Completion and Reduction Orders

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Basic Notions for Term Rewrite Systems (TRSs)

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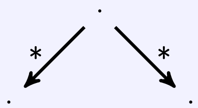
- TRS $\mathcal{R}$ is **confluent** if 

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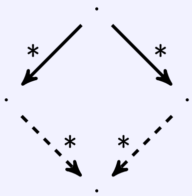

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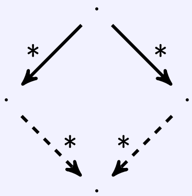
- TRS $\mathcal{R}$ is **confluent** if 

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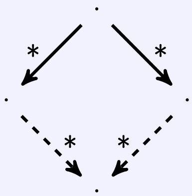

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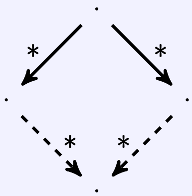

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Fact

$s \approx_{\mathcal{E}} t \iff s{\downarrow_{\mathcal{R}}} = t{\downarrow_{\mathcal{R}}}$ if $\mathcal{R}$ is complete presentation of $\mathcal{E}$

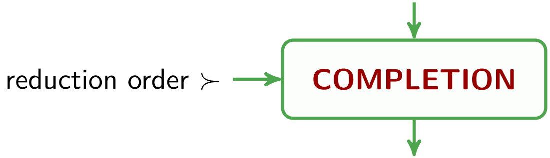
Knuth-Bendix Completion Procedure (1970)

input: equational system $\mathcal{E}$ and reduction order $>$
output: complete presentation $\mathcal{R}$ of $\mathcal{E}$‘

```
R := ∅; C := E;
while C ≠ ∅ do
    choose s ≈ t ∈ C;
    C := C \ {s ≈ t};
    normalize s and t to s' and t' with respect to R;
    if s' ≯ t' and s' ≠ t' and t' ≯ s' then failure; fi;
    S := {s' → t', t' → s'} ∩ >;
    C := C ∪ CP(R, S) ∪ CP(S, R) ∪ CP(S);
    R := R ∪ S
od
```

$$\mathcal{E} = \left\{ \begin{array}{c} \mathsf{0} + x \approx x \\ (-x) + x \approx \mathsf{0} \\ (x + y) + z \approx x + (y + z) \end{array} \right\}$$

$$\mathcal{E} = \left\{ \begin{array}{c} 0 + x \approx x \\ (-x) + x \approx 0 \\ (x + y) + z \approx x + (y + z) \end{array} \right\}$$



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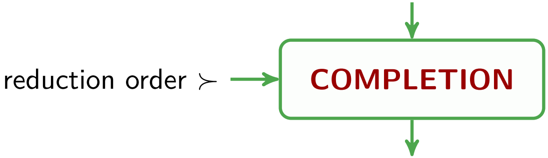

$$\mathcal{R} = \left\{ \begin{array}{rr} 0 + x \to x & -(-x) \to x \\ x + 0 \to x & x + ((-x) + y) \to y \\ (-x) + x \to 0 & (-x) + (x + y) \to y \\ x + (-x) \to 0 & -(x + y) \to (-y) + (-x) \\ -0 \to 0 & (x + y) + z \to x + (y + z) \end{array} \right\}$$

Goal: Complete Commuting Group Endomorphisms Automatically

consider equation system known as CGE_2:

$$\mathsf{e} + x \approx x \qquad \mathsf{f}(x + y) \approx \mathsf{f}(x) + \mathsf{f}(y)$$

$$\mathsf{i}(x) + x \approx \mathsf{e} \qquad \mathsf{g}(x + y) \approx \mathsf{g}(x) + \mathsf{g}(y)$$

$$(x + y) + z \approx x + (y + z) \qquad \mathsf{f}(x) + \mathsf{g}(y) \approx \mathsf{g}(y) + \mathsf{f}(x)$$

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\end{aligned}
$$

CGE_2 admits 20-rule complete TRS (Stump and Löchner, 2006)

$$
\begin{aligned}
\mathsf{e} + x &\to x & \mathsf{f}(\mathsf{e}) &\to \mathsf{e} & \mathsf{i}(x+y) &\to \mathsf{i}(y) + \mathsf{i}(x) \\
x + \mathsf{e} &\to x & \mathsf{g}(\mathsf{e}) &\to \mathsf{e} & \mathsf{f}(x) + \mathsf{f}(y) &\to \mathsf{f}(x+y) \\
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Modern Completion Tools

$$\mathcal{E} = \left\{ \begin{array}{rlrl} \mathsf{e} + x & \approx x & \mathsf{f}(x+y) & \approx \mathsf{f}(x) + \mathsf{f}(y) \\ \mathsf{i}(x) + x & \approx \mathsf{e} & \mathsf{g}(x+y) & \approx \mathsf{g}(x) + \mathsf{g}(y) \\ (x+y)+z & \approx x + (y+z) & \mathsf{f}(x) + \mathsf{g}(y) & \approx \approx \mathsf{g}(y) + \mathsf{f}(x) \end{array} \right\}$$

termination tool/predicate → **COMPLETION**

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Tools and Approaches

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- good: great termination proving power
- bad: orientation of equations consumes considerable time

Contents

Our Approach

simply use powerful reduction orders to complete CGE_2

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Rest of Talk

1. **termination:** semantic labeling as order extension
2. **confluence:** new critical pair criterion
3. **completion:** Sato and Winkler's method and maximal completion

Termination

How To Prove Termination?

complete presentation of CGE_2:

$$
\begin{array}{rlr}
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\end{array}
$$

it is not orientable by KBO, LPO, matrix interpretations, ...

Semantic Labeling (Zantema 1995)

Definition (labeled terms)

let $\mathcal{M}$ be algebra, t term, and α assignment

$$\mathsf{lab}(t,\alpha) = \begin{cases} x & \text{if } t \text{ is variable} \\ f_a(\mathsf{lab}(t_1,\alpha),\ldots,\mathsf{lab}(t_n,\alpha)) & \text{if } t = f(t_1,\ldots,t_n) \text{ and } f^\sharp \in \mathcal{G} \\ f(\mathsf{lab}(t_1,\alpha),\ldots,\mathsf{lab}(t_n,\alpha)) & \text{if } t = f(t_1,\ldots,t_n) \text{ and } f^\sharp \notin \mathcal{G} \end{cases}$$

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- $\mathsf{lab}_{\mathcal{M}}(\mathsf{f}(\mathsf{g}(\mathsf{f}(x))),\alpha) = \mathsf{f}_0(\mathsf{g}(\mathsf{f}_2(x)))$

because $[\alpha]_{\mathcal{M}}(\mathsf{f}^\sharp(\mathsf{g}(\mathsf{f}(x)))) = 0$ and $[\alpha]_{\mathcal{M}}(\mathsf{f}^\sharp(x)) = 2$

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Theorem (Zantema 1995)

if $\mathcal{R} \subseteq \geqslant_{\mathcal{M}}$ then: $\mathcal{R}$ is terminating $\iff$ $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}(>)$ is terminating

Termination Proof by Semantic Labeling

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2. termination of $\mathcal{R}_{\mathsf{lab}} \cup \mathcal{D}\mathsf{ec}(>)$

$$\begin{aligned} \mathsf{f}_1(\mathsf{f}_a(x)) &\to \mathsf{f}_0(\mathsf{g}(\mathsf{f}_a(x))) && (a \in \mathbb{N}) \\ \mathsf{f}_a(x) &\to \mathsf{f}_b(x) && (a, b \in \mathbb{N} \text{ with } a > b) \end{aligned}$$

   is shown by LPO with precedence: $\cdots \succ \mathsf{f}_2 \succ \mathsf{f}_1 \succ \mathsf{f}_0 \succ \mathsf{g}$

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3. hence, $\mathcal{R}$ is terminating

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Remark

$\succ^{\mathcal{M}}_{\mathsf{mpo}}$ is very similar to monotonic semantic path order

Example of Order Extension $\succ_{\mathsf{lpo}}^{\mathcal{M}}$

consider TRS

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$$

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4. hence, TRS is terminating

Termination of TRS for CGE_2

$$
\begin{array}{rlr}
\mathsf{e} + x \to x & \mathsf{f}(\mathsf{e}) \to \mathsf{e} & \mathsf{i}(x + y) \to \mathsf{i}(y) + \mathsf{i}(x) \\
x + \mathsf{e} \to x & \mathsf{g}(\mathsf{e}) \to \mathsf{e} & \mathsf{f}(x) + \mathsf{f}(y) \to \mathsf{f}(x + y) \\
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x + (\mathsf{i}(x) + y) \to y & \mathsf{i}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{i}(x)) & \mathsf{f}(x) + (\mathsf{f}(y) + z) \to \mathsf{f}(x + y) + z \\
\mathsf{i}(x) + (x + y) \to y & \mathsf{i}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{i}(x)) & \mathsf{g}(x) + (\mathsf{g}(y) + z) \to \mathsf{g}(x + y) + z \\
\multicolumn{2}{c}{(x + y) + z \to x + (y + z)} & \mathsf{f}(y) + (\mathsf{g}(x) + z) \to \mathsf{g}(x) + (\mathsf{f}(y) + z)
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\end{array}
$$

termination is shown by KBO extended by algebra $\mathcal{M}$ on $\mathbb{N}$ with

$$\mathsf{e}_\mathcal{M} = 0 \quad \mathsf{f}_\mathcal{M}(x) = 0 \quad \mathsf{g}_\mathcal{M}(x) = 1 \quad \mathsf{i}_\mathcal{M}(x) = x \quad x +_\mathcal{M} y = x + y \quad x +^\sharp_\mathcal{M} y = x$$

$$w_0 = w(\mathsf{g}) = w(\mathsf{f}) = w(\mathsf{e}) = 1 \quad w(\mathsf{i}) = w(+_a) = 0$$

$$\mathsf{i} \succ \mathsf{f} \succ \cdots \succ +_2 \succ +_1 \succ +_0 \succ \mathsf{e} \succ \mathsf{g}$$

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	LPO	KBO	ELPO	EKBO	ELPO+EKBO
# termination proofs	144	102	247	136	272

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Open Question

what about dependency graphs?

