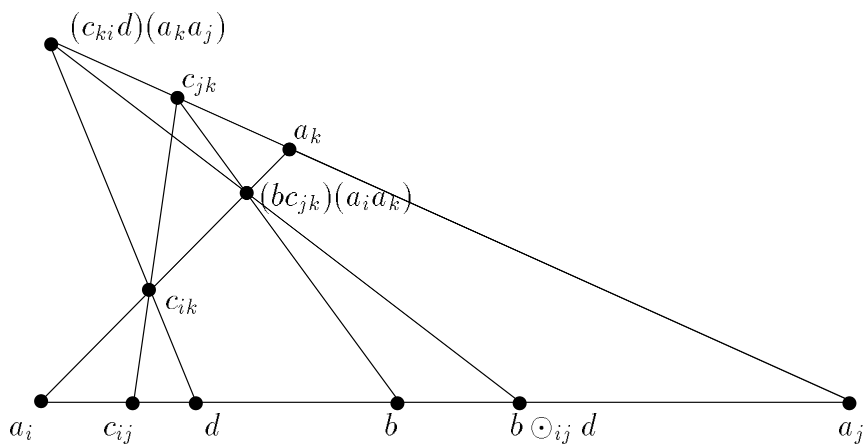

Figure 2: The geometric meaning of $\odot_{ij}$.

the (i, j)-coordinate of α and $P(i, j, k, h)$ is the composition of the two perspective isomorphisms with axes c_{jh} and c_{ik}. Lemma 6.1 of [10] guarantees that one can work with the fixed (i, j)-coordinates of L-numbers instead of them, since given $i, j \in \mathbb{N}_n$ the correspondence between α and $(\alpha)_{ij}$ is a bijection. Moreover, this bijection between L-numbers under the multiplication and addition defined in [10] and L_{ij} under $\odot_{ij}$ and $\oplus_{ij}$ is a ring isomorphism, as it can be deduced from Lemmas 6.2, 6.3, Theorem 6.1 and the appendix to Chapter 6, Part II of [10]. Freese, in [4], is the first one to use $\odot_{ij}$ and $\oplus_{ij}$, instead of multiplication and addition of L-numbers, and essentially the definition of an n-frame presented here.

In the context of the first example, L_{ij} is the set of all points x on the line $a_i \vee a_j$, $(x \vee a_i = a_i \vee a_j)$, different from a_i, $(x \wedge a_j = e)$, $b \otimes_{ijk} d$ is by definition the intersection of the lines $b \vee d$ and $a_i \vee a_j$, for $b \in L_{ij}, d \in L_{jk}$, while $b \odot_{ij} d$ and $b \oplus_{ij} d$, $b, d \in L_{ij}$ are the (von Staudt) product and sum, see Figures 2 and 3, of b and d on the line $a_i \vee a_j$, where a_i plays the role of zero, c_{ij} is the unit and a_j is infinity. Some projective geometry is required to verify this assertion.

The following theorem of [10] justifies the terminology of multiplication and addition, and validates the connection between projective geometry and modular lattices.

Theorem 2.6 [Von Neumann]. *Let $C = [c_{ij}]$ be a modular n-frame in a modular lattice $\mathbf{L}$, where $n \geq 4$. Then $\mathbf{R}_{ij} = \langle L_{ij}, \oplus_{ij}, \odot_{ij}, a_i, c_{ij} \rangle$ is a ring for all distinct $i, j \in \mathbb{N}_n$. Moreover, all rings $\mathbf{R}_{ij}$ are isomorphic.*

In view of the last statement of the previous theorem the choice of indices i, j in $\mathbf{R}_{ij}$ is inessential. So, $\mathbf{R}_{12} = \langle L_{12}, \oplus_{12}, \odot_{12}, a_1, c_{12} \rangle$ is called the *ring associated with*

