Three termination problems

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  but no global understanding:

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- Three unrelated termination problems : partial specific answers known,
  but no global understanding: can some general tools be useful?

- Plan :

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    1. The Polish Algorithm for Left-Selfdistributivity

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- The associativity law

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  - return YES.

- Remember : in Polish, associativity is $\begin{cases} x\,y\,z*\,* \\ x\,y*z\,* \end{cases}$.

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So $\boldsymbol{t}_2 = \boldsymbol{t}'_2$, hence $\boldsymbol{t}_0$ and $\boldsymbol{t}'_0$ are A-equivalent.

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- "Theorem".—

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- "Theorem".— The Polish Algorithm works for associativity.

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- "Theorem".— The Polish Algorithm works for associativity.
(In particular, it terminates.)

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\boldsymbol{t}'_0 &= x\,x*x\,x\,x*** \\[1ex]
\boldsymbol{t}_1 &= x\,x*x\,x**x\,x\,x*** \\
\boldsymbol{t}'_1 &= x\,x*x\,x\,x*** & \qquad (= \boldsymbol{t}'_0) \\[1ex]
\boldsymbol{t}_2 &= x\,x*x\,x**x\,x\,x*** & \qquad (= \boldsymbol{t}_1) \\
\boldsymbol{t}'_2 &= x\,x*x\,x*x\,x** \\[1ex]
\boldsymbol{t}_3 &= x\,x*x\,x**x\,x\,x*** & \qquad (= \boldsymbol{t}_2) \\
\boldsymbol{t}'_3 &= x\,x*x\,x**x\,x*x\,x***
\end{aligned}$$

- Left-selfdistributivity (LD) : $x*(y*z) = (x*y)*(x*z)$,

  i.e., in Polish, $\begin{cases} x\,y\,z**\\ x\,y*x\,z** \end{cases}$ compare with associativity $\begin{cases} x\,y\,z**\\ x\,y*x* \end{cases}$

- Polish Algorithm: the same as for associativity.

- Example: $\boldsymbol{t} = x*((x*x)*(x*x))$, $\boldsymbol{t}' = (x*x)*(x*(x*x))$, i.e., in Polish,

$$\begin{aligned}
\boldsymbol{t}_0 &= x\,x\,x*x\,x***\\
\boldsymbol{t}'_0 &= x\,x*x\,x\,x***\\[1ex]
\boldsymbol{t}_1 &= x\,x*x\,x**x\,x\,x*** \\
\boldsymbol{t}'_1 &= x\,x*x\,x\,x*** && (= \boldsymbol{t}'_0)\\[1ex]
\boldsymbol{t}_2 &= x\,x*x\,x**x\,x\,x*** && (= \boldsymbol{t}_1)\\
\boldsymbol{t}'_2 &= x\,x*x\,x*x\,x**\\[1ex]
\boldsymbol{t}_3 &= x\,x*x\,x**x\,x\,x*** && (= \boldsymbol{t}_2)\\
\boldsymbol{t}'_3 &= x\,x*x\,x**x\,x*x\,x***
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$$
\begin{aligned}
\boldsymbol{t}_0 &= x\,x\,x*x\,x*\,*\,* \\
\boldsymbol{t}'_0 &= x\,x*x\,x\,x*\,*\,* \\[4pt]
\boldsymbol{t}_1 &= x\,x*x\,x*\,*x\,x\,x*\,*\,* \\
\boldsymbol{t}'_1 &= x\,x*x\,x\,x*\,*\,* && (= \boldsymbol{t}'_0) \\[4pt]
\boldsymbol{t}_2 &= x\,x*x\,x*\,*x\,x\,x*\,*\,* && (= \boldsymbol{t}_1) \\
\boldsymbol{t}'_2 &= x\,x*x\,x*x\,x*\,* \\[4pt]
\boldsymbol{t}_3 &= x\,x*x\,x*\,*x\,x\,x*\,*\,* && (= \boldsymbol{t}_2) \\
\boldsymbol{t}'_3 &= x\,x*x\,x*\,*x\,x*x\,x*\,*\,* \\[4pt]
\boldsymbol{t}_4 &= x\,x*x\,x*\,*x\,x*x\,x*\,*\,* \\
\boldsymbol{t}'_4 &= x\,x*x\,x*\,*x\,x*x\,x*\,*\,* && (= \boldsymbol{t}'_3)
\end{aligned}
$$

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- Example: $\boldsymbol{t} = x*((x*x)*(x*x))$, $\boldsymbol{t}' = (x*x)*(x*(x*x))$, i.e., in Polish,

$$\begin{aligned}
\boldsymbol{t}_0 &= xxx*xx***\\
\boldsymbol{t}'_0 &= xx*xxx***\\[1ex]
\boldsymbol{t}_1 &= xx*xx**xxx***\\
\boldsymbol{t}'_1 &= xx*xxx*** && (= \boldsymbol{t}'_0)\\[1ex]
\boldsymbol{t}_2 &= xx*xx**xxx*** && (= \boldsymbol{t}_1)\\
\boldsymbol{t}'_2 &= xx*xx*xx**\\[1ex]
\boldsymbol{t}_3 &= xx*xx**xxx*** && (= \boldsymbol{t}_2)\\
\boldsymbol{t}'_3 &= xx*xx**xx*xx***\\[1ex]
\boldsymbol{t}_4 &= xx*xx**xx*xx***\\
\boldsymbol{t}'_4 &= xx*xx**xx*xx*** && (= \boldsymbol{t}'_3)
\end{aligned}$$

So $\boldsymbol{t}_4 = \boldsymbol{t}'_4$, hence $\boldsymbol{t}_0$ and $\boldsymbol{t}'_0$ are LD-equivalent.

- Conjecture.— The Polish Algorithm works for left-selfdistributivity.

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- Known.— (i) If it terminates, the Polish Algorithm works for left-selfdistributivity.

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  (ii) The smallest counter-example to termination (if any) is huge.

- A true (but infinite) rewrite system.

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- Alphabet: `a`, `b`, `A`, `B`

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- Example:

$$\begin{aligned} \boldsymbol{w}_0 &= \mathtt{a}\underline{\mathtt{abA}}\mathtt{bbAA} \\ \boldsymbol{w}_1 &= \mathtt{aBabbbAA} \end{aligned}$$

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$$\begin{aligned} \boldsymbol{w}_0 &= \mathtt{a}\underline{\mathtt{abA}}\mathtt{bbAA} \\ \boldsymbol{w}_1 &= \mathtt{aB}\underline{\mathtt{abbbA}}\mathtt{A} \\ \boldsymbol{w}_2 &= \mathtt{aBBaaabA} \end{aligned}$$

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$$
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\boldsymbol{w}_3 &= \mathtt{aBBaaBab},
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draw the grid:

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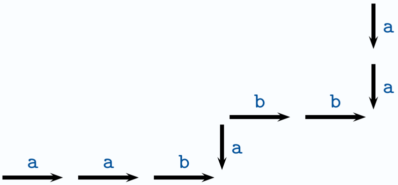
draw the grid:



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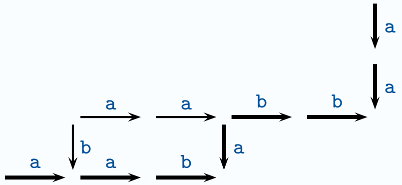

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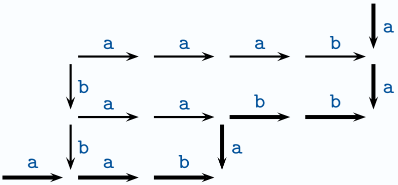

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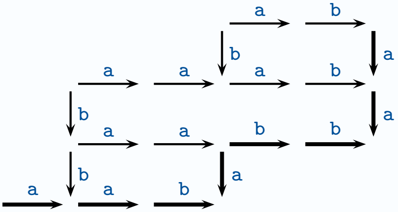

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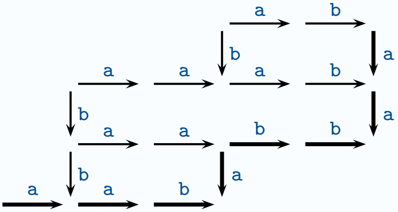

□

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  so far: case of "3-strand" braids; now: case of "4-strand" braids

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  so far: case of "3-strand" braids; now: case of "4-strand" braids
  (case of "n strand" braids entirely similar for every n).