Confluence

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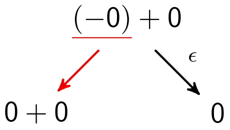

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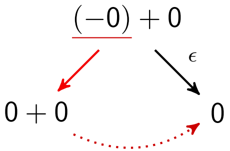

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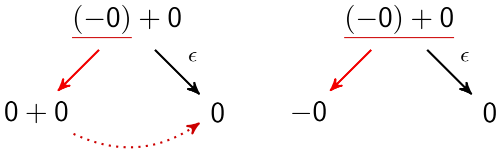

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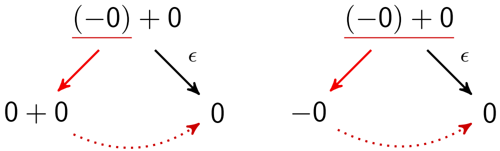

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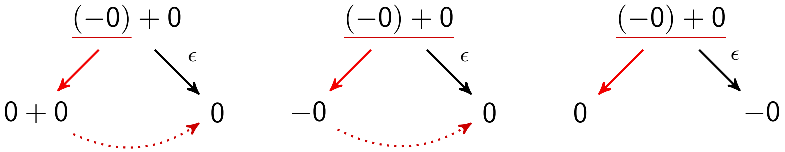

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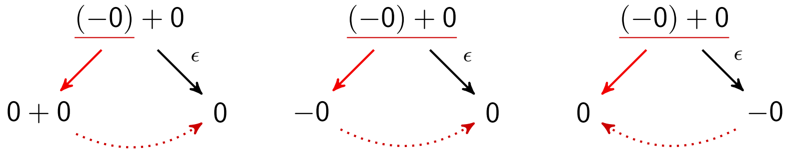

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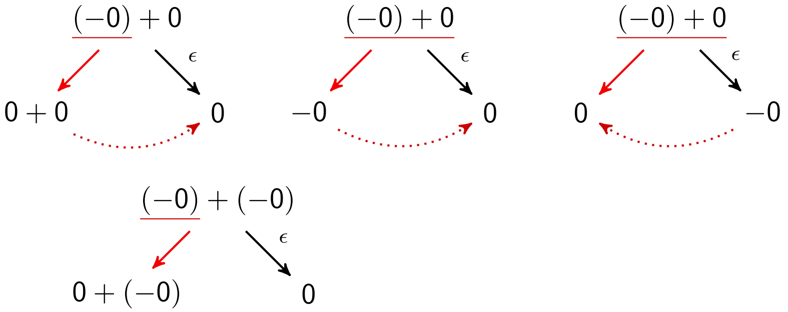

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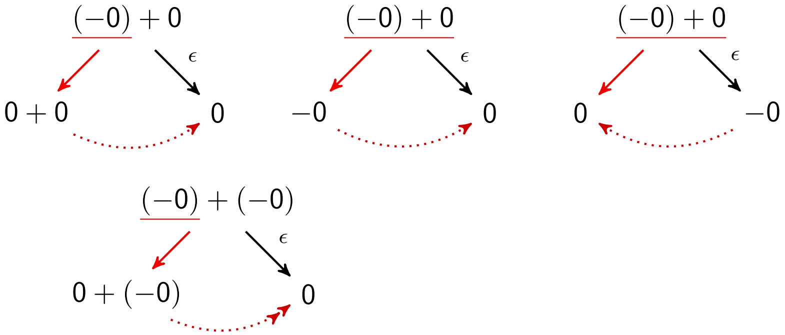

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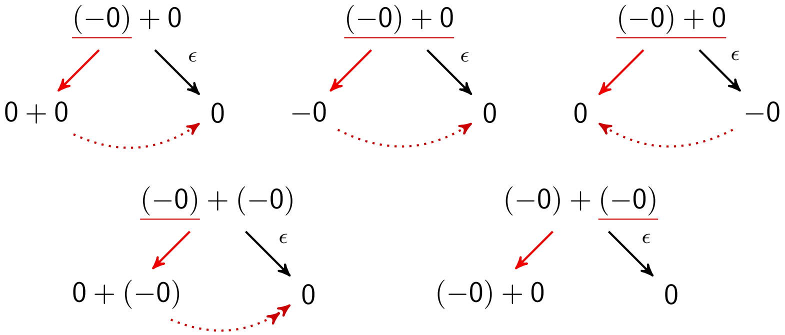

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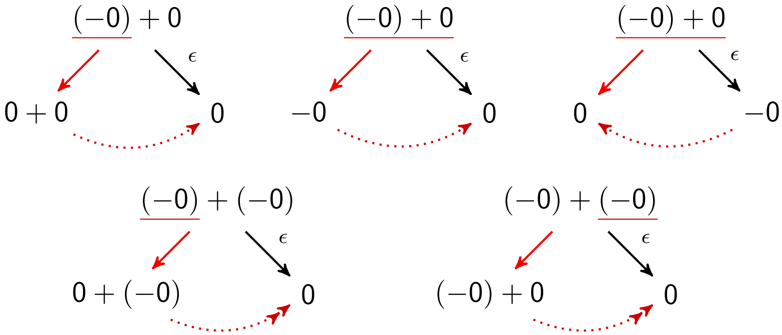

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ARS $\mathcal{B}$ is **rewrite strategy** for ARS $\mathcal{A}$ if $\to_{\mathcal{B}} \subseteq \to_{\mathcal{A}}^{+}$ and $\mathsf{NF}(\mathcal{A}) = \mathsf{NF}(\mathcal{B})$

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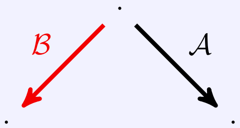

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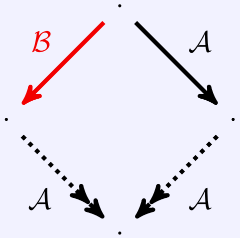

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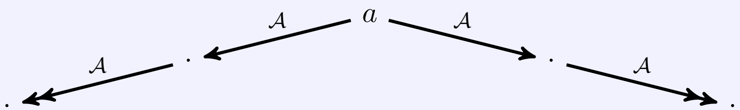

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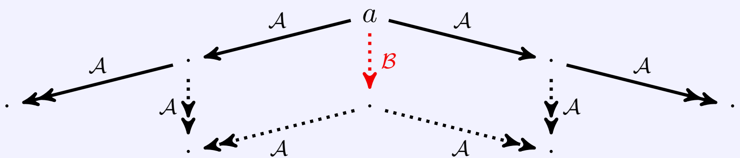

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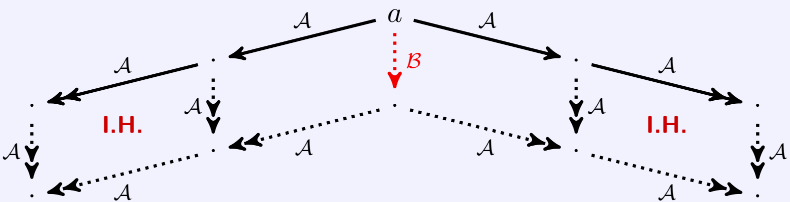

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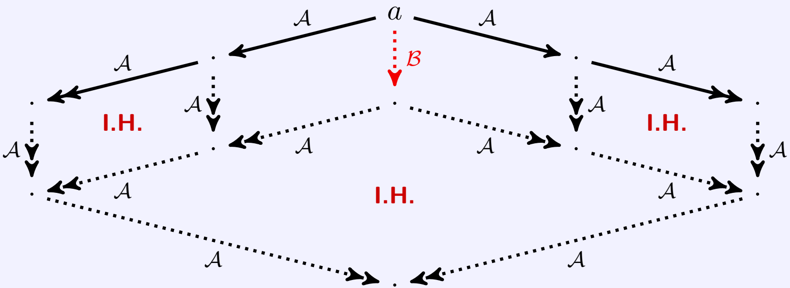

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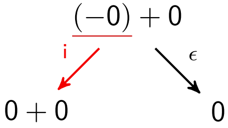

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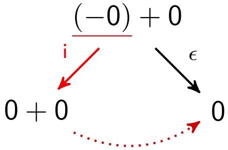

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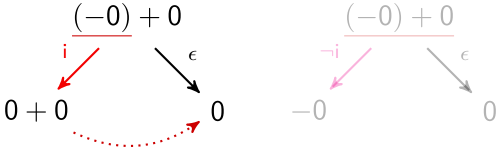

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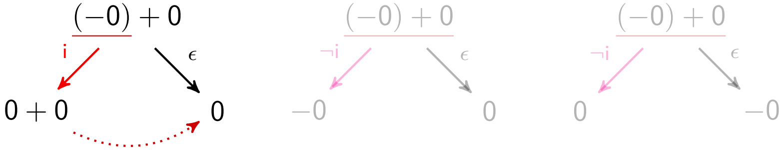

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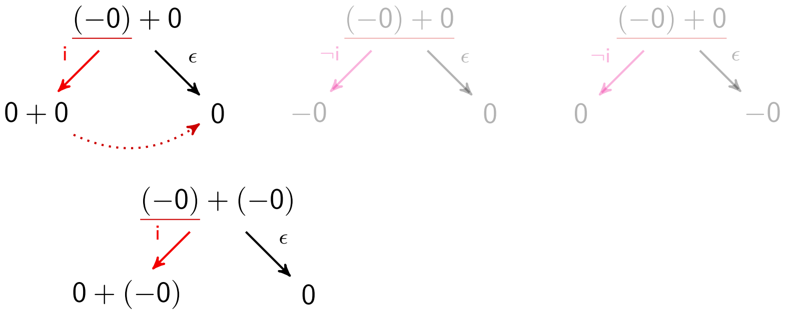