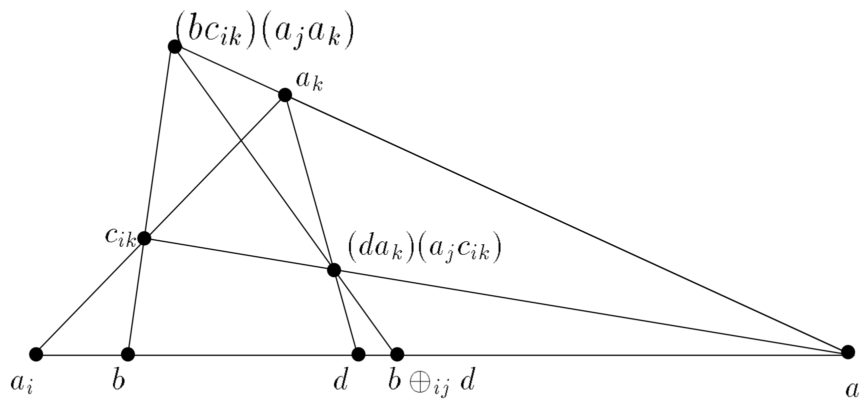

Figure 3: The geometric meaning of $\oplus_{ij}$.

the modular n-frame C of **L**.

For a vector space, **V**, denote by $L(\mathbf{V})$ the set of all subspaces of **V**. It is well known that $\mathbf{L}(\mathbf{V}) = \langle L(\mathbf{V}), \wedge, \vee \rangle$ is a modular lattice, where meet is intersection and the join of two subspaces is the subspace generated by their union.

The following results of [5] make use of the definition of a modular n-frame.

Lemma 2.7 [Lipshitz]**.** *Let* **V** *be an infinite-dimensional vector space. Then,*

i) $\mathbf{L}(\mathbf{V})$ *contains a 4-frame, C, where e is the least element of* $\mathbf{L}(\mathbf{V})$ *and*

ii) *Any countable semigroup is a subsemigroup of the multiplicative semigroup of the ring associated with C.*

Theorem 2.8 [Lipshitz]**.** *The word problem for modular lattices is undecidable.*

3 Distributive residuated lattices.

A residuated lattice is called *distributive* if its lattice reduct is distributive. Obviously the class of distributive residuated lattices is a variety and is denoted by $\mathcal{DRL}$.

We modify the definition of a modular n-frame, to suit our purposes.

Definition 3.1. A *residuated-lattice n-frame* (or just *n-frame*) in a residuated lattice **L** is an $n \times n$ matrix, $C = [c_{ij}]$, $c_{ij} \in \mathbf{L}$, (set $a_i = c_{ii}$), such that:

i) $a_ia_j = a_ja_i$, for all $i, j \in \mathbb{N}_n$;

ii) $\prod A_1 \wedge \prod A_2 = \prod(A_1 \cap A_2)$, for all $A_1, A_2 \subseteq \{a_1, a_2, ..., a_n\}$, where $\prod \emptyset = e$;

iii) $a_i^2 = a_i$, for all $i \in \mathbb{N}_n$;

iv) $c_{ij}c_{jk} \wedge a_i a_k = c_{ik}$, for all distinct triples $i, j, k \in \mathbb{N}_n$;

v) $c_{ij} = c_{ji}$, for all $i, j \in \mathbb{N}_n$;

vi) $c_{ij}a_j = a_i a_j$, for all $i, j \in \mathbb{N}_n$;

vii) $c_{ij} \wedge a_j = e$, for all distinct $i, j \in \mathbb{N}_n$.

It is clear that if multiplication is replaced by join, the conditions in the definition reduce to the ones in Definition 2.1.

Definition 3.2. i) An element a of a residuated lattice $\mathbf{L}$ is called *modular* if $c(b \wedge a) = cb \wedge a$ and $(a \wedge b)c = a \wedge bc$ for all elements b, c of L, such that $c \leq a$.

ii) An n-frame of a residuated lattice is called *modular* if $\prod A$ is modular, for all $A \subseteq \{a_1, a_2, ..., a_n\}$.

Definition 3.3. Let $[c_{ij}]$ be an n-frame in a residuated lattice $\mathbf{L}$. Define

i) $L_{ij} = \{x \in \mathbf{L} \mid xa_j = a_i a_j \text{ and } x \wedge a_j = e\}$, for all distinct $i, j \in \mathbb{N}_n$;

ii) $b \otimes_{ijk} d = bd \wedge a_i a_k$, for all $b \in L_{ij}, d \in L_{jk}$;

iii) $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, for all $b, d \in L_{ij}$ and for all distinct triples i, j, k.