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- Alphabet: `a`, `b`, `c`, `A`, `B`, `C`.

- This is the braid handle reduction procedure;
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- Rewrite rules:
  - $\mathtt{aA} \longrightarrow \varepsilon$, $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{bB} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, $\mathtt{cC} \longrightarrow \varepsilon$, $\mathtt{Cc} \longrightarrow \varepsilon$,

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- This is the braid handle reduction procedure;
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  (case of "n strand" braids entirely similar for every n).

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  - $\mathtt{aA} \longrightarrow \varepsilon$, $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{bB} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, $\mathtt{cC} \longrightarrow \varepsilon$, $\mathtt{Cc} \longrightarrow \varepsilon$, (as above)
  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$:

- This is the braid handle reduction procedure;
  so far: case of "3-strand" braids; now: case of "4-strand" braids
  (case of "$\boldsymbol{n}$ strand" braids entirely similar for every $\boldsymbol{n}$).

- Alphabet: $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{A}, \mathtt{B}, \mathtt{C}$.
- Rewrite rules:
  - $\mathtt{aA} \longrightarrow \boldsymbol{\varepsilon}$, $\mathtt{Aa} \longrightarrow \boldsymbol{\varepsilon}$, $\mathtt{bB} \longrightarrow \boldsymbol{\varepsilon}$, $\mathtt{Bb} \longrightarrow \boldsymbol{\varepsilon}$, $\mathtt{cC} \longrightarrow \boldsymbol{\varepsilon}$, $\mathtt{Cc} \longrightarrow \boldsymbol{\varepsilon}$, (as above)
  - for $\boldsymbol{w}$ in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}\boldsymbol{w}\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(\boldsymbol{w})$,

- This is the braid handle reduction procedure;
  so far: case of "3-strand" braids; now: case of "4-strand" braids
  (case of "n strand" braids entirely similar for every n).

- Alphabet: $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{A}, \mathtt{B}, \mathtt{C}$.
- Rewrite rules:
  - $\mathtt{aA} \longrightarrow \varepsilon$, $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{bB} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, $\mathtt{cC} \longrightarrow \varepsilon$, $\mathtt{Cc} \longrightarrow \varepsilon$, (as above)
  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,

- This is the braid handle reduction procedure;
  so far: case of "3-strand" braids; now: case of "4-strand" braids
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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,

- This is the braid handle reduction procedure;
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- Rewrite rules:
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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,

- This is the braid handle reduction procedure;
  so far: case of "3-strand" braids; now: case of "4-strand" braids
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- Rewrite rules:
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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,
  - for w in $\{\mathtt{c}\}^*$ or $\{\mathtt{C}\}^*$: $\mathtt{b}w\mathtt{B} \longrightarrow \phi_{\mathtt{b}}(w)$, $\mathtt{B}w\mathtt{b} \longrightarrow \phi_{\mathtt{B}}(w)$,

- This is the braid handle reduction procedure;
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  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,
  - for w in $\{\mathtt{c}\}^*$ or $\{\mathtt{C}\}^*$: $\mathtt{b}w\mathtt{B} \longrightarrow \phi_{\mathtt{b}}(w)$, $\mathtt{B}w\mathtt{b} \longrightarrow \phi_{\mathtt{B}}(w)$,
    with $\phi_{\mathtt{b}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{Cbc}$ and $\mathtt{C} \longrightarrow \mathtt{CBc}$,
    and $\phi_{\mathtt{B}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{cbC}$ and $\mathtt{C} \longrightarrow \mathtt{cBC}$.

- This is the braid handle reduction procedure;
  so far: case of "3-strand" braids; now: case of "4-strand" braids
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  - $\mathtt{aA} \longrightarrow \varepsilon$, $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{bB} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, $\mathtt{cC} \longrightarrow \varepsilon$, $\mathtt{Cc} \longrightarrow \varepsilon$, (as above)
  - for w in $\{\mathtt{b}, \mathtt{c}, \mathtt{C}\}^*$ or $\{\mathtt{B}, \mathtt{c}, \mathtt{C}\}^*$: $\mathtt{a}w\mathtt{A} \longrightarrow \phi_{\mathtt{a}}(w)$, $\mathtt{A}w\mathtt{a} \longrightarrow \phi_{\mathtt{A}}(w)$,
    with $\phi_{\mathtt{a}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{Bab}$ and $\mathtt{B} \longrightarrow \mathtt{BAb}$,
    and $\phi_{\mathtt{A}}(w)$ obtained from w by $\mathtt{b} \longrightarrow \mathtt{baB}$ and $\mathtt{B} \longrightarrow \mathtt{bAB}$,
  - for w in $\{\mathtt{c}\}^*$ or $\{\mathtt{C}\}^*$: $\mathtt{b}w\mathtt{B} \longrightarrow \phi_{\mathtt{b}}(w)$, $\mathtt{B}w\mathtt{b} \longrightarrow \phi_{\mathtt{B}}(w)$,
    with $\phi_{\mathtt{b}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{Cbc}$ and $\mathtt{C} \longrightarrow \mathtt{CBc}$,
    and $\phi_{\mathtt{B}}(w)$ obtained from w by $\mathtt{c} \longrightarrow \mathtt{cbC}$ and $\mathtt{C} \longrightarrow \mathtt{cBC}$.

- Remark.— $\mathtt{ab}^i\mathtt{A} \longrightarrow (\mathtt{Bab})^i \longrightarrow \mathtt{Ba}^i\mathtt{b}$: extends the 3-strand case.

- Example:

- Example:

  `abcbABABCBA`

- Example:

  `abcbABABCBA`

- Example:

  abcbABABCBA
  BabcBabBABCBA

- Example:

  `abcbABABCBA`
  `BabcBabBABCBA`
  `BabcBaABCBA`

- Example:

  ```
  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
  ```

- Example:

  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
  BaCbcBCBA

- Example:

  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
  BaCbcBCBA
  BaCCbcCBA

- Example:

  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
  BaCbcBCBA
  BaCCbcCBA
  BaCCbBA

- Example:

```
abcbABABCBA
BabcBabBABCBA
BabcBaABCBA
BabcBBCBA
BaCbcBCBA
BaCCbcCBA
BaCCbBA
BaCCA
```

- Example:

```
abcbABABCBA
BabcBabBABCBA
BabcBaABCBA
BabcBBCBA
BaCbcBCBA
BaCCbcCBA
BaCCbBA
BaCCA
BCC
```

- Example:

  `abcbABABCBA`
  `BabcBabBABCBA`
  `BabcBaABCBA`
  `BabcBBCBA`
  `BaCbcBCBA`
  `BaCCbcCBA`
  `BaCCbBA`
  `BaCCA`
  `BCC`

$\rightsquigarrow$ Terminates: the final word does not contain both `a` and `A`

- Example:

  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
  BaCbcBCBA
  BaCCbcCBA
  BaCCbBA
  BaCCA
  BCC

⇝ Terminates: the final word does not contain both a and A
(by the way: contains neither a nor A, and not both b and B.)

- Example:

  abcbABABCBA
  BabcBabBABCBA
  BabcBaABCBA
  BabcBBCBA
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⇝ Terminates: the final word does not contain both a and A
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- Theorem.— Handle reduction always terminates in exponential time

- Example:

  abcbABABCBA
  BabcBabBABCBA
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  BaCbcBCBA
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  BCC

$\rightsquigarrow$ Terminates: the final word does not contain both a and A
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- **Theorem.**— Handle reduction always terminates in exponential time
(and *id.* for n-strand version).

