Rule Labeling for Confluence of Left-Linear Term Rewrite Systems

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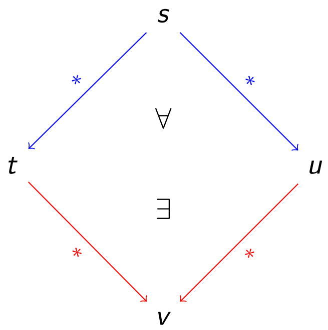
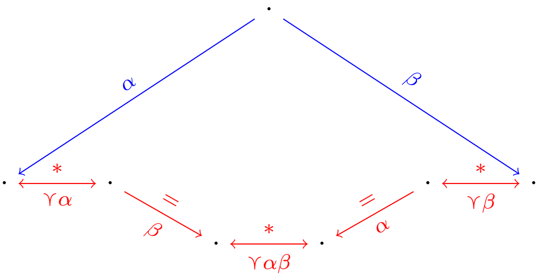
Confluence



Decreasing Diagrams



Theorem (van Oostrom 2008)

Let $\succ$ be a well-founded order on $\mathcal{L}$. Define $\curlyvee\alpha = \{\beta \in \mathcal{L} \mid \alpha \succ \beta\}$. Then $\to$ is confluent if $\to = \bigcup_{\alpha\in\mathcal{L}}(\xrightarrow[\alpha]{})$ is locally decreasing:

$$\forall \alpha, \beta \in \mathcal{L}\colon \; \xleftarrow[\alpha]{} \cdot \xrightarrow[\beta]{} \subseteq \xleftrightarrow[\curlyvee\alpha]{*} \cdot \xrightarrow[\beta]{=} \cdot \xleftrightarrow[\curlyvee\alpha\beta]{*} \cdot \xleftarrow[\alpha]{=} \cdot \xleftrightarrow[\curlyvee\beta]{*}$$

Contents

Labeling TRSs

- idea: label $s \to t$ using a labeling function $\ell(s \to t)$
- proceed as in critical pair lemma

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- we consider left-linear TRSs only

Rule Labeling

- $\ell(s \xrightarrow[l \to r]{} t) = \ell(l \to r)$

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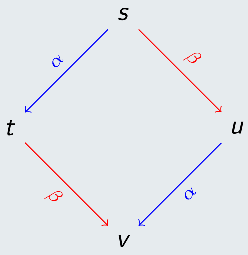
Decreasing diagrams for TRSs

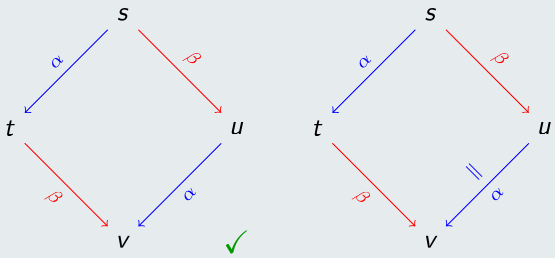


(a) parallel

Rule Labeling

- $\ell(s \xrightarrow[l \to r]{} t) = \ell(l \to r)$

Decreasing diagrams for TRSs

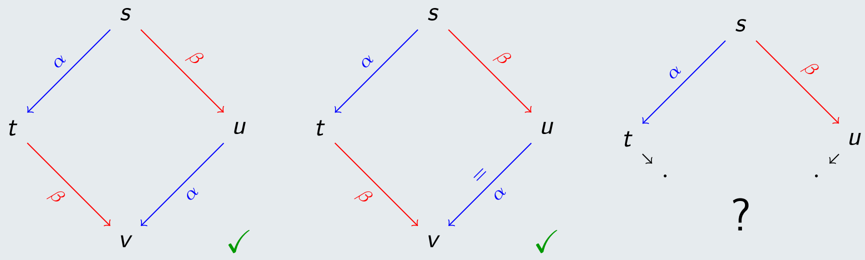


(a) parallel (b) variable

Rule Labeling

- $\ell(s \xrightarrow[l \to r]{} t) = \ell(l \to r)$

Decreasing diagrams for linear TRSs



(a) parallel (b) variable (c) critical

Rule Labeling

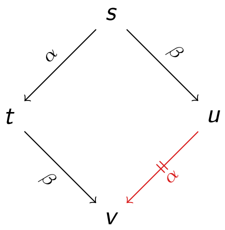
Corollary (van Oostrom 2008)

A linear TRS is confluent if its critical pairs can be joined decreasingly using the rule labeling heuristic.