The definition of $\odot_{ij}$ doesn't depend on the choice of k, as is shown in the lemma below.

Lemma 3.4. *Let $c = [c_{ij}]$ be a modular 4-frame in a residuated lattice $\mathbf{L}$.*

i) *If $b \in L_{ij}$, then $b \leq a_i a_j$;*

ii) *If $b \in L_{ij}$ and $d \in L_{jk}$, then $b \otimes_{ijk} d \in L_{ik}$, for all distinct triples $i, j, k \in \mathbb{N}_n$;*

iii) *If $b \in L_{ij}$, $d \in L_{jk}$ and $f \in L_{kl}$, then $(b \otimes_{ijk} d) \otimes_{ikl} f = b \otimes_{ijl} (d \otimes_{jkl} f)$, for all distinct quadruples $i, j, k, l \in \mathbb{N}_n$;*

iv) *If $b, d \in L_{ij}$, then $(b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d) = (b \otimes_{ijl} c_{jl}) \otimes_{ilj} (c_{li} \otimes_{lij} d)$, for all distinct quadruples $i, j, k, l \in \mathbb{N}_4$.*

Proof. i) $b = be \leq ba_j = a_i a_j$.
ii) We first show that $(b \otimes_{ijk} d)a_k = a_i a_k$.

$$\begin{array}{rcll}
(b \otimes_{ijk} d)a_k & = & (a_i a_k \wedge bd)a_k, & \\
& = & bda_k \wedge a_i a_k, & a_i a_k \text{ is modular and } a_k \leq a_i a_k, \text{ since } e \leq a_i \\
& = & ba_j a_k \wedge a_i a_k, & da_k = a_j a_k, \text{ since } d \in L_{jk} \\
& = & a_i a_j a_k \wedge a_i a_k, & ba_j = a_i a_j, \text{ since } b \in L_{ij} \\
& = & a_i a_k, & \text{(ii) of Def. 3.1}
\end{array}$$

To prove $b \otimes_{ijk} d \in L_{ij}$ we also need to show that $(b \otimes_{ijk} d) \wedge a_k = e$.

$$\begin{array}{rcll}
(b \otimes_{ijk} d) \wedge a_k & = & bd \wedge a_i a_k \wedge a_k & \\
& = & bd \wedge a_k, & a_k \leq a_i a_k, \text{ since } e \leq a_i \\
& \leq & bd \wedge a_j a_k, & a_k \leq a_j a_k, \text{ since } e \leq a_i \\
& = & (b \wedge a_j a_k)d, & a_j a_k \text{ is modular and } d \leq a_j a_k, \text{ by (i)} \\
& = & (b \wedge a_j a_k \wedge a_i a_j)d, & b \wedge a_i a_j = b, \text{ by (i)} \\
& = & (b \wedge a_j)d, & a_j a_k \wedge a_i a_j = a_j, \text{ by (ii) of Def. 3.1} \\
& = & d, & b \wedge a_j = e, \text{ since } b \in L_{ij}
\end{array}$$

So, $(b \otimes_{ijk} d) \wedge a_k = b \otimes_{ijk} d \wedge a_k \wedge a_k \leq d \wedge a_k = e$, since $d \in L_{jk}$. Moreover,

$$e = ee \wedge ee \wedge e \leq bd \wedge a_i a_k \wedge a_k = (b \otimes_{ijk} d) \wedge a_k.$$

Thus, $(b \otimes_{ijk} d) \wedge a_k = e$.

iii) Since $b \in L_{ij}$, $d \in L_{jk}$ and $f \in L_{kl}$, by (ii) we get, $b \otimes_{ijk} d \in L_{ik}$ and $d \otimes_{jkl} f \in L_{jl}$; thus, $(b \otimes_{ijk} d) \otimes_{ikl} f$, $b \otimes_{ijl} (d \otimes_{jkl} f) \in L_{il}$.