- Example:

  $\mathtt{\underline{abcbA}BABCBA}$
  $\mathtt{BabcBa\underline{bB}ABCBA}$
  $\mathtt{BabcB\underline{aA}BCBA}$
  $\mathtt{Ba\underline{bcB}BCBA}$
  $\mathtt{BaC\underline{bcB}CBA}$
  $\mathtt{BaCCb\underline{cC}BA}$
  $\mathtt{BaCC\underline{bB}A}$
  $\mathtt{\underline{BaCCA}}$
  $\mathtt{BCC}$

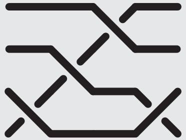
$\rightsquigarrow$ Terminates: the final word does not contain both $\mathtt{a}$ and $\mathtt{A}$
(by the way: contains neither $\mathtt{a}$ nor $\mathtt{A}$, and not both $\mathtt{b}$ and $\mathtt{B}$.)

- Theorem.— Handle reduction always terminates in exponential time
(and *id.* for n-strand version).

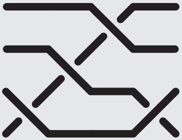
- Experimental evidence.— It terminates in quadratic time (for every n).

- A $\mathbf{4}$-strand braid diagram

- A 4-strand braid diagram



- A 4-strand braid diagram = 2D-projection of a 3D-figure:



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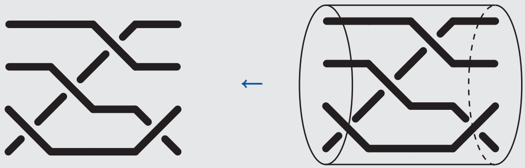

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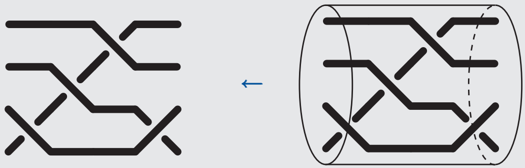
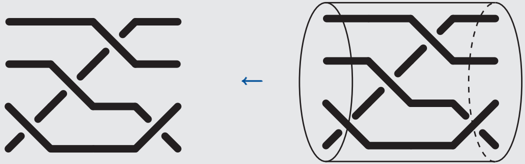


- isotopy = move the strands but keep the ends fixed:

- A 4-strand braid diagram = 2D-projection of a 3D-figure:



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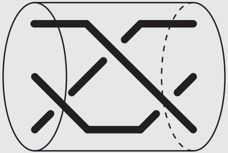

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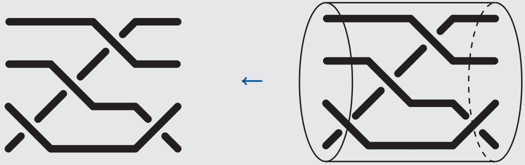

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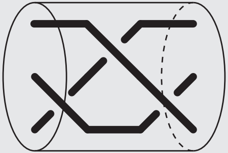
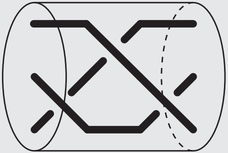
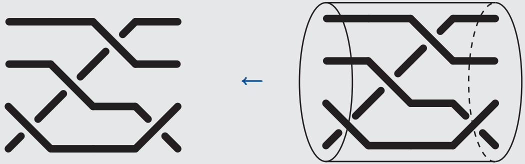


isotopic to

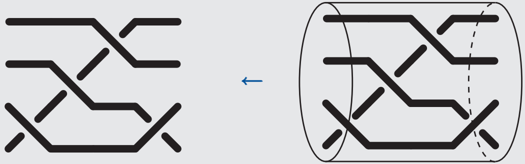


- A $\mathbf{4}$-strand braid diagram = 2D-projection of a 3D-figure:

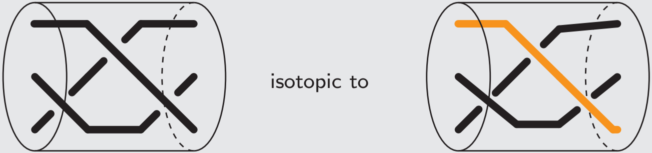


- isotopy = move the strands but keep the ends fixed:

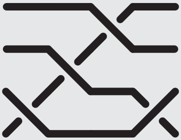
- A $\mathbf{4}$-strand braid diagram = 2D-projection of a 3D-figure:



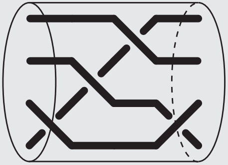
- isotopy = move the strands but keep the ends fixed:



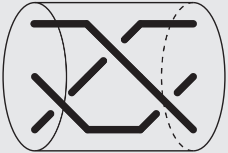
- A 4-strand braid diagram = 2D-projection of a 3D-figure:



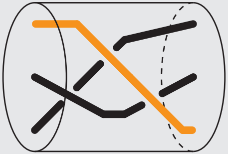
$\leftarrow$



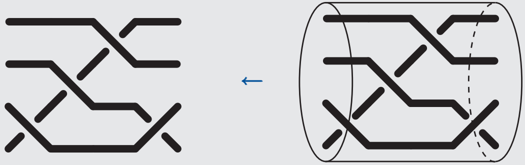
- isotopy = move the strands but keep the ends fixed:



isotopic to



- A 4-strand braid diagram = 2D-projection of a 3D-figure:



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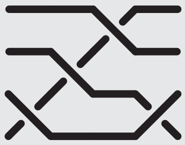
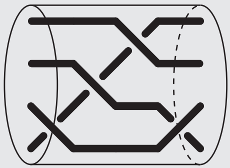
- A 4-strand braid diagram  = 2D-projection of a 3D-figure:



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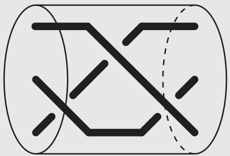
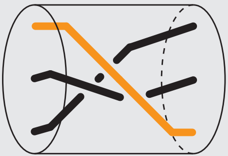


isotopic to



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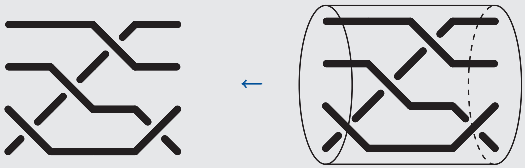

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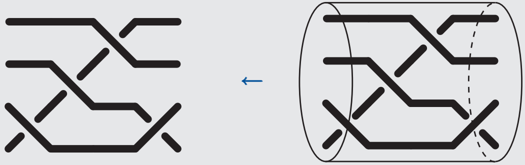

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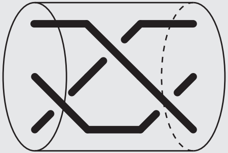

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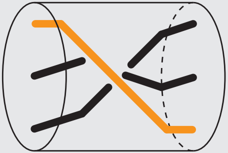

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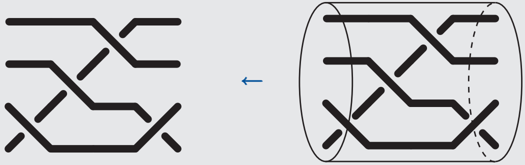

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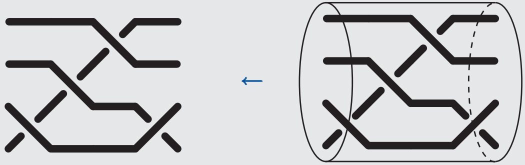

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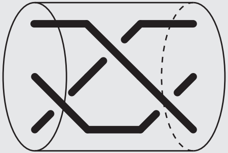
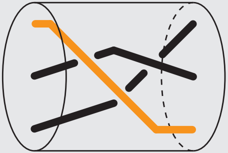
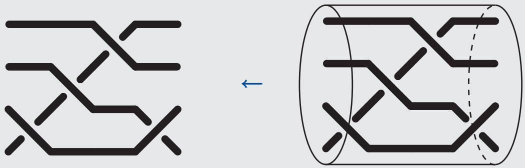