Example for Prime Critical Pair Criterion

consider terminating TRS $\mathcal{R}$:

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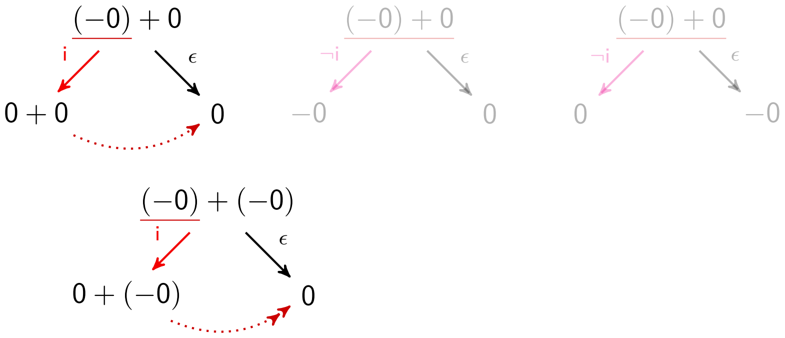

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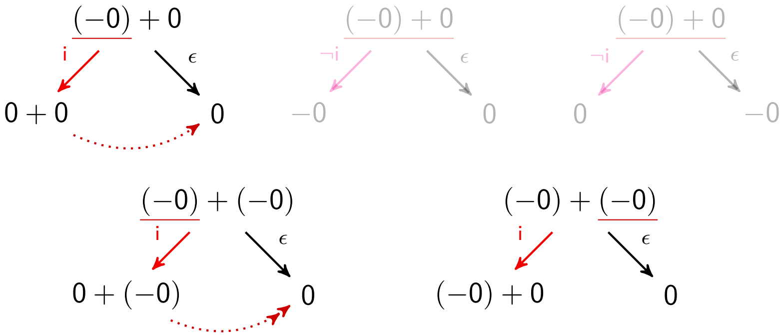

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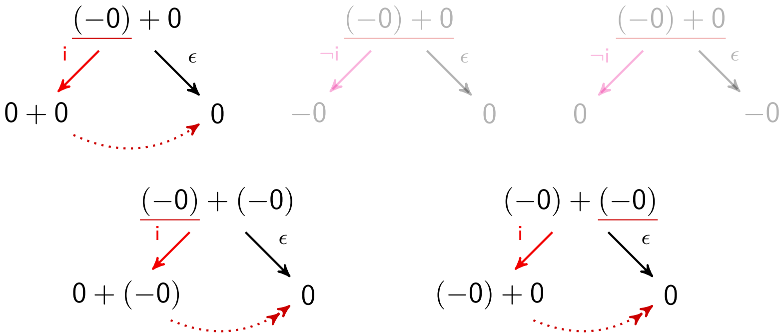

Leftmost Innermost Critical Pairs (New)

Theorem

terminating TRS $\mathcal{R}$ is confluent if and only if ${}_{\mathcal{R}}\xleftarrow{\mathsf{li}} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$

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Proof.

$\xrightarrow{\mathsf{li}}_{\mathcal{R}}$ is rewrite strategy and ${}_{\mathcal{R}}\xleftarrow{\mathsf{li}} \cdot \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ □

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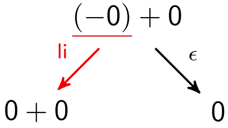

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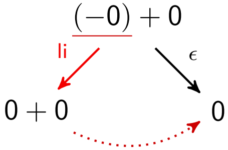

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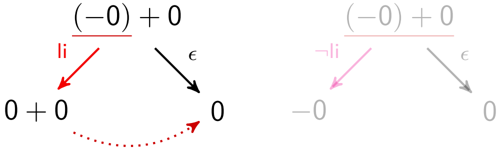

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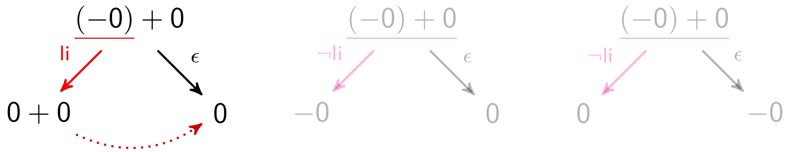

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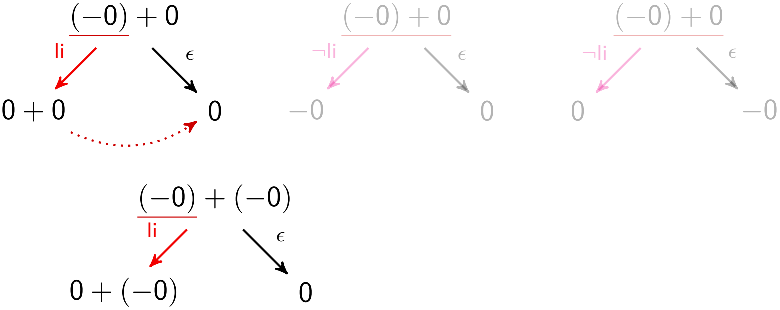

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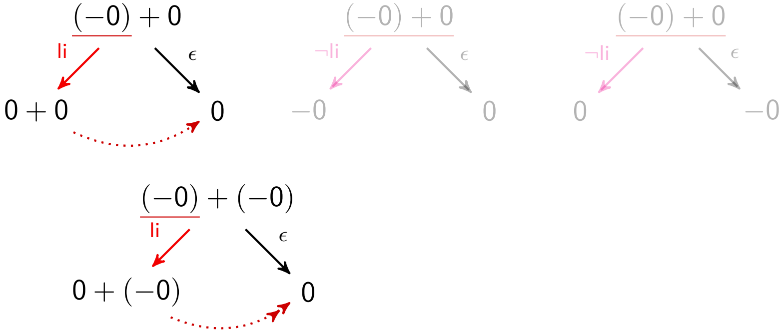

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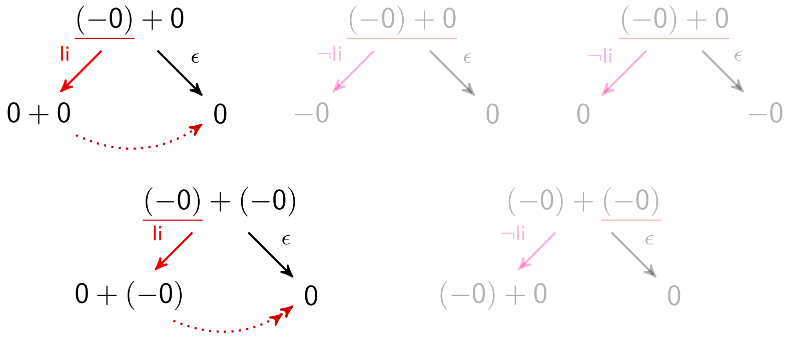

Confluence of 20-rule TRS for CGE_2

$$
\begin{array}{rlr}
\mathsf{e} + x \to x & \mathsf{f}(\mathsf{e}) \to \mathsf{e} & \mathsf{i}(x+y) \to \mathsf{i}(y) + \mathsf{i}(x) \\
x + \mathsf{e} \to x & \mathsf{g}(\mathsf{e}) \to \mathsf{e} & \mathsf{f}(x) + \mathsf{f}(y) \to \mathsf{f}(x+y) \\
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TRS is confluent, as all leftmost innermost critical pairs are joinable

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- TRS admits 115 critical peaks

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Note

- TRS admits 115 critical peaks
- 18 critical peaks are discarded by prime / leftmost innermost critical pairs

Outermost Strategy Cannot be Used

consider terminating TRS

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}$$

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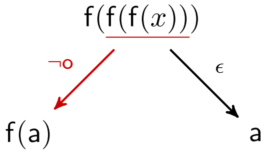

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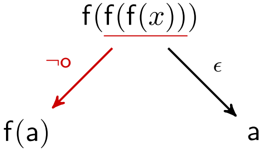

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- $\xleftarrow{\mathsf{o}} \rtimes \xrightarrow{\epsilon}$ is empty

Summary for Confluence

Results

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OK innermost critical pairs = prime critical pairs

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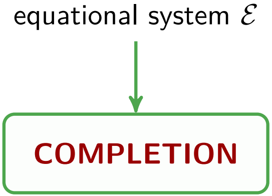
Future Work

- any other useful strategy?
- to make variants for ordered rewriting, AC rewriting, ...