$$\begin{array}{rcll}
(b \otimes_{ijk} d) \otimes_{ikl} f & = & (bd \wedge a_i a_k)f \wedge a_i a_l & \\
& = & (bd \wedge a_i a_j a_k \wedge a_i a_k a_l)f \wedge a_i a_l, & \text{(ii) of Def. 3.1} \\
& = & (bd \wedge a_i a_k a_l)f \wedge a_i a_l, & bd \leq a_i a_j a_j a_k = a_i a_j a_k, \\
& & & \text{by (i), since } b \in L_{ij}, d \in L_{jk} \\
& & & \text{and } a_j^2 = a_j \\
& = & bdf \wedge a_i a_k a_l \wedge a_i a_l, & a_i a_k a_l \text{ is modular and} \\
& & & f \leq a_k a_l \leq a_i a_k a_l, \\
& & & \text{since } f \in L_{kl} \\
& = & bdf \wedge a_i a_l, & \text{(ii) of Def. 3.1}
\end{array}$$

Similarly, $b \otimes_{ijl} (d \otimes_{jkl} f) = bdf \wedge a_i a_l$, so

$$(b \otimes_{ijk} d) \otimes_{ikl} f = b \otimes_{ijl} (d \otimes_{jkl} f).$$

iv) First note that condition (iv) of the definition of an n-frame can be written as

$c_{rs} = c_{rt} \otimes_{rts} c_{ts}$, for all distinct triples $r, t, s \in \mathbb{N}_n$.

$$\begin{aligned}(a \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} b) &= (a \otimes_{ijk} (c_{jl} \otimes_{jlk} c_{lk})) \otimes_{ikj} (c_{ki} \otimes_{kij} b) \\ &= ((a \otimes_{ijl} c_{jl}) \otimes_{ilk} c_{lk}) \otimes_{ikj} (c_{ki} \otimes_{kij} b) \\ &= (a \otimes_{ijl} c_{jl}) \otimes_{ilj} (c_{lk} \otimes_{lkj} (c_{ki} \otimes_{kij} b)) \\ &= (a \otimes_{ijl} c_{jl}) \otimes_{ilj} ((c_{lk} \otimes_{lki} c_{ki}) \otimes_{lij} b) \\ &= (a \otimes_{ijl} c_{jl}) \otimes_{ilj} (c_{li} \otimes_{lij} b)\end{aligned}$$

Thus the definition of $\odot_{ij}$ is independent of k. ∎

Lemma 3.5. *Let $c = [c_{ij}]$ be a modular 4-frame in a residuated lattice $\mathbf{L}$.*

i) *If $b, d \in L_{ij}$, then $b \odot_{ij} d \in L_{ij}$, for all distinct $i, j \in \mathbb{N}_n$.*

ii) *If $b, d, f \in L_{12}$, then $(b \odot_{12} d) \odot_{12} f = b \odot_{12} (d \odot_{12} f)$.*

Proof. i) Since $b \in L_{ij}$, $c_{jk} \in L_{jk}$ and $c_{ki} \in L_{ki}$, $d \in L_{ij}$, we have $b \otimes_{ijk} c_{jk} \in L_{ik}$ and $c_{ki} \otimes_{kij} d \in L_{kj}$. So,

$$b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d) \in L_{ij}.$$

ii)
$$\begin{aligned}(b \odot_{12} d) \odot_{12} f &= \{[(b \otimes_{123} c_{23}) \otimes_{132} (c_{31} \otimes_{312} d)] \otimes_{123} c_{23}\} \otimes_{132} (c_{31} \otimes_{312} f) \\ &= \{[(b \otimes_{124} c_{24}) \otimes_{142} (c_{41} \otimes_{412} d)] \otimes_{123} c_{23}\} \otimes_{132} (c_{31} \otimes_{312} f) \\ &= \{(b \otimes_{124} c_{24}) \otimes_{143} [(c_{41} \otimes_{412} d) \otimes_{423} c_{23}]\} \otimes_{132} (c_{31} \otimes_{312} f) \\ &= (b \otimes_{124} c_{24}) \otimes_{142} \{[(c_{41} \otimes_{412} d) \otimes_{423} c_{23}] \otimes_{432} (c_{31} \otimes_{312} f)\} \\ &= (b \otimes_{124} c_{24}) \otimes_{142} \{[c_{41} \otimes_{413} (d \otimes_{123} c_{23})] \otimes_{432} (c_{31} \otimes_{312} f)\} \\ &= (b \otimes_{124} c_{24}) \otimes_{142} \{c_{41} \otimes_{412} [(d \otimes_{123} c_{23}) \otimes_{132} (c_{31} \otimes_{312} f)]\} \\ &= (b \otimes_{123} c_{23}) \otimes_{132} \{c_{31} \otimes_{312} [(d \otimes_{123} c_{23}) \otimes_{132} (c_{31} \otimes_{312} f)]\} \\ &= b \odot_{12} (d \odot_{12} f)\end{aligned}$$