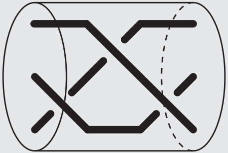
isotopic to



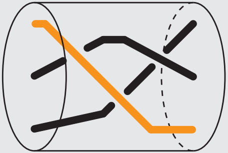
- A 4-strand braid diagram = 2D-projection of a 3D-figure:



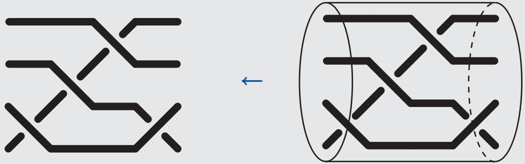
- isotopy = move the strands but keep the ends fixed:



isotopic to

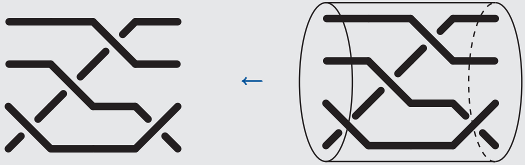


- A $\mathbf{4}$-strand braid diagram $=$ 2D-projection of a 3D-figure:

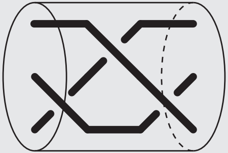


- isotopy $=$ move the strands but keep the ends fixed:

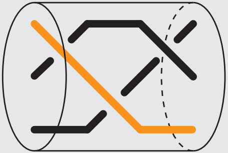
- A $\mathbf{4}$-strand braid diagram $=$ 2D-projection of a 3D-figure:



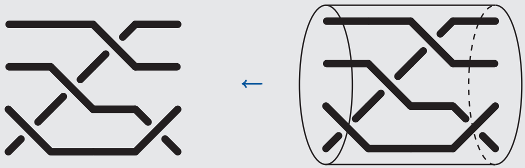
- isotopy $=$ move the strands but keep the ends fixed:



isotopic to



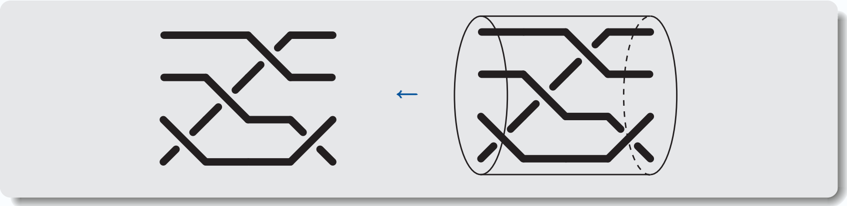
- A 4-strand braid diagram = 2D-projection of a 3D-figure:



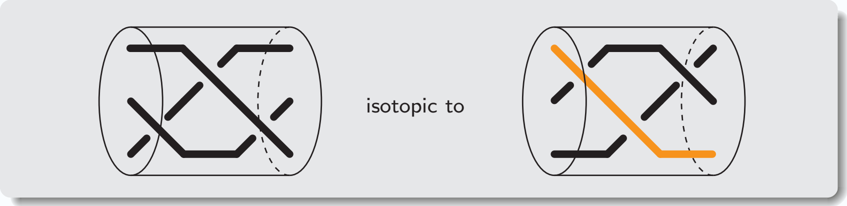
- isotopy = move the strands but keep the ends fixed:

- a braid := an isotopy class $\rightsquigarrow$ represented by 2D-diagram,

- A 4-strand braid diagram = 2D-projection of a 3D-figure:

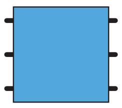


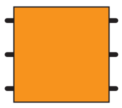
- isotopy = move the strands but keep the ends fixed:



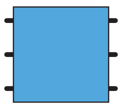
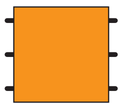
- a braid := an isotopy class $\rightsquigarrow$ represented by 2D-diagram,
  but different 2D-diagrams may give rise to the same braid.

- Product of two braids:





- Product of two braids:

 $*$  $:=$

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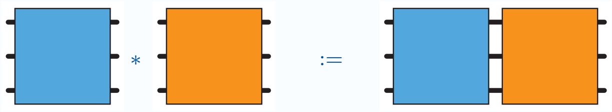

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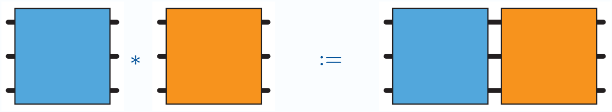
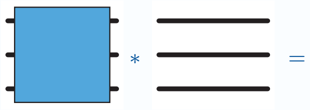


- Then well-defined with respect to isotopy), associative, admits a unit:



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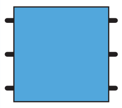
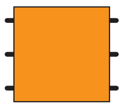
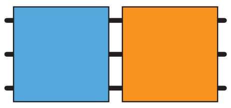
 $*$  $:=$ 

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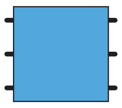
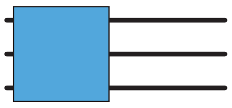
 $*$  $=$ 

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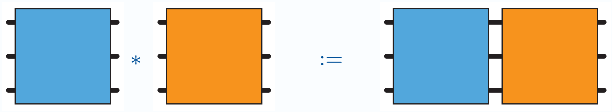

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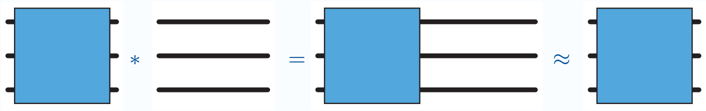

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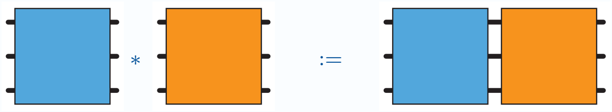

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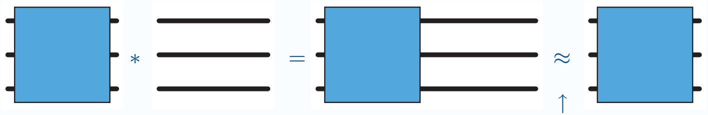
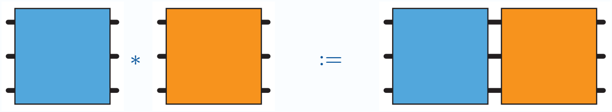


isotopic to

- Product of two braids:



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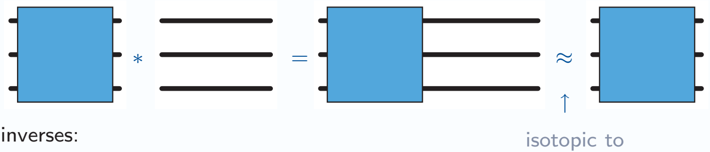

and inverses:

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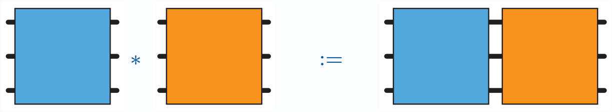

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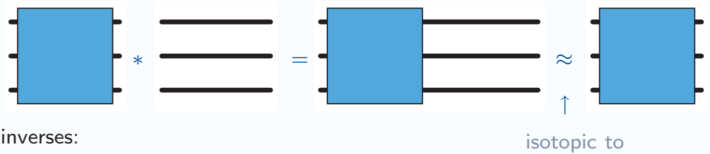

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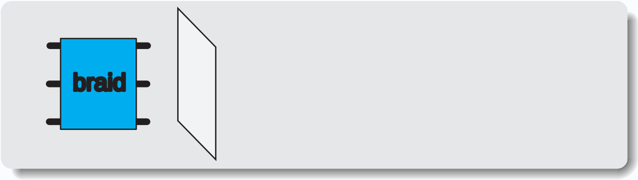