Completion

Knuth-Bendix Completion (1970)



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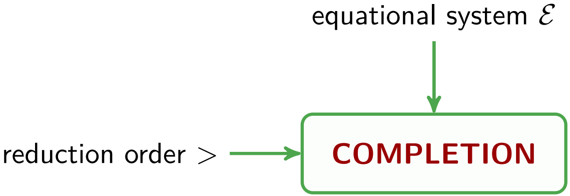

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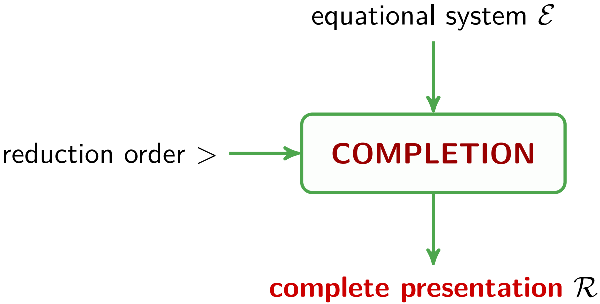

Knuth-Bendix Completion Procedure (1970)

---

input: equational system $\mathcal{E}$ and reduction order $>$
output: complete presentation $\mathcal{R}$ of $\mathcal{E}$'

$\mathcal{R} := \varnothing$; $C := \mathcal{E}$;
`while` $C \neq \varnothing$ `do`
  choose $s \approx t \in C$;
  $C := C \setminus \{s \approx t\}$;
  normalize s and t to s' and t' with respect to $\mathcal{R}$;
  `if` $s' \not> t'$ and $s' \neq t'$ and $t' \not> s'$ `then` *failure*; `fi`;
  $\mathcal{S} := \{s' \to t', t' \to s'\} \cap >$;
  $C := C \cup \mathsf{CP}(\mathcal{R}, \mathcal{S}) \cup \mathsf{CP}(\mathcal{S}, \mathcal{R}) \cup \mathsf{CP}(\mathcal{S})$;
  $\mathcal{R} := \mathcal{R} \cup \mathcal{S}$
`od`

Definition (abstract completion, Bachmair, Dershowitz and Hsiang, 1986)

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delete $$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

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Theorem

$\mathcal{R}_n$ is complete presentation of $\mathcal{E}_0$ if

$$(\mathcal{E}_0, \mathcal{R}_0) \vdash \cdots \vdash (\mathcal{E}_n, \mathcal{R}_n) \text{ with } \mathcal{R}_0 = \mathcal{E}_n = \varnothing \quad \text{and} \quad \mathrm{CP}(\mathcal{R}) \subseteq \mathcal{E}_0 \cup \cdots \cup \mathcal{E}_n$$

use LPO with $+ \succ \mathsf{s} \succ \mathsf{p}$ to complete:

$$\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

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$$\mathsf{p}(\mathsf{s}(x)) \approx x$$

use LPO with $+ \succ \mathsf{s} \succ \mathsf{p}$ to complete:

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

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$$\mathsf{p}(\mathsf{s}(x)) \to x$$

$$\mathsf{p}(\mathsf{s}(\mathsf{p}(x)))$$

$$\swarrow \quad \searrow$$

$$x$$

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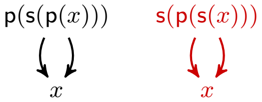

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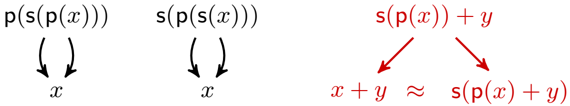
$x + y \approx \mathsf{s}(\mathsf{p}(x) + y)$



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$$\mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \rightrightarrows x$$

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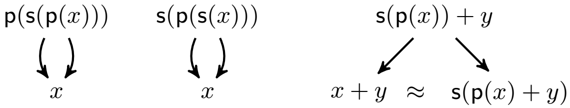

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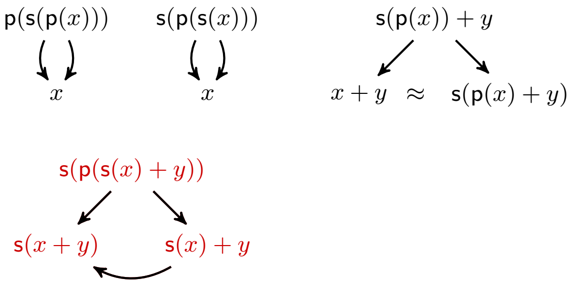

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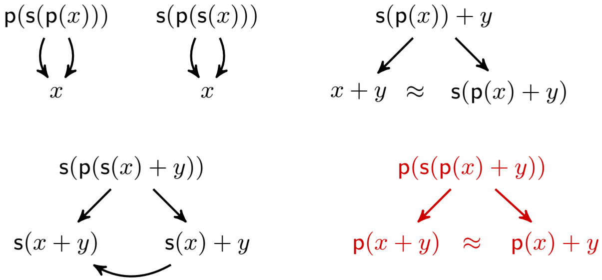

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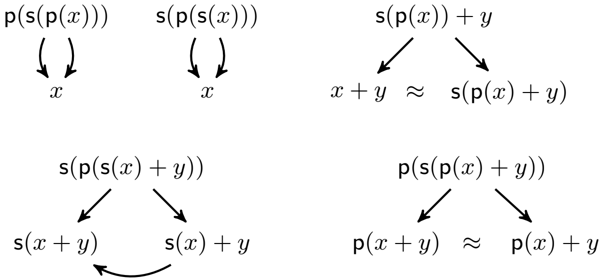

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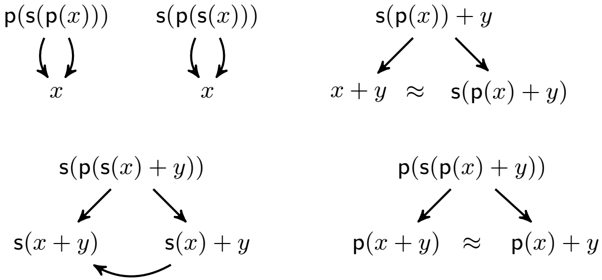

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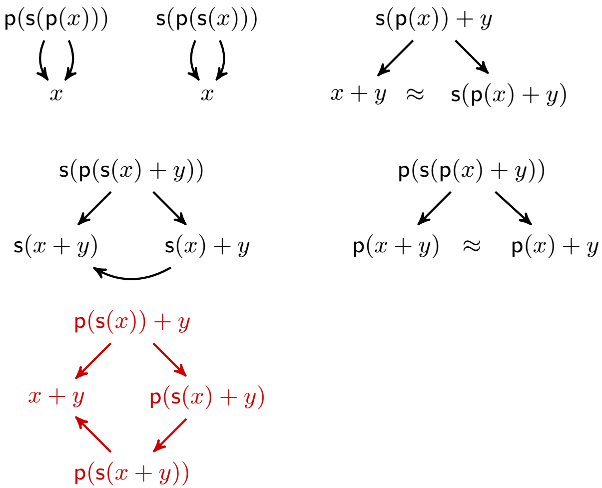

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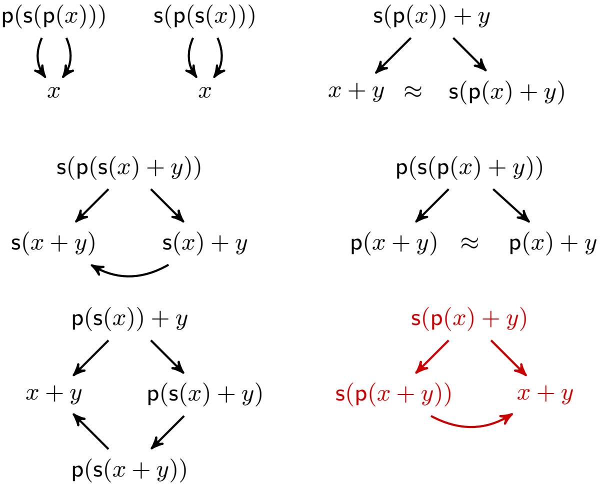

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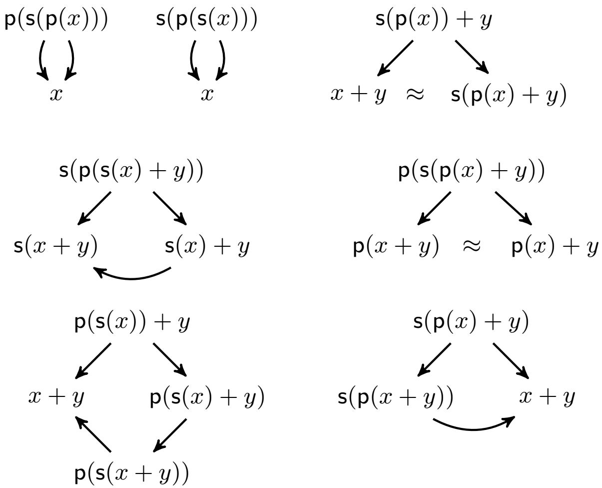
complete TRS



Completion with Inter-Reduction

TRS

$$\begin{aligned}
\mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\
\mathsf{s}(\mathsf{p}(x)) &\to x \\
\mathsf{p}(\mathsf{s}(x)) &\to x \\
\mathsf{s}(\mathsf{p}(x) + y) &\to x + y \\
\mathsf{p}(x) + y &\to \mathsf{p}(x + y)
\end{aligned}$$

is complete but **not reduced**

Completion with Inter-Reduction

TRS

$$\begin{aligned} \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\ \mathsf{s}(\mathsf{p}(x)) &\to x \\ \mathsf{p}(\mathsf{s}(x)) &\to x \\ \mathsf{s}(\mathsf{p}(x) + y) &\to x + y \\ \mathsf{p}(x) + y &\to \mathsf{p}(x + y) \end{aligned}$$

is complete but **not reduced**

Canonical TRSs

Definition

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- TRS $\mathcal{R}$ is reduced if for every rule $\ell \to r \in \mathcal{R}$

  $r \in \mathsf{NF}(\mathcal{R})$ and $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ (modulo renaming)

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- complete TRS is canonical if it is reduced

Theorem (Ballantyne 1980?; Métivier, 1983)

canonical presentations $\mathcal{R}$ and $\mathcal{S}$ of $\mathcal{E}$ are identical if

$$\mathcal{R} \subseteq \succ \text{ and } \mathcal{S} \subseteq \succ \qquad \text{for some reduction order } \succ$$

Definition (abstract completion, Bachmair, Dershowitz and Hsiang, 1986)

delete $$\frac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$

deduce $$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \underset{\mathcal{R}}{\longleftarrow} \cdot \underset{\mathcal{R}}{\longrightarrow} t$$

orient $$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}} \quad \text{if } s > t$$

$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \to s\}} \quad \text{if } t > s$$

simplify $$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \underset{\mathcal{R}}{\longrightarrow} u$$

$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}} \quad \text{if } t \underset{\mathcal{R}}{\longrightarrow} u$$