A fact that establishes the associativity of $\odot_{12}$. ∎

Corollary 3.6. *Let $C = [c_{ij}]$ be a modular residuated-lattice 4-frame in a residuated lattice $\mathbf{L}$. Then, $\mathbf{S}_{12} = \langle L_{12}, \odot_{12} \rangle$ is a semigroup, called the semigroup associated with the 4-frame C.*

Lemma 3.7. *Let $\mathbf{L}$ be a distributive residuated lattice, with a top element, T, and a bottom element, B. If $a, \tilde{a}, \in L$, $a^2 \le a$, $a\tilde{a} \le \tilde{a}$, $\tilde{a}a \le \tilde{a}$, $a \wedge \tilde{a} = B$ and $a \vee \tilde{a} = T$, then, a is modular.*

Proof. Let $b, c \in L, c \le a$. Then, $(a \wedge b)c \le ac \le a^2 \le a$ and $(a \wedge b)c \le bc$; thus, $(a \wedge b)c \le a \wedge bc$. On the other hand,

$$\begin{aligned}a \wedge bc &= a \wedge (b \wedge T)c = a \wedge (b \wedge (a \vee \tilde{a}))c \\ &= a \wedge ((b \wedge a) \vee (b \wedge \tilde{a}))c = a \wedge ((b \wedge a)c \vee (b \wedge \tilde{a})c) \\ &\le a \wedge ((b \wedge a)c \vee \tilde{a}a) \le (a \wedge (b \wedge a)c) \vee (a \wedge \tilde{a})\end{aligned}$$

$$\leq \quad (b \wedge a)c \vee B = (b \wedge a)c$$

Thus, $a \wedge bc = (b \wedge a)c$. Similarly, we get the other condition $a \wedge cb = c(a \wedge b)$. ∎

Let $\mathbf{V}$ be a vector space. For $A, B \in L_{\mathbf{V}} = \mathcal{P}(V)$, the power set of $\mathbf{V}$, let $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $AB = \{a+b \,|\, a \in A, y \in B\}$, $A \setminus B = B/A = \{c | \{c\}A \subseteq B\}$ and $e = \{0_{\mathbf{V}}\}$. It's easy to see that $\mathbf{L_V} = \langle L_{\mathbf{V}}, \wedge, \vee, \cdot, e, \setminus, / \rangle$ is a distributive residuated lattice. Moreover, $L(\mathbf{V})$ is a subset of $L_{\mathbf{V}}$, but $\mathbf{L}(\mathbf{V})$ is not a sublattice of the lattice reduct of $\mathbf{L_V}$. Nevertheless, a subset A of V is in $L(\mathbf{V})$ if and only if $e \leq A$ and $AA = A$. Additionally, $\wedge_{\mathbf{L}(\mathbf{V})} = \wedge_{\mathbf{L_V}}$ and $\vee_{\mathbf{L}(\mathbf{V})} = \cdot_{\mathbf{L_V}}$.

Definition 3.8. If $\mathbf{S} = \langle S, \bullet \rangle$, $S = \langle x_1, x_2, ..., x_n | \, r_1^\bullet(\overline{x}) = s_1^\bullet(\overline{x}), ..., r_k^\bullet(\overline{x}) = s_k^\bullet(\overline{x}) \rangle$, is a finitely presented semigroup and $\mathcal{V}$ is a variety of residuated lattices, let $\mathbf{L}(\mathbf{S}, \mathcal{V})$ be the residuated lattice in $\mathcal{V}$ with the presentation described below:

Generators:

$$x'_1, x'_2, ..., x'_n, \; c_{ij} \; (i, j \in \mathbb{N}_4), \; \top, \bot \;\text{ and }\; \widetilde{\prod A} \;\; (A \in \mathcal{A}(C) = \mathcal{P}(\{a_1, a_2, a_3, a_4\})).$$