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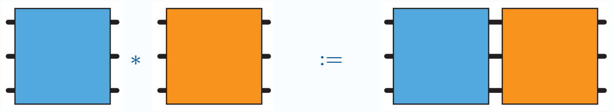

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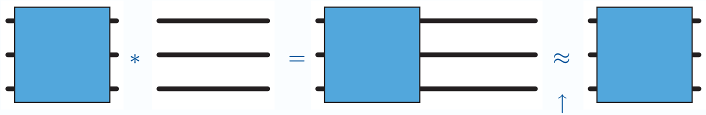

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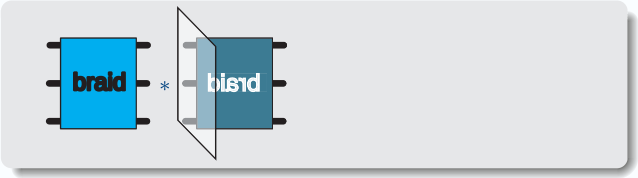

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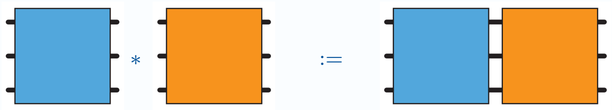

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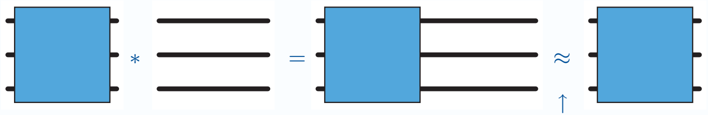

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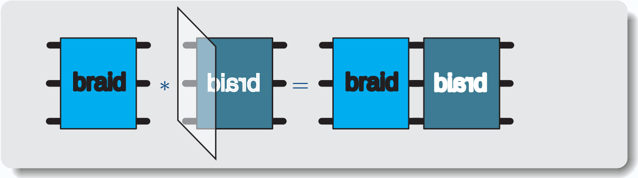

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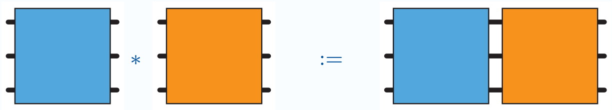

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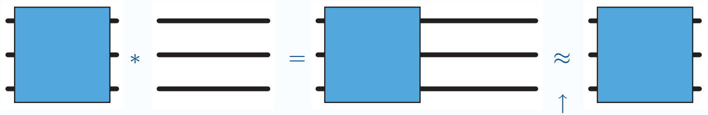

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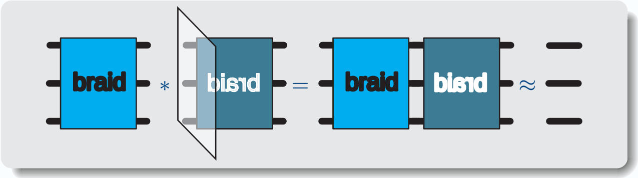

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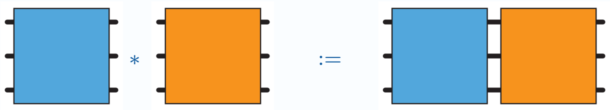

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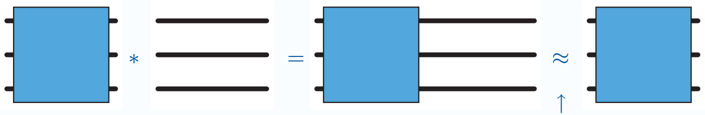

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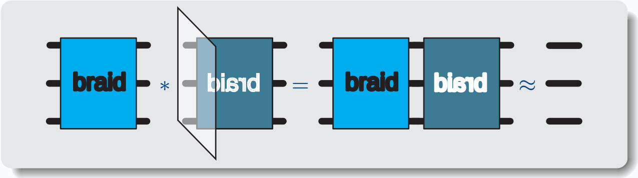
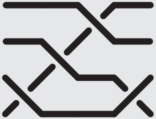


⇝ For each n, the group B_n of n-strand braids (E.Artin, 1925).

- Artin generators of B_n:



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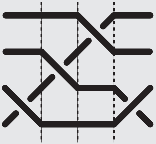

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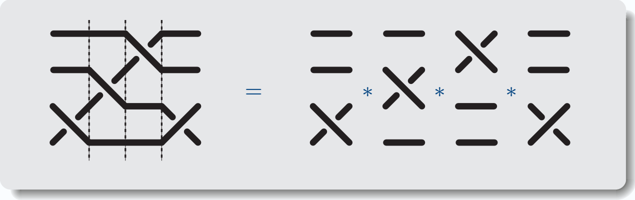

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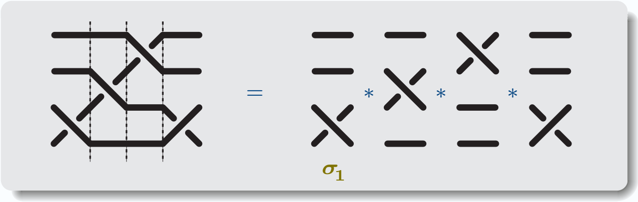

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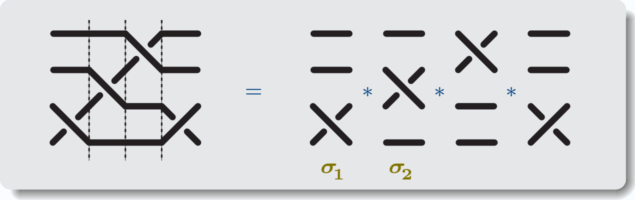

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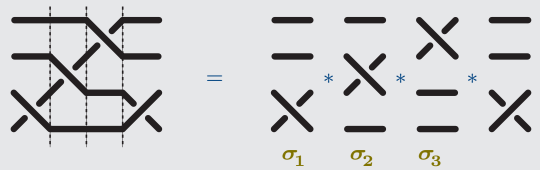

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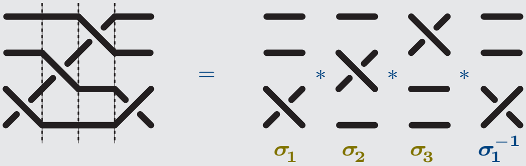

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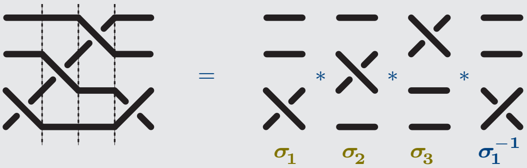

- Theorem (Artin): The group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$,

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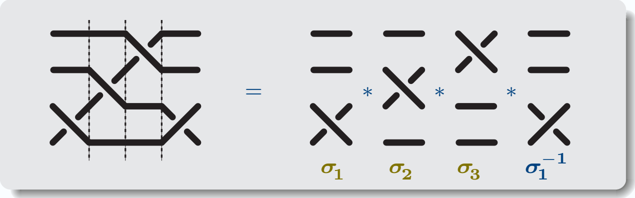

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subject to $\left\{ \begin{array}{ll} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i-j| = 1, \\ & \end{array} \right.$

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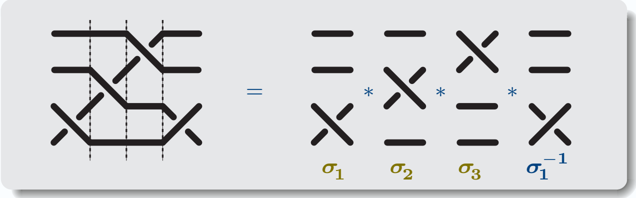

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$$\begin{cases} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2. \end{cases}$$