Contents

Idea

- problematic case



Idea

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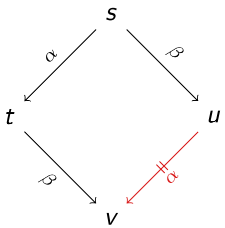
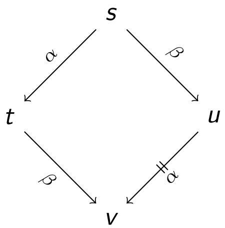


- label parallel rewrite steps ($\color{red}{\twoheadrightarrow\!\!\!\!\!\!\parallel}$)

Idea

- problematic case

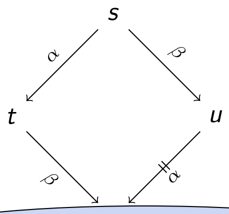


- label parallel rewrite steps ($\nrightarrow\!\!\!\!\!\!\parallel$)
- use set of labels of used rules as label, order by $\succ_{\mathsf{mul}}$

$$\ell^{\parallel}(s \overset{P}{\nrightarrow\!\!\!\!\!\!\parallel} t) = \{\ell(s \to s[t|_p]_p) \mid p \in P\}$$

Idea

- problematic case



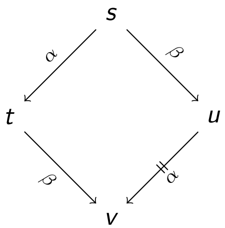
step of $s \nrightarrow t$ at position p

- label parallel rewrite steps ($\nrightarrow$)
- use set of labels of used rules as label, order by $\succ_{\mathsf{mul}}$

$$\ell^{\|}(s \overset{P}{\nrightarrow} t) = \{\ell(s \to s[t|_p]_p) \mid p \in P\}$$

Idea

- problematic case



- label parallel rewrite steps ($\nrightarrow$)
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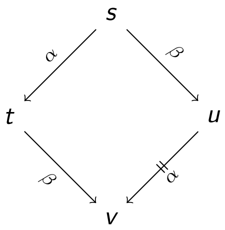
$$\ell^{\|}(s \overset{P}{\nrightarrow} t) = \{\ell(s \to s[t|_p]_p) \mid p \in P\}$$

- for orthogonal $t \mathrel{\underset{\Gamma}{\nleftarrow}} s \mathrel{\underset{\Delta}{\nrightarrow}} t$ we have $t \mathrel{\underset{\Delta'}{\nrightarrow}} v \mathrel{\underset{\Gamma'}{\nleftarrow}} t$ with $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$

Idea

- problematic case



- label parallel rewrite steps ($\mathrel{\|\!\!\!\rightarrow}$)
- use set of labels of used rules as label, order by $\succ_{\mathsf{mul}}$

$$\ell^{\|}(s \overset{P}{\mathrel{\|\!\!\!\rightarrow}} t) = \{\ell(s \to s[t|_p]_p) \mid p \in P\}$$

- for orthogonal $t \underset{\Gamma}{\mathrel{\leftarrow\!\!\!\|}} s \underset{\Delta}{\mathrel{\|\!\!\!\rightarrow}} t$ we have $t \underset{\Delta'}{\mathrel{\|\!\!\!\rightarrow}} v \underset{\Gamma'}{\mathrel{\leftarrow\!\!\!\|}} t$ with $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$
- for non-orthogonal steps, consider parallel critical pairs

Homogeneity

- $\underset{\Gamma}{\nrightarrow\!\!\!\!\!\!\rightarrow}$ is homogeneous if $\#\Gamma \leqslant 1$