Relations:

i) Equations (i)-(vii) of Definition 3.1 (for $n = 4$);

ii) $x'_i a_2 = a_1 a_2$ and $x'_1 \wedge a_2 = e$, for all $i \in \mathbb{N}_n$;

iii) $r_i^{\odot_{12}}(\overline{x}') = s_i^{\odot_{12}}(\overline{x}')$, for all $i \in \mathbb{N}_k$, where $t^{\odot_{12}}$ denotes the evaluation of t in the semigroup associated with the 4-frame $[c_{ij}]$;

iv) $\bot^2 = \bot$, $\top^2 = \top = \top/\bot = \bot/\bot = \bot \setminus \top = \bot \setminus \bot$, $\bot \leq e \leq \top$ and $\bot \leq x \leq \top$, $\bot x = x \bot = \bot$, for every generator x;

v) $x^2 \leq x$, $x\tilde{x} \leq \tilde{x}$, $\tilde{x}x \leq \tilde{x}$, $x \wedge \tilde{x} = \bot$ and $x \vee \tilde{x} = \top$, for all x of the form $\prod A, A \in \mathcal{A}(C)$.

Let $R(\overline{x})$ denote the conjunction $\bigwedge_{i \in \mathbb{N}_k} r_i(\overline{x}) = s_i(\overline{x})$ of the relations of $\mathbf{S}$ and $R'(\overline{x}', C, \overline{\mathcal{A}}(C), \bot, \top)$ of the relations of $\mathbf{L}(\mathbf{S}, \mathcal{V})$. ($\overline{x}$ denotes $(x_1, x_2, ..., x_n)$)

Lemma 3.9. *For every semigroup $\mathbf{S}$, $\mathbf{L}(\mathbf{S}, \mathcal{V})$ has a bounded lattice reduct and $\bot$, $\top$ are the bottom and top elements.*

Proof. We will prove that $\bot \leq w \leq \top$, for every word w in the generators. We first prove that "$\bot \leq w$ and $\bot w = w \bot = \bot$" for every word, w, using induction on the complexity of w. Properties stated in Lemma 1.2 will be used without reference. The statement is true for the generators and for e, by (iv) in the relations of $\mathbf{L}(\mathbf{S}, \mathcal{V})$. If the statement is true for words u, v, i.e. $\bot \leq u, v$ and $\bot u = u \bot = \bot v = v \bot = \bot$ then:

- $\bot(u \vee v) = \bot u \vee \bot v = \bot$ and $(u \vee v)\bot = \bot$. Also, $\bot \leq u \vee v$.
- $\bot \leq u \wedge v$ and $\bot = \bot\bot \leq \bot(u \wedge v) \leq \bot u \wedge \bot v = \bot$, while $(u \wedge v)\bot = \bot$ is proven in a similar way.
- $\bot \leq \bot\bot \leq uv$ and $\bot uv = \bot v = \bot$, while the other products equal $\bot$, also.
- Since $\bot v = \bot \leq u$, we have $\bot \leq u/v$; thus $\bot = \bot\bot \leq \bot(u/v)$ and $\bot \leq (u/v)\bot$.

Moreover, $\bot(u/v) \leq (\bot u)/v = \bot/v \leq \bot/\bot = \top$, so $u/v \leq \bot \setminus \top = \bot \setminus \bot = \bot/\bot$; hence $\bot(u/v) \leq \bot$ and $(u/v)\bot \leq \bot$. Thus, $\bot(u/v) = \bot$ and $(u/v)\bot = \bot$. For left division we work analogously. Consequently, $\bot$ is the bottom element of $\mathbf{L}$ and, by (12) of Lemma 1.2, $\top = \bot/\bot$ is the top element of $\mathbf{L}$. ∎

The following lemma is a well known fact from the theory of Universal Algebra fact.