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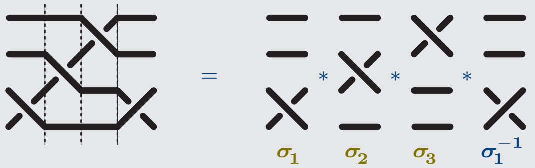

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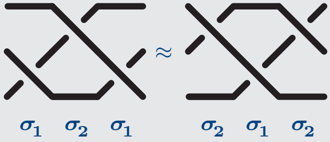

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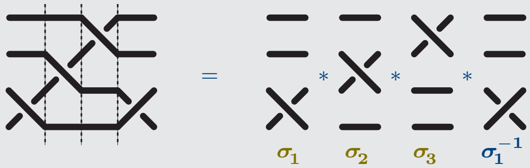

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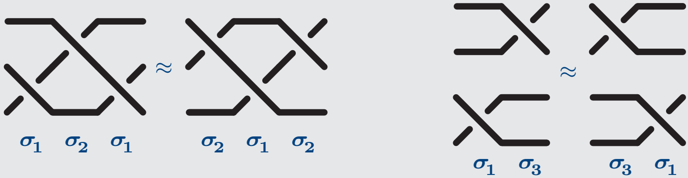

- A $\boldsymbol{\sigma}_i$-handle:

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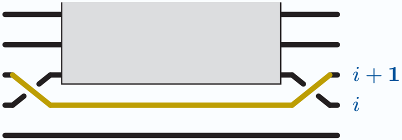

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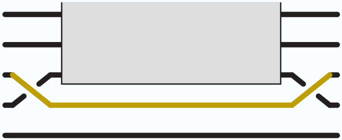

- Reducing a handle:

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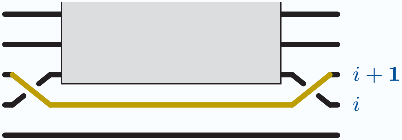

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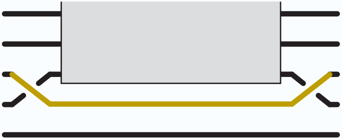

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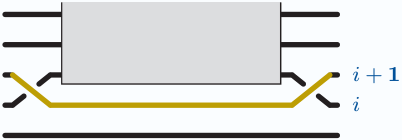

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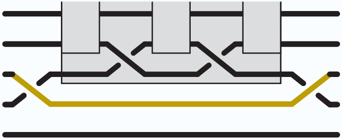

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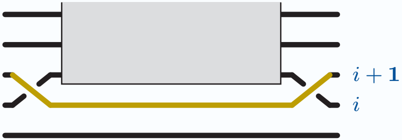
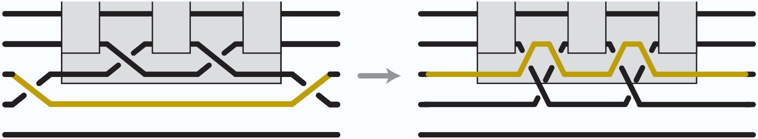


- Reducing a handle:



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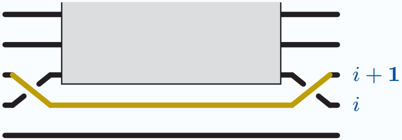

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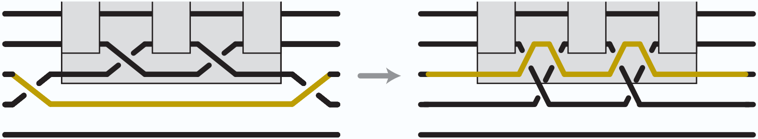

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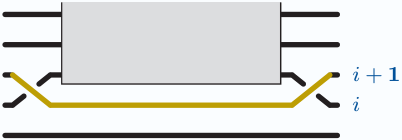

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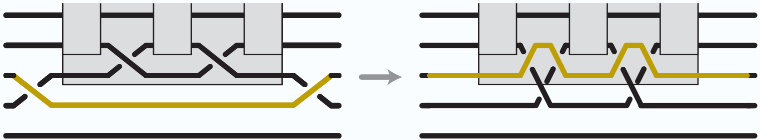

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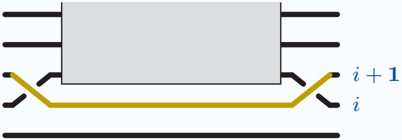

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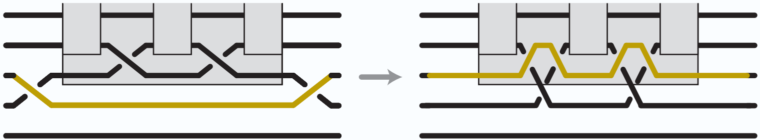

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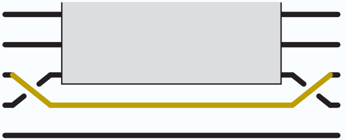

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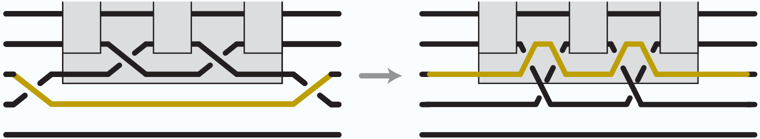

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- **Theorem**.— Every sequence of handle reductions terminates.

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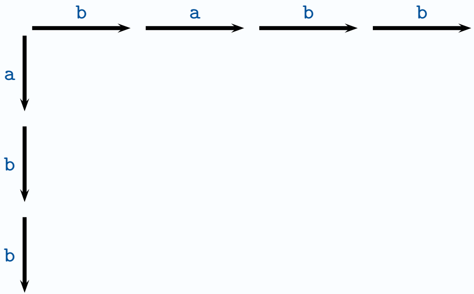

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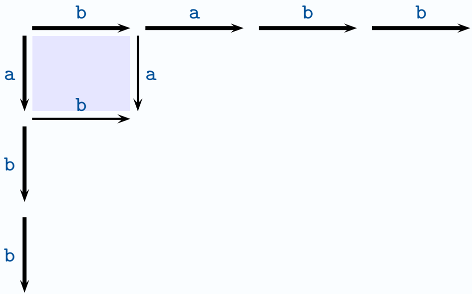

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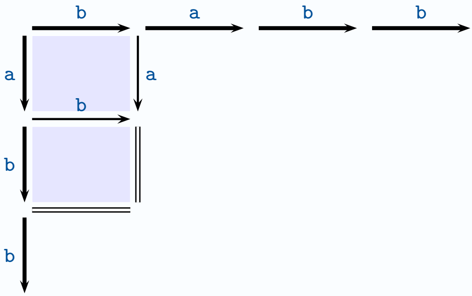

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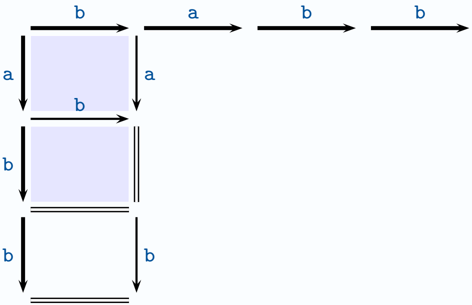

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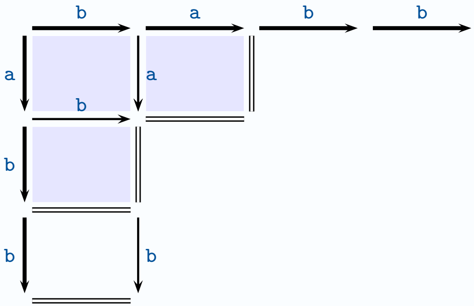

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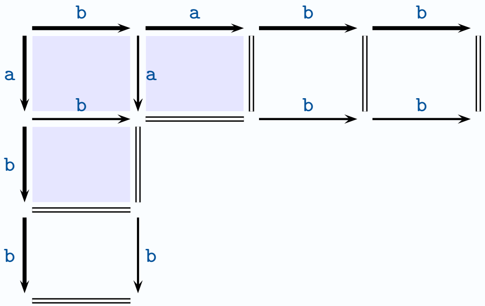