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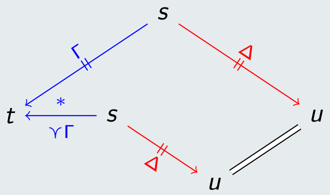
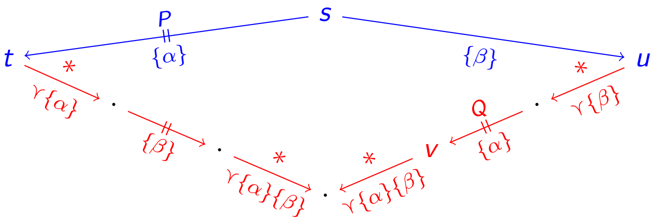
- if $\#\Gamma > 1$ then $\Gamma \succ_{\mathsf{mul}} \{\alpha\}$ for all $\alpha \in \Gamma$

Lemma

If $\#\Gamma > 1$ then $t \underset{\Gamma}{\leftarrow\!\!\!\nparallel} s \underset{\Delta}{\nparallel\!\!\!\rightarrow} u$ can be joined decreasingly:



Main Theorem



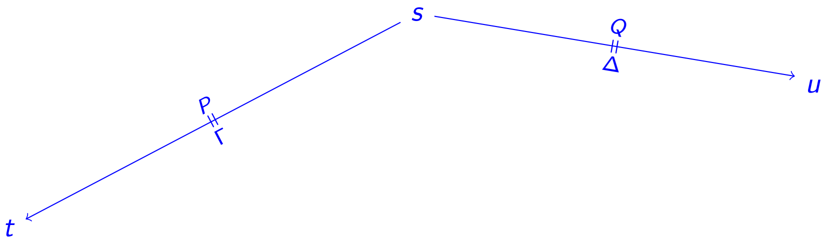
Theorem

A left-linear TRS $\mathcal{R}$ is confluent if all its homogeneous parallel critical peaks $t \xleftarrow[\{\alpha\}]{P\ \parallel} s \xrightarrow[\{\beta\}]{} u$ *can be joined decreasingly as*

$$t \xrightarrow[\curlyvee\{\alpha\}]{*} \cdot \xrightarrow[\{\beta\}]{\parallel} \cdot \xrightarrow[\curlyvee\{\alpha\}\{\beta\}]{*} \cdot \xleftarrow[\curlyvee\{\alpha\}\{\beta\}]{*} v \xleftarrow[\{\alpha\}]{Q\ \parallel} \cdot \xleftarrow[\curlyvee\{\beta\}]{*} u$$

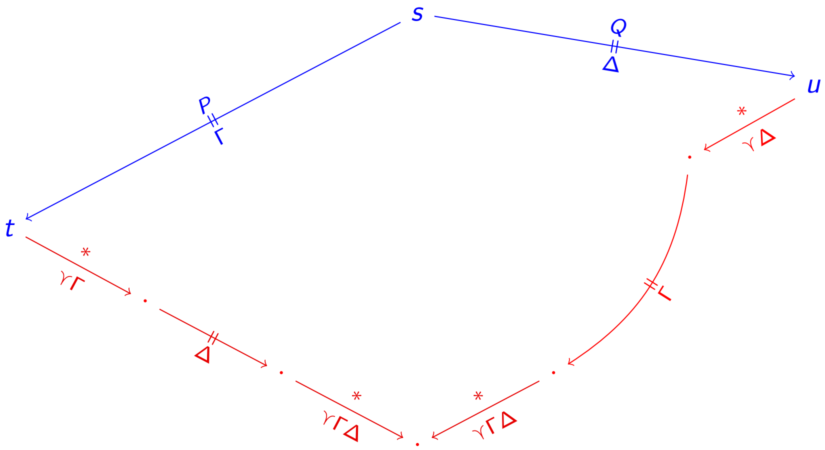
such that $\mathcal{V}\mathrm{ar}(v|_Q) \subseteq \mathcal{V}\mathrm{ar}(s|_P)$.