Lemma 3.10. *Let $A = \langle \overline{x} | R(\overline{x}) \rangle$, be a finite presentation of an algebra $\mathbf{A}$ in a variety $\mathcal{V}$ where $\overline{x} = (x_1, ..., x_n)$, $n \in \mathbf{N}$ is the sequence of generators and $R(\overline{x})$ the conjunction of the relations. Also, let r, s be n-ary semigroup terms; then the following are equivalent:*

i) *$\mathbf{A}$ satisfies $r^{\mathbf{A}}(\overline{x}) = s^{\mathbf{A}}(\overline{x})$.*

ii) *For every algebra $\mathbf{B}$ in $\mathcal{V}$, if there exist elements $y_1, ..., y_n \in B$, such that $R(\overline{y})$ holds in $\mathbf{B}$, then $\mathbf{B}$ satisfies $r^{\mathbf{B}}(\overline{y}) = s^{\mathbf{B}}(\overline{y})$.*

Proof. For the non-trivial direction, note that the natural epimorphism from the free algebra of $\mathcal{V}$ on $\overline{x}$ to the subalgebra of $\mathbf{B}$ generated by $\overline{y}$, $F_{\mathcal{V}}(\overline{x}) \longrightarrow Sg_{\mathbf{B}}(\overline{y})$, $x_i \mapsto y_i$, factors through $\mathbf{F}_{\mathcal{V}}(\overline{x})/R(\overline{x}) \cong \mathbf{A}$. So, $f : A \longrightarrow Sg_{\mathbf{B}}(\overline{y}) \subseteq B$, $x_i \mapsto y_i$ is a homomorphism. Since $r^{\mathbf{A}}(\overline{x}) = s^{\mathbf{A}}(\overline{x})$, we get

$$r^{\mathbf{B}}(\overline{y}) = r^{\mathbf{B}}(f(\overline{x})) = f(r^{\mathbf{A}}(\overline{x})) = f(s^{\mathbf{A}}(\overline{x})) = s^{\mathbf{B}}(f(\overline{x})) = s^{\mathbf{B}}(\overline{y})$$

in $\mathbf{B}$. ∎

Lemma 3.11. *Let $\mathbf{S}$ be a semigroup, r, s semigroup terms and $\mathcal{V}$ a variety of distributive residuated lattices. If $\mathbf{S}$ satisfies $r^{\bullet}(\overline{x}) = s^{\bullet}(\overline{x})$ then $\mathbf{L}(\mathbf{S}, \mathcal{V})$ satisfies $r^{\odot_{12}}(\overline{x}') = s^{\odot_{12}}(\overline{x}')$.*

Proof. $C = [c_{ij}]$ is a 4-frame in $\mathbf{L}(\mathbf{S}, \mathcal{V})$, by (i) of $R'(\overline{x}', C, \widetilde{\mathcal{A}}(C), \bot, \top)$ and $x_i' \in L_{12}$, by (ii). Moreover, by (v) and Lemma 3.7, $\prod A$ is modular, for all $A \in \mathcal{A}(C)$, hence C is modular. By Corollary 3.6, $\langle L_{12}, \odot_{12} \rangle$ is a semigroup and, by (iii), it satisfies $R(\overline{x}')$; thus, by Lemma 3.10, it also satisfies $r^{\odot_{12}}(\overline{x}') = s^{\odot_{12}}(\overline{x}')$. ∎

We can now prove the main theorem.

Theorem 3.12. *Let $\mathcal{V}$ be a variety of distributive residuated lattices, containing $\mathbf{L_V}$, for some infinite-dimensional vector space $\mathbf{V}$. Then, there is a finitely presented residuated lattice in $\mathcal{V}$, with undecidable word problem.*

Proof. Let $\mathbf{S} = \langle S, \bullet \rangle$, $S = \langle x_1, x_2, ..., x_n | \, r_1^\bullet(\overline{x}) = s_1^\bullet(\overline{x}), ..., r_k^\bullet(\overline{x}) = s_k^\bullet(\overline{x}) \rangle$, be a finitely presented semigroup with undecidable word problem (see [6]) and consider $\mathbf{L}(\mathbf{S}, \mathcal{V})$.