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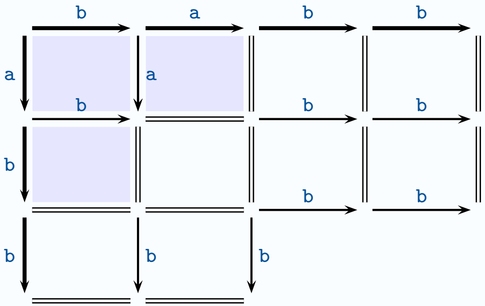

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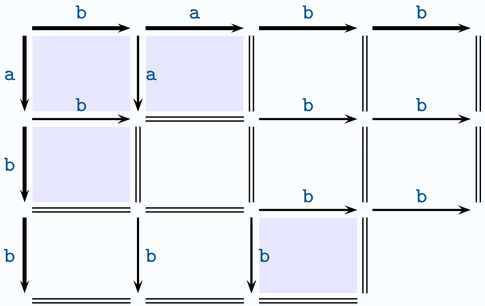

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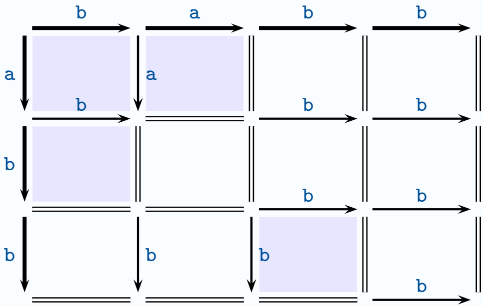

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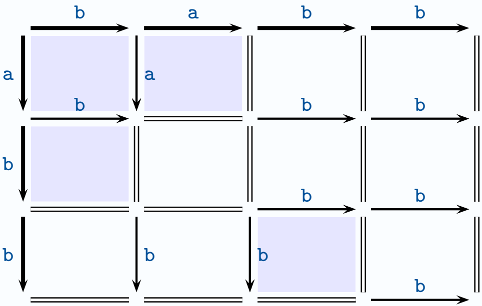

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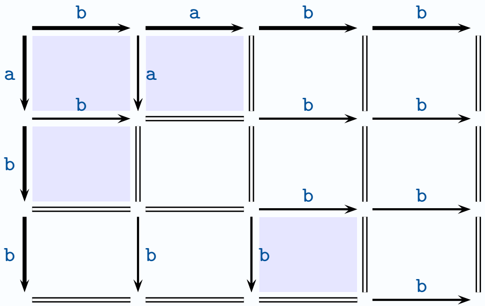

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- Reversing grid:

The real problem III

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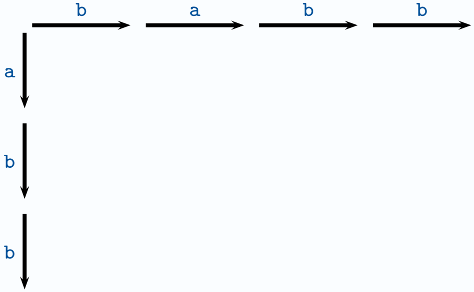

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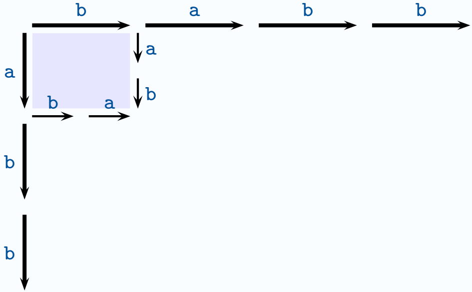

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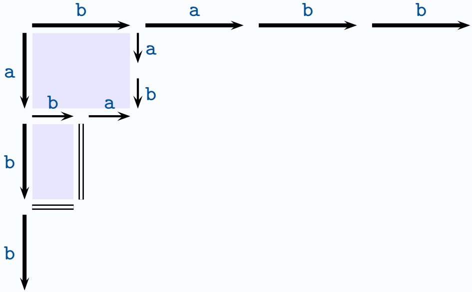

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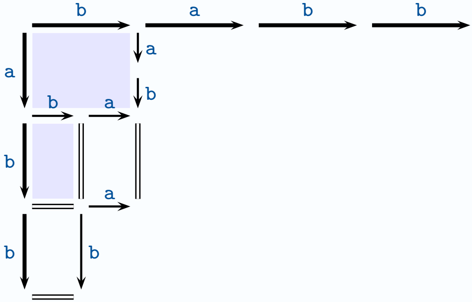

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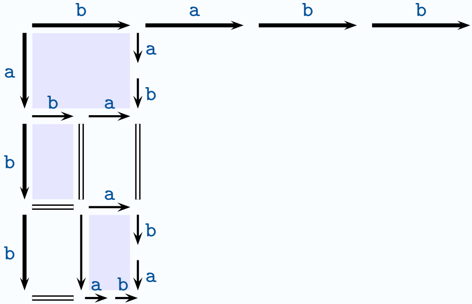

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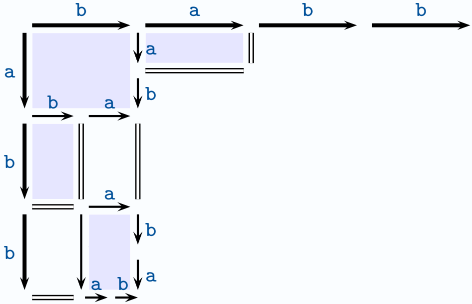

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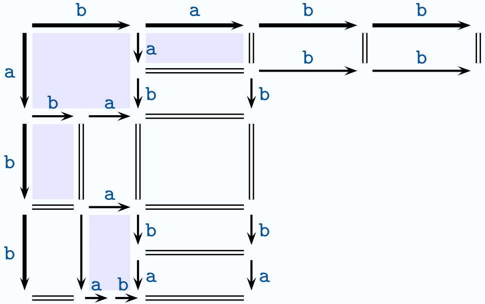

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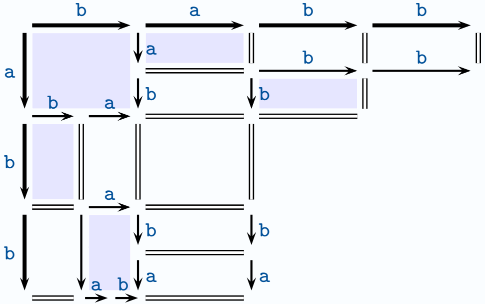

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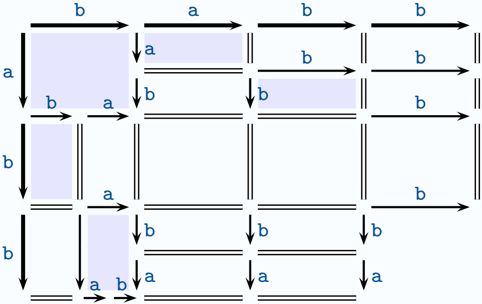

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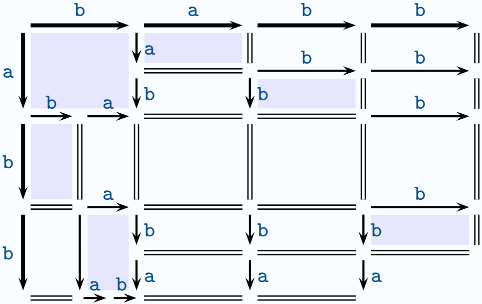

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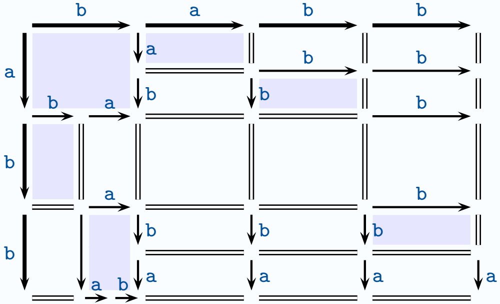

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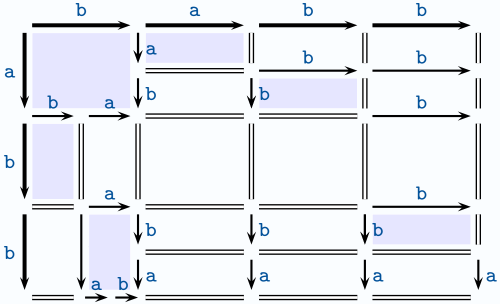

- Theorem.— Reversing terminates in quadratic time (in this specific case).