Proof



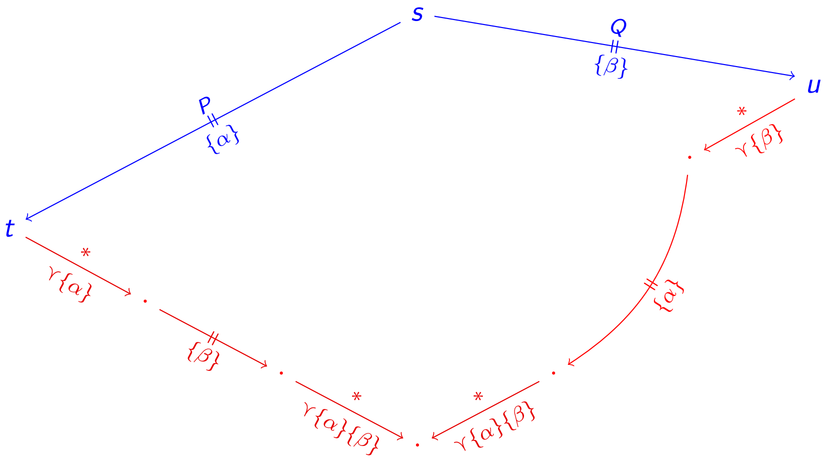
- show local decreasingness

Proof



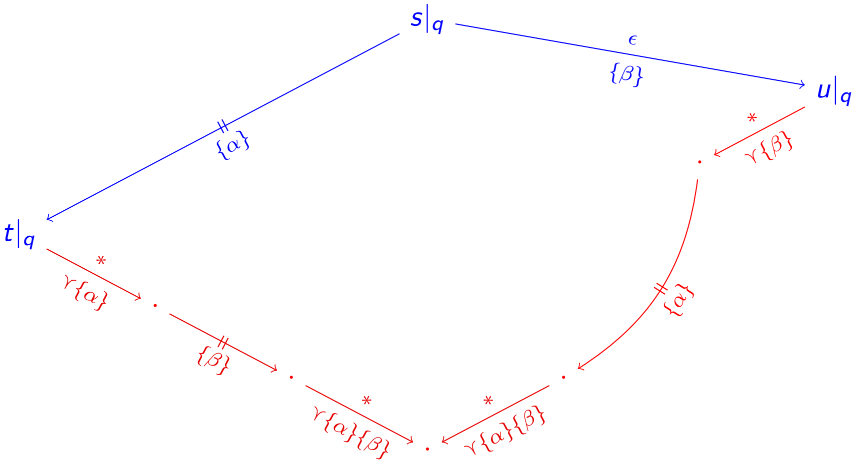
- next: homogeneity

Proof



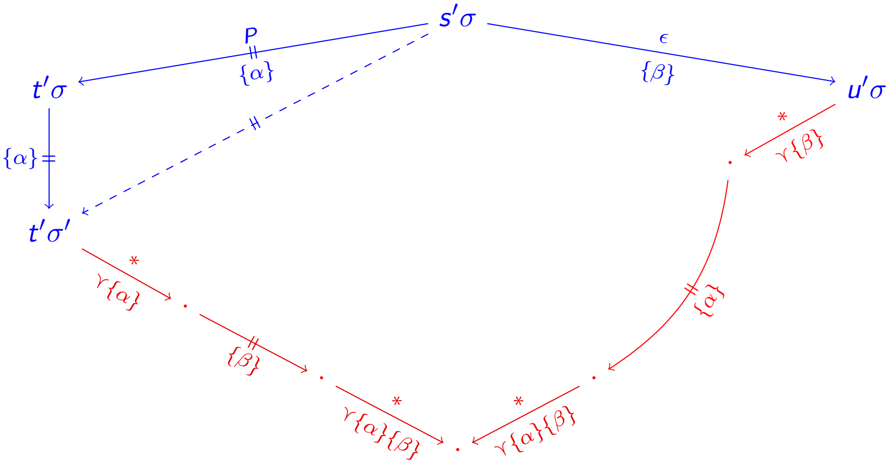
- next: consider $q \in \min(P, Q)$ (w.l.o.g. $q \in Q$)