We will show that, for every pair r, s of semigroup words, $\mathbf{S}$ satisfies $r^\bullet(\overline{x}) = s^\bullet(\overline{x})$ if and only if $\mathbf{L}(\mathbf{S}, \mathcal{V})$ satisfies $r^{\odot_{12}}(\overline{x}') = s^{\odot_{12}}(\overline{x}')$. Since one direction follows from Lemma 3.11, suppose that $\mathbf{S}$ does not satisfy $r(\overline{x}) = s(\overline{x})$. By Lemma 2.7, $\mathbf{S}$ is embeddable, via f, say, into the multiplicative semigroup of the ring, $\mathbf{R}$, associated with a modular 4-frame $\hat{C}$ in the modular lattice $\mathbf{L}(\mathbf{V})$. So, $r^{\mathbf{R}}(f(\overline{x})) = s^{\mathbf{R}}(f(\overline{x}))$ is false in $\mathbf{R}$, where $f(\overline{x}) = (f(x_1), ..., f(x_n))$, thus also false in $\mathbf{L}(\mathbf{V})$, if viewed as a lattice equation. Since, as noted before, $\wedge_{\mathbf{L}(\mathbf{V})} = \wedge_{\mathbf{L_V}}$ and $\vee_{\mathbf{L}(\mathbf{V})} = \cdot_{\mathbf{L_V}}$, $\mathbf{L_V}$ fails $r^{\mathbf{R}}(f(\overline{x})) = s^{\mathbf{R}}(f(\overline{x}))$, where the latter is considered a residuated lattice equation ($r^{\mathbf{L_V}}(f(\overline{x})) = s^{\mathbf{L_V}}(f(\overline{x}))$). On the other hand, if we view the above mentioned modular 4-frame as a residuated lattice 4-frame and take $\emptyset$ as $\bot$, V as $\top$ and $V - x$ as $\overline{x}$, for all $\overline{x} \in \widetilde{\mathcal{A}}(\hat{C})$, it follows that $\mathbf{L_V}$ satisfies $R'(f(\overline{x}), \hat{C}, \widetilde{\mathcal{A}}(\hat{C}), \bot, \top)$. Indeed, (i) and (iv) of $R'(f(\overline{x}), \hat{C}, \widetilde{\mathcal{A}}(\hat{C}), \bot, \top)$ are obvious, while (ii) is true, since $f(x_i)$ is a member of the multiplicative semigroup of $\mathbf{R}$ and this semigroup plays the role of L_{12}. Condition (iii) holds because it holds in $\mathbf{S}$ for $\overline{x}$ and (v) is very easy to check. So, for $\overline{y} = f(\overline{x})$, $\mathbf{L_V}$ satisfies $R'(\overline{y}, C, \widetilde{\mathcal{A}}(C), \bot, \top)$, but not $r^{\mathbf{L_V}}(\overline{y}) = s^{\mathbf{L_V}}(\overline{y})$; hence, by Lemma 3.10, $\mathbf{L}(\mathbf{S}, \mathcal{V})$ fails $r^{\odot_{12}}(\overline{x}') = s^{\odot_{12}}(\overline{x}')$.

If the word problem for $\mathbf{L}(\mathbf{S}, \mathcal{V})$ were decidable then the one for $\mathbf{S}$ would be decidable, too. Thus, $\mathcal{V}$ has an undecidable word problem. ■

Corollary 3.13. *If $\mathcal{V}$ is a variety such that $\mathtt{HSP}(\mathbf{L_V}) \subseteq \mathcal{V} \subseteq \mathcal{DRL}$, for some infinite-dimensional vector space $\mathbf{V}$, then $\mathcal{V}$ has an undecidable quasi-equational theory, namely, there is no algorithm that decides whether a quasi-equation in the language of residuated lattices is valid in $\mathcal{V}$ or not.*

Corollary 3.14. *The word problem and quasi-equational theory for (commutative) distributive residuated lattices is undecidable.*

As pointed out by one of the referees, results in [7] inply the consequence of Theorem 3.12 in the commutative case. Moreover, the result for $\mathcal{DRL}$ alone can be proved in a much more simple and direct way; [11] contains a discussion on decidability and residuated lattices. Thus the novelty of the result in this paper lies in the non-commutative varieties different from $\mathcal{DRL}$.

The equational theory of $\mathcal{RL}$ is known to be decidable (see [11]). It is an open problem, though, whether the same is true for $\mathcal{DRL}$ or for other subvarieties of $\mathcal{RL}$.

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