Theorem

$\mathcal{R}_n$ *is complete presentation of* $\mathcal{E}_0$ *if*

$$(\mathcal{E}_0, \mathcal{R}_0) \vdash \cdots \vdash (\mathcal{E}_n, \mathcal{R}_n) \text{ with } \mathcal{R}_0 = \mathcal{E}_n = \varnothing \quad \text{and} \quad \mathsf{CP}(\mathcal{R}) \subseteq \mathcal{E}_0 \cup \cdots \cup \mathcal{E}_n$$

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delete
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orient
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$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \to s\}} \quad \text{if } t > s$$

simplify
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \xrightarrow[\mathcal{R}]{} u$$
$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}} \quad \text{if } t \xrightarrow[\mathcal{R}]{} u$$

collapse
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \quad \text{if } t \xrightarrow[\mathcal{R}]{} u$$

Theorem

$\mathcal{R}_n$ is complete presentation of $\mathcal{E}_0$ if

$$(\mathcal{E}_0, \mathcal{R}_0) \vdash \cdots \vdash (\mathcal{E}_n, \mathcal{R}_n) \textit{ with } \mathcal{R}_0 = \mathcal{E}_n = \varnothing \quad \textit{and} \quad \mathrm{CP}(\mathcal{R}) \subseteq \mathcal{E}_0 \cup \cdots \cup \mathcal{E}_n$$

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$$\frac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \quad \text{if } t \xrightarrow[\mathcal{R}]{} u$$

compose
$$\frac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}} \quad \text{if } t \xrightarrow[\mathcal{R}]{} u$$

Theorem

$\mathcal{R}_n$ is complete presentation of $\mathcal{E}_0$ if

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Completion with Inter-Reduction

$$\begin{aligned}
\mathsf{s}(x) + y &\to \mathsf{s}(x + y) \\
\mathsf{s}(\mathsf{p}(x)) &\to x \\
\mathsf{p}(\mathsf{s}(x)) &\to x \\
\mathsf{s}(\mathsf{p}(x) + y) &\to x + y \\
\mathsf{p}(x) + y &\to \mathsf{p}(x + y)
\end{aligned}$$

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\end{aligned}$$

Completion with Inter-Reduction

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

$$\mathsf{s}(\mathsf{p}(x)) \to x$$

$$\mathsf{p}(\mathsf{s}(x)) \to x$$

$$\mathsf{s}(\mathsf{p}(x + y)) \approx x + y$$

$$\mathsf{p}(x) + y \to \mathsf{p}(x + y)$$

Completion with Inter-Reduction

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canonical TRS

When To Find Suitable Reduction Order?

$$\mathcal{E} = \left\{ \begin{array}{c} \mathsf{0} + x \approx x \\ (-x) + x \approx \mathsf{0} \\ (x + y) + z \approx x + (y + z) \end{array} \right\}$$

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LPO with $- \succ + \succ 0$ → **COMPLETION**

When To Find Suitable Reduction Order?

$$\mathcal{E} = \left\{ \begin{array}{c} 0 + x \approx x \\ (-x) + x \approx 0 \\ (x+y)+z \approx x + (y+z) \end{array} \right\}$$

↓

LPO with $- \succ + \succ 0$ → **COMPLETION**

↓

$$\mathcal{R} = \left\{ \begin{array}{rlrl} 0 + x & \to x & -(-x) & \to x \\ x + 0 & \to x & x + ((-x)+y) & \to y \\ (-x) + x & \to 0 & (-x) + (x+y) & \to y \\ x + (-x) & \to 0 & -(x+y) & \to (-y) + (-x) \\ -0 & \to 0 & (x+y)+z & \to x + (y+z) \end{array} \right\}$$

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- difficult to find suitable reduction order before performing completion

When To Find Suitable Reduction Order?

$$\mathcal{E} = \left\{ \begin{array}{r@{\ }c@{\ }l} 0 + x &\approx& x \\ (-x) + x &\approx& 0 \\ (x + y) + z &\approx& x + (y + z) \end{array} \right\}$$

↓

LPO with $- \succ + \succ 0$ → **COMPLETION**

↓

$$\mathcal{R} = \left\{ \begin{array}{r@{\ }c@{\ }l r@{\ }c@{\ }l} 0 + x &\to& x & -(-x) &\to& x \\ x + 0 &\to& x & x + ((-x) + y) &\to& y \\ (-x) + x &\to& 0 & (-x) + (x + y) &\to& y \\ x + (-x) &\to& 0 & -(x + y) &\to& (-y) + (-x) \\ -0 &\to& 0 & (x + y) + z &\to& x + (y + z) \end{array} \right\}$$

- difficult to find suitable reduction order before performing completion
- why not find suitable reduction order **during** completion?

Quiz: Orient Some Equations to Construct Complete Presentation

$$
\begin{aligned}
\mathsf{s}(x) + y &\approx \mathsf{s}(x + y) \\
\mathsf{s}(\mathsf{p}(x)) &\approx x \\
\mathsf{p}(\mathsf{s}(x)) &\approx x \\
\mathsf{s}(\mathsf{p}(x) + y) &\approx \mathsf{s}(x + y) \\
x + y &\approx \mathsf{s}(\mathsf{p}(x) + y) \\
\mathsf{p}(x + y) &\approx \mathsf{p}(x) + y \\
\mathsf{p}((\mathsf{s}(x) + y) + z &\approx (x + y) + z
\end{aligned}
$$

NB. these are valid equations of $\{\mathsf{s}(x) + y \approx \mathsf{s}(x + y),\ \mathsf{s}(\mathsf{p}(x)) \approx x,\ \mathsf{p}(\mathsf{s}(x)) \approx x\}$

Quiz: Orient Some Equations to Construct Complete Presentation

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$$\mathsf{s}(\mathsf{p}(x)) \to x$$
$$\mathsf{p}(\mathsf{s}(x)) \to x$$
$$\mathsf{s}(\mathsf{p}(x) + y) \approx \mathsf{s}(x + y)$$
$$x + y \approx \mathsf{s}(\mathsf{p}(x) + y)$$
$$\mathsf{p}(x + y) \leftarrow \mathsf{p}(x) + y$$
$$\mathsf{p}((\mathsf{s}(x) + y) + z \approx (x + y) + z$$

NB. these are valid equations of $\{\mathsf{s}(x) + y \approx \mathsf{s}(x + y),\ \mathsf{s}(\mathsf{p}(x)) \approx x,\ \mathsf{p}(\mathsf{s}(x)) \approx x\}$

Sato and Winkler's Method

Problem

input: equational system $\mathcal{E}$ and decidable class $\mathcal{RO}$ of reduction orders

output: complete presentation $\mathcal{R}$ of $\mathcal{E}$ such that

$$\mathcal{R} \subseteq (\mathcal{E} \cup \mathcal{E}^{-1}) \cap \succ \quad \text{for some } \succ \text{ in } \mathcal{RO}$$

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choose pair $(\mathcal{R}, \succ)$ that **minimizes** $|\mathcal{R}|$ subject to

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Rationale

Sato and Winkler's Method

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Rationale

- attempts to find **canonical** (i.e. reduced complete) TRS

Sato and Winkler's Method

Problem

input: equational system $\mathcal{E}$ and decidable class $\mathcal{RO}$ of reduction orders

output: complete presentation $\mathcal{R}$ of $\mathcal{E}$ such that

$$\mathcal{R} \subseteq (\mathcal{E} \cup \mathcal{E}^{-1}) \cap \succ \quad \text{for some } \succ \text{ in } \mathcal{RO}$$

Heuristics (Sato and Winkler, 2015)

choose pair $(\mathcal{R}, \succ)$ that **minimizes** $|\mathcal{R}|$ subject to

- $\mathcal{R} \subseteq (\mathcal{E} \cup \mathcal{E}^{-1}) \cap \succ$, and
- all nontrivial equations $s \approx t$ in $\mathcal{E}$ are reducible, i.e. $s \notin \mathsf{NF}(\mathcal{R})$ or $t \notin \mathsf{NF}(\mathcal{R})$

Rationale

- attempts to find **canonical** (i.e. reduced complete) TRS
- redundant equations in $\mathcal{E}$ increase accuracy of the method

```
plus(s(x),y) == s(plus(x,y))
s(p(x)) == x
p(s(x)) == x
p(plus(s(x0),x1)) == plus(x0,x1)
plus(x0,x1) == s(plus(p(x0),x1))
p(plus(x0,x1)) == plus(p(x0),x1)
p(plus(plus(s(x0),x1),x2)) == plus(plus(x0,x1),x2)
```

```
plus(s(x),y) <- s(plus(x,y))
s(p(x)) -> x
p(s(x)) -> x
p(plus(s(x0),x1)) == plus(x0,x1)
plus(x0,x1) == s(plus(p(x0),x1))
p(plus(x0,x1)) -> plus(p(x0),x1)
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```