Proof



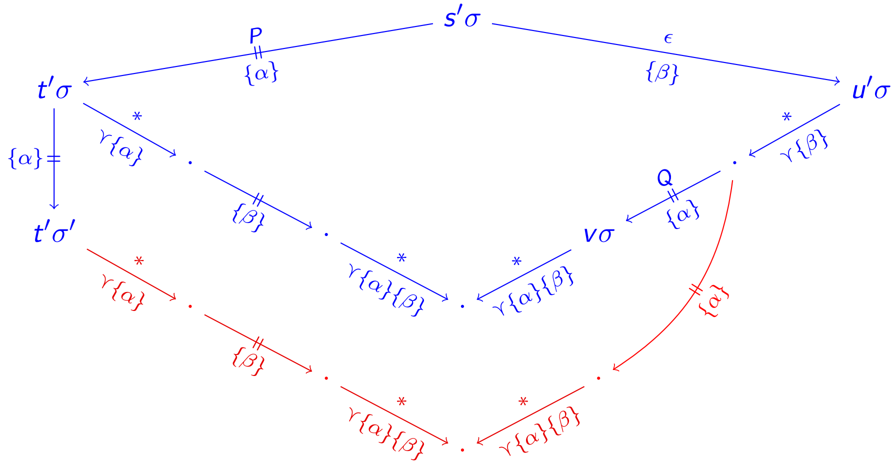
- next: instance of parallel critical pair

Proof



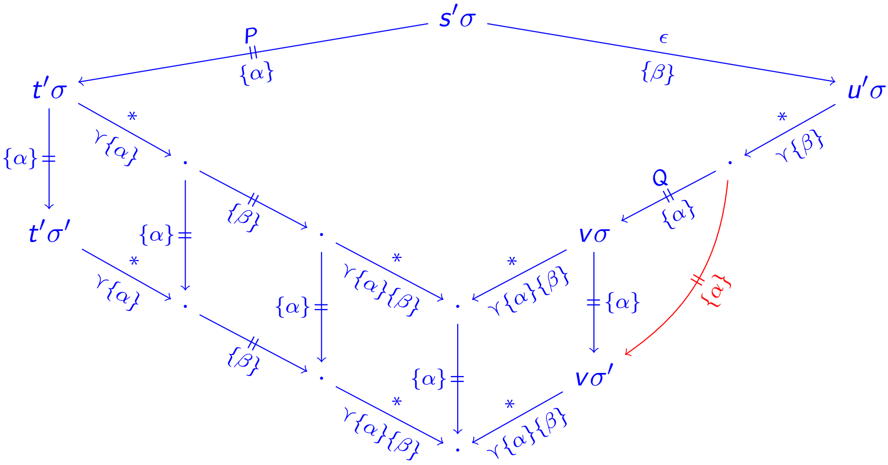
- next: assumption

Proof



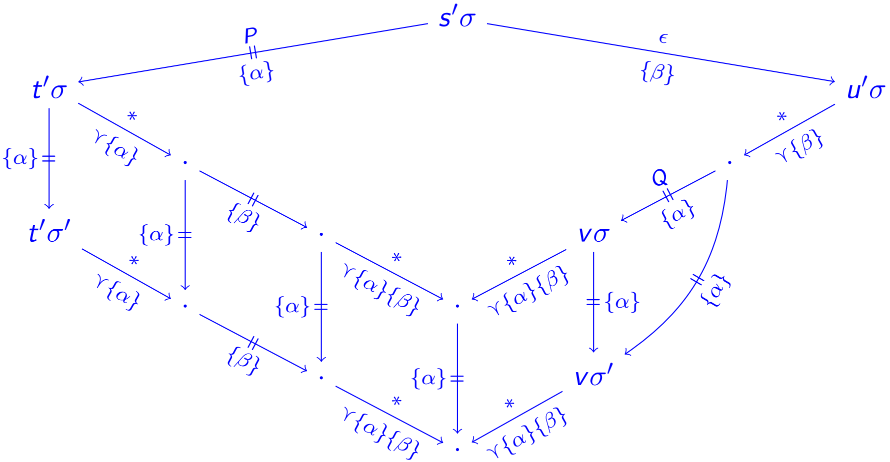
- next: apply $\sigma \twoheadrightarrow\!\!\!\!\!\!+ \sigma'$

Proof



- next: $\mathcal{V}\mathrm{ar}(v|_Q) \subseteq \mathcal{V}\mathrm{ar}(s'|_P)$

Proof



- q.e.d.