- Proof:

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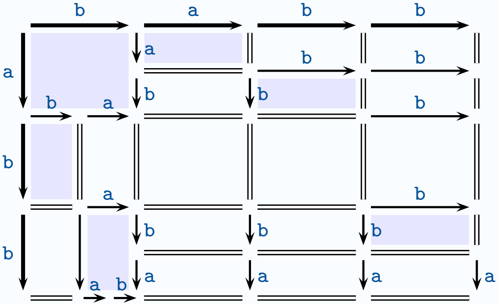

- **Theorem.—** Reversing terminates in quadratic time (in this specific case).

- Proof: Return to the baby case

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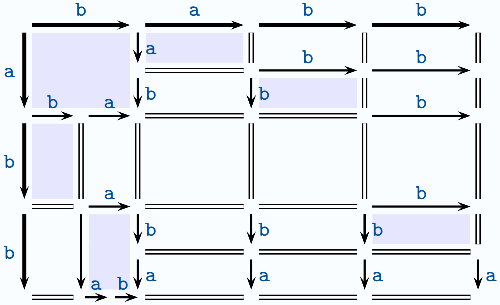
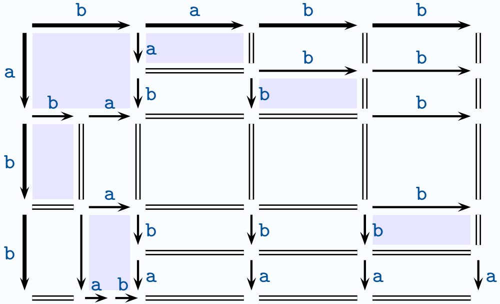


- Theorem.— Reversing terminates in quadratic time (in this specific case).

- Proof: Return to the baby case = find a (finite) set of words S that includes the alphabet and closed under reversing.

- Reversing grid: same, but possibly smaller and smaller arrows.



- Theorem.— Reversing terminates in quadratic time (in this specific case).

- Proof: Return to the baby case = find a (finite) set of words $\boldsymbol{S}$ that includes the alphabet and closed under reversing.

for all $\boldsymbol{u}, \boldsymbol{v}$ in $\boldsymbol{S}$, exist $\boldsymbol{u}', \boldsymbol{v}'$ in $\boldsymbol{S}$ s.t. $\exists$ reversing grid

$$\begin{tikzcd} \cdot \arrow[r, "\boldsymbol{v}"] \arrow[d, "\boldsymbol{u}"'] & \cdot \arrow[d, "\boldsymbol{u}'"] \\ \cdot \arrow[r, "\boldsymbol{v}'"'] & \cdot \end{tikzcd}$$

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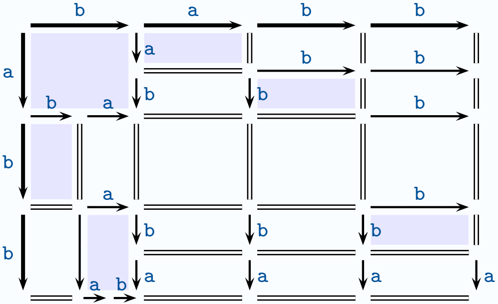

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Here: works with $\boldsymbol{S} = \{\mathtt{a}, \mathtt{b}, \mathtt{ab}, \mathtt{ba}\}$. $\square$

- Always like that?

- Always like that? Not really...

- Example 3:
Alphabet `a`, `b`, `A`, `B`,

- Always like that? Not really...

- Example 3:

Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$, rules $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$,

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$\rightsquigarrow$ Here : terminating in quadratic time and linear space

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$\rightsquigarrow$ Here : non-terminating

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- Example 5:

Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$,

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Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$, rules $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, plus $\mathtt{Ab} \longrightarrow \mathtt{abA}$, $\mathtt{Ba} \longrightarrow \mathtt{aBA}$

Start with $\mathtt{Bab}$: $\underbrace{\underline{\mathtt{Bab}}}_{w} \longrightarrow \mathtt{aB\underline{Ab}} \longrightarrow \underbrace{\mathtt{a\underline{Bab}A}}_{\mathtt{a}w\mathtt{A}} \longrightarrow \mathtt{aaB\underline{Ab}A} \longrightarrow \underbrace{\mathtt{aa\underline{Ba}bAA}}_{\mathtt{a}^2 w \mathtt{A}^2} \longrightarrow \mathtt{aaaB\underline{Ab}AA} \longrightarrow \mathtt{aaa\underline{Ba}bAAA} \longrightarrow \mathtt{aaaaB\underline{Ab}AAA}$

⇝ Here : non-terminating

- Example 5:

Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$, rules $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, plus $\mathtt{Ba} \longrightarrow \varepsilon$, $\mathtt{Ab} \longrightarrow \mathtt{abab}^2\mathtt{ab}^2\mathtt{abab}$

⇝ Here : terminating in cubic time and quadratic space

- Always like that? Not really...

- Example 3:

Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$, rules $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, plus $\mathtt{Ab} \longrightarrow \underbrace{\mathtt{baba}\ldots}_{m\,\text{letters}}\underbrace{\ldots\mathtt{BABA}}_{m\,\text{letters}}$, $\mathtt{Ba} \longrightarrow \underbrace{\mathtt{abab}\ldots}_{m\,\text{letters}}\underbrace{\ldots\mathtt{ABAB}}_{m\,\text{letters}}$.

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Alphabet $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$, rules $\mathtt{Aa} \longrightarrow \varepsilon$, $\mathtt{Bb} \longrightarrow \varepsilon$, plus $\mathtt{Ab} \longrightarrow \mathtt{abA}$, $\mathtt{Ba} \longrightarrow \mathtt{aBA}$

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- Definition.— Assume (A, R) semigroup presentation and, for all $s \neq t$ in A, there is exactly one relation $s... = t...$ in R, say $sC(s,t) = tC(t,s)$. Then reversing is the rewrite system on $A \cup \overline{A}$ (a copy of A, here : capitalized letters) with rules $\overline{s}s \to \varepsilon$ and $\overline{s}t \to C(s,t)\overline{C(t,s)}$ for $s \neq t$ in A.

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- Example 5 = reversing for the ordered group: $\langle \mathtt{a}, \mathtt{b} \mid \mathtt{a} = \mathtt{babab}^2\mathtt{ab}^2\mathtt{abab}\rangle$.

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  - if reversing is complete for $(\boldsymbol{A}, \boldsymbol{R})$, then it is terminating
    iff any two elements of the monoid $\langle \boldsymbol{A} \mid \boldsymbol{R} \rangle^{+}$ admit a common right-multiple.

- Question.— What are YOU say about reversing?

For the Polish Algorithm:

• P. Dehornoy, Braids and selfdistributivity, Progress in math. vol 192, Birkhaüser 2000 (Chapter VIII)

• O. Deiser, Notes on the Polish Algorithm, deiser@tum.de (Technishe Universität München)

For Handle Reduction of braids:

• P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, Braid ordering, Math. Surveys and Monographs vol. 148, Amer. Math. Soc. 2008 (Chapter V)

For reversing associated with a semigroup presentation:

• P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, J. Michel, Foundations of Garside Theory, submitted www.math.unicaen.fr/∼dehornoy/ (Chapter II)