```
¦ ELPO with interpretations on N
¦
¦   plus_A(x1,x2) = x1 + x2 + 1
¦   s_A(x1) = x1 + 1
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¦   plus#_A(x1,x2) = x1
¦   s#_A(x1) = x1
¦   p#_A(x1) = x1
¦
¦ and precedence:
¦
¦ p > s > plus
```

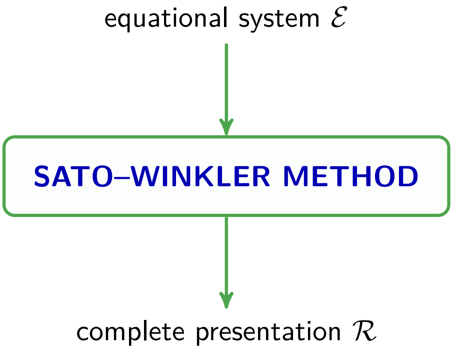
New Procedure: Maximal Completion with Inter-Reduction

equational system $\mathcal{E}$

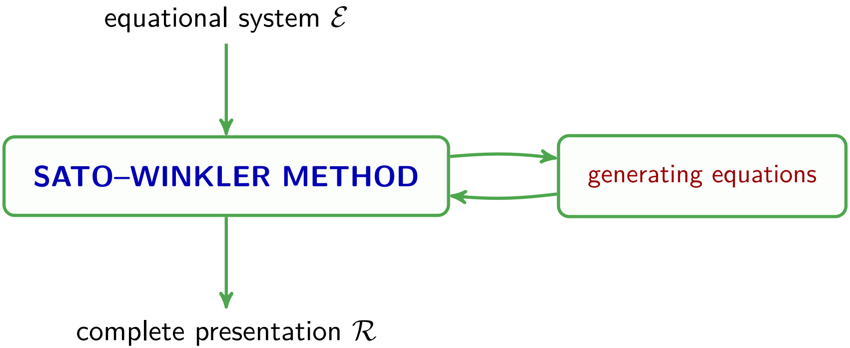
↓

SATO–WINKLER METHOD

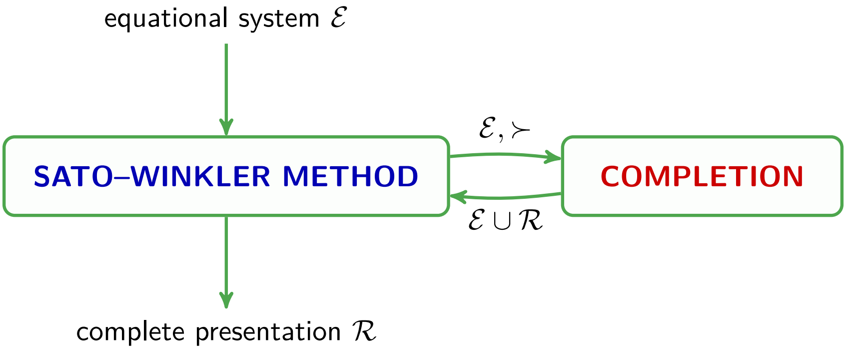
New Procedure: Maximal Completion with Inter-Reduction



New Procedure: Maximal Completion with Inter-Reduction



New Procedure: Maximal Completion with Inter-Reduction



Maximal Completion with Inter-Reduction

- let $\mathcal{O}(\mathcal{E})$ be result of Sato and Winkler's method
- let $\psi(\mathcal{E}, \succ)$ be result of deduce-free completion on $(\mathcal{E}, \varnothing)$ with respect to $\succ$

Idea

find canonical presentation by SW method $\mathcal{O}$, generating equations by completion ψ

Maximal Completion with Inter-Reduction

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Definition

$$\varphi(\mathcal{E}) = \begin{cases} \mathcal{R} & \text{if } \mathcal{R} \text{ is complete for } \mathcal{E} \\ \varphi(\mathcal{E} \cup \mathcal{S}(\mathcal{E})) & \text{otherwise} \end{cases}$$

where, $(\mathcal{P}, \succ) = \mathcal{O}(\mathcal{E})$, $(\mathcal{E}', \mathcal{R}) = \psi(\mathcal{E}, \succ)$, and $\mathcal{S}(\mathcal{E})$ is subset of $\mathcal{E}' \cup \mathcal{R} \cup \mathrm{CP}(\mathcal{R})\downarrow_{\mathcal{R}}$

Maximal Completion with Inter-Reduction

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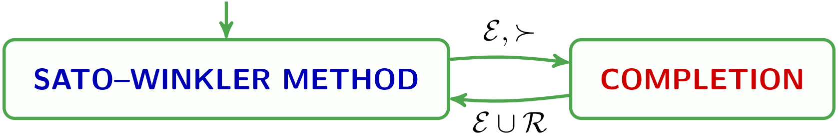
Theorem

$\varphi(\mathcal{E})$ is complete TRS for $\mathcal{E}$ if $\varphi(\mathcal{E})$ is defined

Example 1: Peano Arithmetic

$$\mathcal{E} = \left\{ \begin{array}{c} \mathsf{s}(x) + y \approx \mathsf{s}(x+y) \\ \mathsf{s}(\mathsf{p}(x)) \approx x \\ \mathsf{p}(\mathsf{s}(x)) \approx x \end{array} \right\}$$



```
plus(s(x),y) == s(plus(x,y))
s(p(x)) == x
p(s(x)) == x
```

```
plus(s(x),y) <- s(plus(x,y))
s(p(x)) -> x
p(s(x)) -> x
```

```
¦ ELPO with interpretations on N
¦
¦   plus_A(x1,x2) = 1
¦   s_A(x1) = x1 + 1
¦   p_A(x1) = x1
¦   s#_A(x1) = 0
¦   p#_A(x1) = x1
¦
¦ and precedence:
¦
¦ p > s > plus
```

```
plus(s(x),y) == s(plus(x,y))
s(p(x)) == x
p(s(x)) == x
p(plus(s(x0),x1)) == plus(x0,x1)
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plus(s(x),y) == s(plus(x,y))
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plus(s(x),y) -> s(plus(x,y))
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```
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plus(x0,x1) == s(plus(p(x0),x1))
```

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plus(s(x),y) == s(plus(x,y))
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p(s(x)) == x
p(plus(s(x0),x1)) == plus(x0,x1)
plus(x0,x1) == s(plus(p(x0),x1))
```

```
plus(s(x),y) <- s(plus(x,y))
s(p(x)) -> x
p(s(x)) -> x
p(plus(s(x0),x1)) -> plus(x0,x1)
plus(x0,x1) == s(plus(p(x0),x1))
```

```
EKBO with interpretations on N

  plus_A(x1,x2) = 1
  s_A(x1) = x1 + 1
  p_A(x1) = x1 + 1
  plus#_A(x1,x2) = x1
  s#_A(x1) = 0

weights

  w0 = 1
  w(plus) = 0
  w(s) = 1
  w(p) = 1

and precedence:

p > s > plus
```

```
plus(s(x),y) == s(plus(x,y))
s(p(x)) == x
p(s(x)) == x
p(plus(s(x0),x1)) == plus(x0,x1)
plus(x0,x1) == s(plus(p(x0),x1))
p(plus(x0,x1)) == plus(p(x0),x1)
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```
plus(s(x),y) <- s(plus(x,y))
s(p(x)) -> x
p(s(x)) -> x
p(plus(s(x0),x1)) == plus(x0,x1)
plus(x0,x1) == s(plus(p(x0),x1))
p(plus(x0,x1)) -> plus(p(x0),x1)
p(plus(plus(s(x0),x1),x2)) == plus(plus(x0,x1),x2)
```

```
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¦
¦   plus_A(x1,x2) = x1 + x2 + 1
¦   s_A(x1) = x1 + 1
¦   p_A(x1) = x1 + 1
¦   plus#_A(x1,x2) = x1
¦   s#_A(x1) = x1
¦   p#_A(x1) = x1
¦
¦ and precedence:
¦
¦ p > s > plus
```

```
YES

(VAR x0 x1 x y)
(RULES
  p(plus(x0,x1)) -> plus(p(x0),x1)
  p(s(x)) -> x
  s(p(x)) -> x
  s(plus(x,y)) -> plus(s(x),y)
)
```

```
(COMMENT
Termination is shown by ELPO with interpretations on N

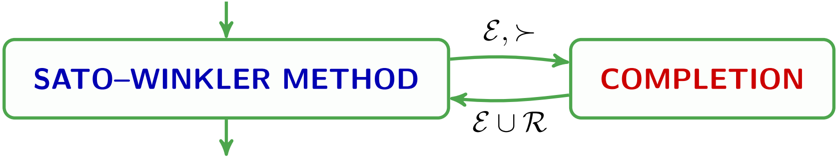
  plus_A(x1,x2) = x1 + x2 + 1
  s_A(x1) = x1 + 1
  p_A(x1) = x1 + 1
  plus#_A(x1,x2) = x1
  s#_A(x1) = x1
  p#_A(x1) = x1

and precedence:

p > s > plus
)
```

Example 1: Peano Arithmetic

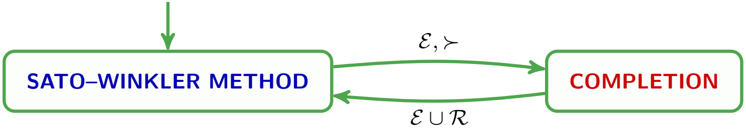
$$\mathcal{E} = \left\{ \begin{array}{c} \mathsf{s}(x) + y \approx \mathsf{s}(x+y) \\ \mathsf{s}(\mathsf{p}(x)) \approx x \\ \mathsf{p}(\mathsf{s}(x)) \approx x \end{array} \right\}$$



$$\mathcal{R} = \left\{ \begin{array}{c} \mathsf{s}(x) + y \to \mathsf{s}(x+y) \\ \mathsf{s}(\mathsf{p}(x)) \to x \\ \mathsf{p}(\mathsf{s}(x)) \to x \\ \mathsf{p}(x) + y \to \mathsf{p}(x+y) \end{array} \right\}$$