Example

TRS

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{c} \qquad \mathsf{h}(x) \to \mathsf{h}(\mathsf{f}(x,x))$$

$$\mathsf{b} \to \mathsf{a} \qquad \mathsf{f}(\mathsf{b},\mathsf{b}) \to \mathsf{c}$$

Example

TRS

$$\mathsf{a} \xrightarrow[1]{} \mathsf{b} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \xrightarrow[1]{} \mathsf{c} \qquad \mathsf{h}(x) \xrightarrow[0]{} \mathsf{h}(\mathsf{f}(x,x))$$

$$\mathsf{b} \xrightarrow[0]{} \mathsf{a} \qquad \mathsf{f}(\mathsf{b},\mathsf{b}) \xrightarrow[2]{} \mathsf{c}$$

Example

TRS

$$\mathsf{a} \underset{1}{\to} \mathsf{b} \qquad \mathsf{f(a,a)} \underset{1}{\to} \mathsf{c} \qquad \mathsf{h}(x) \underset{0}{\to} \mathsf{h}(\mathsf{f}(x,x))$$

$$\mathsf{b} \underset{0}{\to} \mathsf{a} \qquad \mathsf{f(b,b)} \underset{2}{\to} \mathsf{c}$$

parallel critical peaks

- $f(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \underset{\{1\}}{\leftarrow\!\!\!\!\!+\!\!\!\!+\!\!-} f(\mathsf{a},\mathsf{a}) \underset{\{1\}}{\longrightarrow} \mathsf{c}$
- $f(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \underset{\{0\}}{\leftarrow\!\!\!\!\!+\!\!\!\!+\!\!-} f(\mathsf{b},\mathsf{b}) \underset{\{2\}}{\longrightarrow} \mathsf{c}$

Example

TRS

$$\mathsf{a} \xrightarrow[1]{} \mathsf{b} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \xrightarrow[1]{} \mathsf{c} \qquad \mathsf{h}(x) \xrightarrow[0]{} \mathsf{h}(\mathsf{f}(x,x))$$

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parallel critical peaks

- $f(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \xleftarrow[\{1\}]{+\!\!\!\!|\!|} f(\mathsf{a},\mathsf{a}) \xrightarrow[\{1\}]{} \mathsf{c} : \; \ldots \xrightarrow[\{0\}]{*} f(\mathsf{a},\mathsf{a}) \xrightarrow[\{1\}]{} \mathsf{c}$
- $f(\{\mathsf{a},\mathsf{b}\},\{\mathsf{a},\mathsf{b}\}) \xleftarrow[\{0\}]{+\!\!\!\!|\!|} f(\mathsf{b},\mathsf{b}) \xrightarrow[\{2\}]{} \mathsf{c} : \; \ldots \xrightarrow[\{0\}]{*} f(\mathsf{a},\mathsf{a}) \xrightarrow[\{1\}]{} \mathsf{c}$

Implementation

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A left-linear TRS $\mathcal{R}$ is confluent if all its homogeneous parallel critical peaks $t \xleftarrow[\{\alpha\}]{P} s \xrightarrow[\{\beta\}]{} u$ *can be joined decreasingly as*

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such that $\mathcal{V}\text{ar}(v|_Q) \subseteq \mathcal{V}\text{ar}(s|_P)$, *and* $\Gamma' \preceq \{\alpha\}$, $\Delta' \preceq \{\beta\}$.

Contents

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this talk

- rule labeling
- for left-linear systems using parallel critical pairs

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further developments

- generalized labelings
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- try to relax variable condition $\mathcal{V}\mathrm{ar}(v|_Q) \subseteq \mathcal{V}\mathrm{ar}(s|_P)$
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- conversion version

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Thank you!

Literature

- V. van Oostrom.
  Confluence by decreasing diagrams – converted.
  In *Proc. 19th RTA*, volume 5117 of *LNCS*, pages 306–320, 2008.
- H. Zankl, B. Felgenhauer, and A. Middeldorp.
  Labelings for decreasing diagrams.
  In *Proc. 22nd RTA*, volume 10 of *LIPIcs*, pages 377–392, 2011.