Example 2: Commuting Group Endomorphisms (CGE_2)

$$
\mathcal{E} = \left\{ \begin{array}{cc}
\mathsf{e} + x \approx x & \mathsf{f}(x+y) \approx \mathsf{f}(x) + \mathsf{f}(y) \\
\mathsf{i}(x) + x \approx \mathsf{e} & \mathsf{g}(x+y) \approx \mathsf{g}(x) + \mathsf{g}(y) \\
(x+y)+z \approx x+(y+z) & \mathsf{f}(x) + \mathsf{g}(y) \approx \approx \mathsf{g}(y) + \mathsf{f}(x)
\end{array} \right\}
$$



```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) <- a(f(x),f(y))
g(a(x,y)) <- a(g(x),g(y))
a(f(x),g(y)) -> a(g(y),f(x))
a(x,a(y,z)) -> a(a(x,y),z)
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x2 + 1
  e_A = 1
  i_A(x1) = x1 + 1
  f_A(x1) = 1
  g_A(x1) = 2
  a#_A(x1,x2) = x2
  e#_A = 0
  f#_A(x1) = 0

weights

  w0 = 1
  w(a) = 0
  w(e) = 1
  w(i) = 1
  w(f) = 1
  w(g) = 1

and precedence:

a > g > f > i > e
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) <- a(f(x),f(y))
g(a(x,y)) <- a(g(x),g(y))
a(f(x),g(y)) <- a(g(y),f(x))
a(x,a(y,z)) <- a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x2
  e_A = 0
  i_A(x1) = x1 + 1
  f_A(x1) = x1 + 2
  g_A(x1) = 1
  a#_A(x1,x2) = x2
  e#_A = 0
  f#_A(x1) = x1
  g#_A(x1) = x1

weights

  w0 = 1
  w(a) = 0
  w(e) = 1
  w(i) = 1
  w(f) = 1
  w(g) = 1

and precedence:

a > g > f > i > e
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) -> a(f(x),f(y))
g(a(x,y)) <- a(g(x),g(y))
a(f(x),g(y)) <- a(g(y),f(x))
a(x,a(y,z)) <- a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 <- a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) <- a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) <- a(g(x1),a(f(x0),x2))
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x1 + x2
  e_A = 0
  i_A(x1) = x1
  f_A(x1) = 0
  g_A(x1) = 1
  a#_A(x1,x2) = x1
  e#_A = 0

weights

  w0 = 1
  w(a) = 0
  w(e) = 1
  w(i) = 1
  w(f) = 0
  w(g) = 1

and precedence:

f > g > a > i > e
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) <- a(f(x),f(y))
g(a(x,y)) -> a(g(x),g(y))
a(f(x),g(y)) -> a(g(y),f(x))
a(x,a(y,z)) <- a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 <- a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) <- a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) -> a(g(x1),a(f(x0),x2))
a(i(e()),x0) -> x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) -> a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) -> x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x1 + x2
  e_A = 1
  i_A(x1) = x1 + 2
  f_A(x1) = x1 + 1
  g_A(x1) = 0
  a#_A(x1,x2) = x1
  e#_A = 0

weights

  w0 = 1
  w(a) = 0
  w(e) = 1
  w(i) = 1
  w(f) = 1
  w(g) = 0

and precedence:

g > a > f > i > e
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() == i(e())
a(x0,e()) == x0
x0 == i(i(x0))
e() == a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() == i(e())
a(x0,e()) == x0
x0 == i(i(x0))
e() == a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) <- a(f(x),f(y))
g(a(x,y)) <- a(g(x),g(y))
a(f(x),g(y)) -> a(g(y),f(x))
a(x,a(y,z)) -> a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() <- a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) <- a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) <- a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) <- a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) -> g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() <- i(e())
a(x0,e()) -> x0
x0 <- i(i(x0))
e() <- a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) -> x1
a(i(a(x0,i(x1))),x0) -> x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) -> f(x1)
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x1 + x2 + 1
  e_A = 2
  i_A(x1) = x1 + 1
  f_A(x1) = 1
  g_A(x1) = 2
  a#_A(x1,x2) = x2
  e#_A = 0
  f#_A(x1) = 0
  g#_A(x1) = x1

weights

  w0 = 1
  w(a) = 0
  w(e) = 2
  w(i) = 1
  w(f) = 1
  w(g) = 1

and precedence:

g > a > f > i > e
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() == i(e())
a(x0,e()) == x0
x0 == i(i(x0))
e() == a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
e() == f(e())
e() == g(e())
a(a(x0,i(x1)),x1) == x0
a(a(x0,x1),i(x1)) == x0
a(i(f(e())),f(x0)) == f(x0)
a(x0,i(a(i(x1),x0))) == x1
a(i(a(x0,x1)),x0) == i(x1)
i(a(x0,i(a(x1,x0)))) == x1
a(i(g(e())),g(x0)) == g(x0)
a(i(f(x0)),f(e())) == f(i(x0))
a(i(f(i(x0))),f(e())) == f(x0)
a(i(g(x0)),g(e())) == g(i(x0))
a(i(g(i(x0))),g(e())) == g(x0)
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() == i(e())
a(x0,e()) == x0
x0 == i(i(x0))
e() == a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
e() == f(e())
e() == g(e())
a(a(x0,i(x1)),x1) == x0
a(a(x0,x1),i(x1)) == x0
a(i(f(e())),f(x0)) == f(x0)
a(x0,i(a(i(x1),x0))) == x1
a(i(a(x0,x1)),x0) == i(x1)
i(a(x0,i(a(x1,x0)))) == x1
a(i(g(e())),g(x0)) == g(x0)
a(i(f(x0)),f(e())) == f(i(x0))
a(i(f(i(x0))),f(e())) == f(x0)
a(i(g(x0)),g(e())) == g(i(x0))
a(i(g(i(x0))),g(e())) == g(x0)
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) <- a(f(x),f(y))
g(a(x,y)) -> a(g(x),g(y))
a(f(x),g(y)) -> a(g(y),f(x))
a(x,a(y,z)) <- a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 <- a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) <- a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) -> a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) -> x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() <- i(e())
a(x0,e()) -> x0
x0 <- i(i(x0))
e() <- a(x0,i(x0))
x0 <- a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
```

```
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) -> f(x1)
e() <- f(e())
e() <- g(e())
a(a(x0,i(x1)),x1) == x0
a(a(x0,x1),i(x1)) == x0
a(i(f(e())),f(x0)) == f(x0)
a(x0,i(a(i(x1),x0))) -> x1
a(i(a(x0,x1)),x0) -> i(x1)
i(a(x0,i(a(x1,x0)))) -> x1
a(i(g(e())),g(x0)) == g(x0)
a(i(f(x0)),f(e())) == f(i(x0))
a(i(f(i(x0))),f(e())) == f(x0)
a(i(g(x0)),g(e())) == g(i(x0))
a(i(g(i(x0))),g(e())) == g(x0)
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x1 + x2
  e_A = 0
  i_A(x1) = x1
  f_A(x1) = 1
  g_A(x1) = 0
  a#_A(x1,x2) = x1
  e#_A = 0

weights

  w0 = 1
  w(a) = 0
  w(e) = 2
  w(i) = 1
  w(f) = 1
  w(g) = 0

and precedence:

g > a > f > i > e
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() == i(e())
a(x0,e()) == x0
x0 == i(i(x0))
e() == a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
e() == f(e())
e() == g(e())
a(a(x0,i(x1)),x1) == x0
a(a(x0,x1),i(x1)) == x0
a(i(f(e())),f(x0)) == f(x0)
a(x0,i(a(i(x1),x0))) == x1
a(i(a(x0,x1)),x0) == i(x1)
i(a(x0,i(a(x1,x0)))) == x1
a(i(g(e())),g(x0)) == g(x0)
a(i(f(x0)),f(e())) == f(i(x0))
a(i(f(i(x0))),f(e())) == f(x0)
a(i(g(x0)),g(e())) == g(i(x0))
a(i(g(i(x0))),g(e())) == g(x0)
g(i(x0)) == i(g(x0))
f(i(x0)) == i(f(x0))
a(x0,i(a(x1,x0))) == i(x1)
a(i(x0),x1) == i(a(i(x1),x0))
a(i(x0),i(x1)) == i(a(x1,x0))
a(x0,x1) == i(a(i(x1),i(x0)))
a(x0,a(x1,i(a(x0,x1)))) == e()
f(x0) == i(f(a(x1,i(a(x0,x1)))))
a(f(a(x0,x1)),i(f(x1))) == f(x0)
i(a(f(x0),i(f(a(x1,x0))))) == f(x1)
a(i(f(a(x0,x1))),f(x0)) == i(f(x1))
a(i(f(x0)),f(x1)) == f(a(i(x0),x1))
a(x0,a(x1,a(i(a(x0,x1)),x2))) == x2
```

```
a(e(),x) == x
a(i(x),x) == e()
f(a(x,y)) == a(f(x),f(y))
g(a(x,y)) == a(g(x),g(y))
a(f(x),g(y)) == a(g(y),f(x))
a(x,a(y,z)) == a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 == a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) == a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) == a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) == a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() == i(e())
a(x0,e()) == x0
x0 == i(i(x0))
e() == a(x0,i(x0))
x0 == a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
e() == f(e())
e() == g(e())
a(a(x0,i(x1)),x1) == x0
a(a(x0,x1),i(x1)) == x0
a(i(f(e())),f(x0)) == f(x0)
a(x0,i(a(i(x1),x0))) == x1
a(i(a(x0,x1)),x0) == i(x1)
i(a(x0,i(a(x1,x0)))) == x1
a(i(g(e())),g(x0)) == g(x0)
a(i(f(x0)),f(e())) == f(i(x0))
a(i(f(i(x0))),f(e())) == f(x0)
a(i(g(x0)),g(e())) == g(i(x0))
a(i(g(i(x0))),g(e())) == g(x0)
g(i(x0)) == i(g(x0))
f(i(x0)) == i(f(x0))
a(x0,i(a(x1,x0))) == i(x1)
a(i(x0),x1) == i(a(i(x1),x0))
a(i(x0),i(x1)) == i(a(x1,x0))
a(x0,x1) == i(a(i(x1),i(x0)))
a(x0,a(x1,i(a(x0,x1)))) == e()
f(x0) == i(f(a(x1,i(a(x0,x1)))))
a(f(a(x0,x1)),i(f(x1))) == f(x0)
i(a(f(x0),i(f(a(x1,x0))))) == f(x1)
a(i(f(a(x0,x1))),f(x0)) == i(f(x1))
a(i(f(x0)),f(x1)) == f(a(i(x0),x1))
a(x0,a(x1,a(i(a(x0,x1)),x2))) == x2
```

```
a(e(),x) -> x
a(i(x),x) -> e()
f(a(x,y)) <- a(f(x),f(y))
g(a(x,y)) <- a(g(x),g(y))
a(f(x),g(y)) -> a(g(y),f(x))
a(x,a(y,z)) <- a(a(x,y),z)
a(x0,x1) == a(a(x0,e()),x1)
e() == a(a(i(a(x0,x1)),x0),x1)
a(x0,e()) == a(a(x0,i(x1)),x1)
a(x0,g(a(x1,x2))) == a(a(x0,g(x1)),g(x2))
a(x0,f(a(x1,x2))) == a(a(x0,f(x1)),f(x2))
a(a(x0,g(x1)),f(x2)) == a(a(x0,f(x2)),g(x1))
x0 <- a(i(x1),a(x1,x0))
a(g(a(x0,x1)),x2) <- a(g(x0),a(g(x1),x2))
a(f(a(x0,x1)),x2) <- a(f(x0),a(f(x1),x2))
a(f(x0),a(g(x1),x2)) -> a(g(x1),a(f(x0),x2))
a(i(e()),x0) == x0
a(i(i(x0)),e()) == x0
f(x0) == a(f(e()),f(x0))
a(i(i(x0)),x1) == a(x0,x1)
f(e()) == a(f(i(x0)),f(x0))
a(i(g(x0)),g(a(x0,x1))) == g(x1)
a(i(a(x0,x1)),a(x0,a(x1,x2))) == x2
a(i(g(x0)),a(f(x1),g(x0))) == f(x1)
f(x0) == a(f(i(x1)),a(f(x1),f(x0)))
f(g(a(x0,x1))) == a(f(g(x0)),f(g(x1)))
a(i(g(x0)),a(g(a(x0,x1)),x2)) == a(g(x1),x2)
a(f(f(x0)),f(g(x1))) == a(f(g(x1)),f(f(x0)))
a(i(g(x0)),a(f(x1),a(g(x0),x2))) == a(f(x1),x2)
e() <- i(e())
a(x0,e()) -> x0
x0 <- i(i(x0))
e() <- a(x0,i(x0))
x0 <- a(x1,a(i(x1),x0))
g(x0) == a(g(x0),g(e()))
```

```
g(x0) == a(g(e()),g(x0))
i(a(i(a(x0,x1)),x0)) == x1
a(i(a(x0,i(x1))),x0) == x1
g(x0) == a(g(i(e())),g(x0))
g(e()) == a(g(i(x0)),g(x0))
a(i(a(x0,i(e()))),a(x0,x1)) == x1
a(i(f(x0)),f(a(x0,x1))) == f(x1)
e() <- f(e())
e() <- g(e())
a(a(x0,i(x1)),x1) == x0
a(a(x0,x1),i(x1)) == x0
a(i(f(e())),f(x0)) == f(x0)
a(x0,i(a(i(x1),x0))) == x1
a(i(a(x0,x1)),x0) == i(x1)
i(a(x0,i(a(x1,x0)))) == x1
a(i(g(e())),g(x0)) == g(x0)
a(i(f(x0)),f(e())) == f(i(x0))
a(i(f(i(x0))),f(e())) == f(x0)
a(i(g(x0)),g(e())) == g(i(x0))
a(i(g(i(x0))),g(e())) == g(x0)
g(i(x0)) <- i(g(x0))
f(i(x0)) <- i(f(x0))
a(x0,i(a(x1,x0))) == i(x1)
a(i(x0),x1) == i(a(i(x1),x0))
a(i(x0),i(x1)) <- i(a(x1,x0))
a(x0,x1) == i(a(i(x1),i(x0)))
a(x0,a(x1,i(a(x0,x1)))) == e()
f(x0) == i(f(a(x1,i(a(x0,x1)))))
a(f(a(x0,x1)),i(f(x1))) == f(x0)
i(a(f(x0),i(f(a(x1,x0))))) == f(x1)
a(i(f(a(x0,x1))),f(x0)) == i(f(x1))
a(i(f(x0)),f(x1)) == f(a(i(x0),x1))
a(x0,a(x1,a(i(a(x0,x1)),x2))) == x2
```

```
EKBO with interpretations on N

  a_A(x1,x2) = x1 + x2
  e_A = 0
  i_A(x1) = x1
  f_A(x1) = x1 + 1
  g_A(x1) = 0
  a#_A(x1,x2) = x1
  e#_A = 0
  f#_A(x1) = 0

weights

  w0 = 1
  w(a) = 0
  w(e) = 1
  w(i) = 0
  w(f) = 1
  w(g) = 1

and precedence:

i > g > a > f > e
```

```
YES

(VAR x1 x0 x2 x y z)
(RULES
  i(a(x1,x0)) -> a(i(x0),i(x1))
  i(g(x0)) -> g(i(x0))
  i(f(x0)) -> f(i(x0))
  g(e()) -> e()
  f(e()) -> e()
  a(x1,a(i(x1),x0)) -> x0
  a(x0,i(x0)) -> e()
  i(i(x0)) -> x0
  i(e()) -> e()
  a(x0,e()) -> x0
  a(f(x0),a(g(x1),x2)) -> a(g(x1),a(f(x0),x2))
  a(f(x0),a(f(x1),x2)) -> a(f(a(x0,x1)),x2)
  a(g(x0),a(g(x1),x2)) -> a(g(a(x0,x1)),x2)
  a(i(x1),a(x1,x0)) -> x0
  a(a(x,y),z) -> a(x,a(y,z))
  a(f(x),g(y)) -> a(g(y),f(x))
  a(g(x),g(y)) -> g(a(x,y))
  a(f(x),f(y)) -> f(a(x,y))
  a(i(x),x) -> e()
  a(e(),x) -> x
)
```

```
(COMMENT
Termination is shown by EKBO with interpretations on N

  a_A(x1,x2) = x1 + x2
  e_A = 0
  i_A(x1) = x1
  f_A(x1) = x1 + 1
  g_A(x1) = 0
  a#_A(x1,x2) = x1
  e#_A = 0
  f#_A(x1) = 0

weights

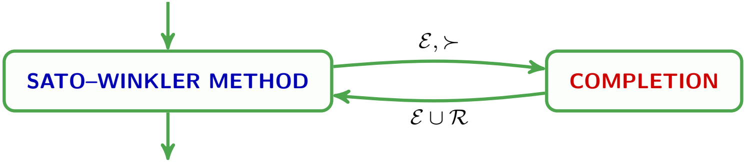
  w0 = 1
  w(a) = 0
  w(e) = 1
  w(i) = 0
  w(f) = 1
  w(g) = 1

and precedence:

i > g > a > f > e
)
```

Example 2: Commuting Group Endomorphisms (CGE_2)

$$\mathcal{E} = \left\{ \begin{array}{cc} \mathsf{e} + x \approx x & \mathsf{f}(x+y) \approx \mathsf{f}(x) + \mathsf{f}(y) \\ \mathsf{i}(x) + x \approx \mathsf{e} & \mathsf{g}(x+y) \approx \mathsf{g}(x) + \mathsf{g}(y) \\ (x+y)+z \approx x + (y+z) & \mathsf{f}(x) + \mathsf{g}(y) \approx \approx \mathsf{g}(y) + \mathsf{f}(x) \end{array} \right\}$$



$$\mathcal{R} = \left\{ \begin{array}{rcc} \mathsf{e} + x \to x & \mathsf{f}(\mathsf{e}) \to \mathsf{e} & \mathsf{i}(x+y) \to \mathsf{i}(y) + \mathsf{i}(x) \\ x + \mathsf{e} \to x & \mathsf{g}(\mathsf{e}) \to \mathsf{e} & \mathsf{f}(x) + \mathsf{f}(y) \to \mathsf{f}(x+y) \\ \mathsf{i}(x) + x \to \mathsf{e} & \mathsf{i}(\mathsf{e}) \to \mathsf{e} & \mathsf{g}(x) + \mathsf{g}(y) \to \mathsf{g}(x+y) \\ x + \mathsf{i}(x) \to \mathsf{e} & \mathsf{i}(\mathsf{i}(x)) \to x & \mathsf{f}(x) + \mathsf{g}(y) \to \mathsf{g}(y) + \mathsf{f}(x) \\ x + (\mathsf{i}(x) + y) \to y & \mathsf{i}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{i}(x)) & \mathsf{f}(x) + (\mathsf{f}(y) + z) \to \mathsf{f}(x+y) + z \\ \mathsf{i}(x) + (x + y) \to y & \mathsf{i}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{i}(x)) & \mathsf{g}(x) + (\mathsf{g}(y) + z) \to \mathsf{g}(x+y) + z \\ (x+y)+z \to x + (y+z) & & \mathsf{f}(y) + (\mathsf{g}(x) + z) \to \mathsf{g}(x) + (\mathsf{f}(y) + z) \end{array} \right\}$$

Experimental Results

- 115 completion problems (taken from problem set of mkbTT)
- 600 seconds timeout
- minimization problems are solved by MaxSMT (Z3)

orders (tool)	LPO	KBO	ELPO	EKBO	ELPO+EKBO	KBCV	MaxcompDP
# completed	82	83	86	86	**96**	86	97

Conclusion

Presented Techniques

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- **termination:** order extension based on semantic labeling

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- **termination:** order extension based on semantic labeling
- **confluence:** characterization by rewrite strategies

Conclusion

Presented Techniques

- **termination:** order extension based on semantic labeling
- **confluence:** characterization by rewrite strategies
- **completion:** maximal completion with inter-reduction
  $=$ Sato and Winkler's method $+$ standard completion

Conclusion

Presented Techniques

- **termination:** order extension based on semantic labeling
- **confluence:** characterization by rewrite strategies
- **completion:** maximal completion with inter-reduction
  = Sato and Winkler's method + standard completion

Future Work

ordered completion