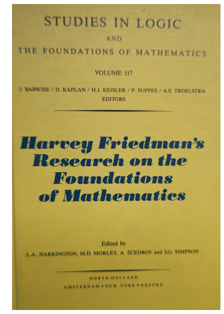
TTP mark 2022: *Higman's lemma with gap condition*



The theme of **TPP 2022** is, *Formal proof of mathematical theorems*. Not only in formal methods, another aim of theorem provers is to formally prove mathematical theorems. However, such efforts are performed individually on each prover, e.g., **Mizar**[1], **Nuprl**[2], **Isabelle/HOL**[3], **Coq**[4], and **Lean**[5]. We hope to discuss about the possiblity to cooperate and unify such efforts, observing the possibility to transfer proofs from a prover to another.

Under such view, **TPP mark 2022** is a formal proof of *Higman's lemma with gap condition*, which is a simpler form of *Kruskal's theorem with gap condition* (**Theorem 4.2** in [1]). Its proof is based on Higman's lemma and MBS (minimal bad sequence), which is lead by *Zorn's lemma*.

Note that in proof archaives, several provers have the proofs of *Krsukal's theorem* and *Higman's lemma*, which you can use as the base of *Higman's lemma with gap condition*.

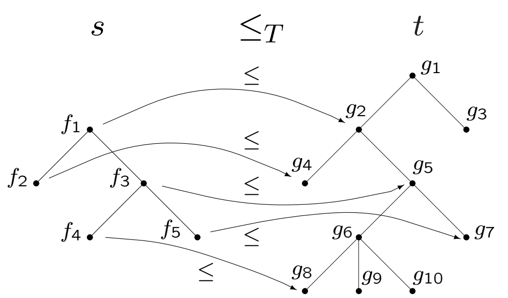
- **Isabelle/HOL**: `https://www.isa-afp.org/entries/Well_Quasi_Orders.html`
- **Coq**: `https://homepages.loria.fr/DLarchey/Kruskal/`, `https://github.com/coq-contribs/higman-s`
- **Others**: **Nuprl** [2], **ACL2** [3], and **LEAN** [4].

Definition 1. $(Q, \preceq)$ *is a* WQO *if, for each infinite sequcence* $(a_i \mid i \in \mathbb{N})$ *in* Q*, there exist* i, j *with* $i < j$ *and* $a_i \preceq a_j$.

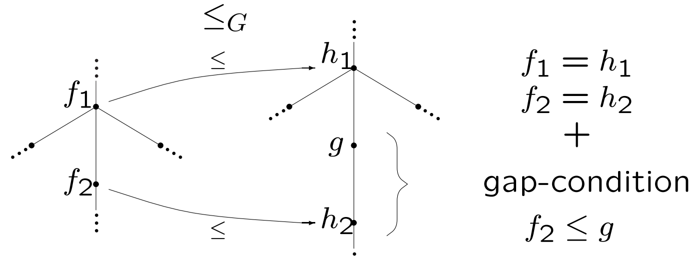
Definition 2. *Let* $\mathcal{T}(Q)$ *be the set of finite rooted trees with the labeling* ℓ *on each node from a QO* $(Q, \preceq)$*. For* $T_1, T_2 \in \mathcal{T}(Q)$ *and* $t, t' \in V(T_1)$*, an injection* $\psi : V(T_1) \to V(T_2)$ *with* $\psi(t \sqcap t') = \psi(t) \sqcap \psi(t')$ *(*$\sqcap$ *with respect to positions) is a* tree embedding *if* $\ell(t) \preceq \ell(\psi(t))$ *for each* $t \in V(T_1)$.

If ψ *further holds that (i)* $\ell(t) = \ell(\psi(t))$ *for each* $t \in V(T_1)$ *and (ii)* $\ell(s) \succeq \ell(t')$ *for* $s \in V(T_2)$ *between* $\psi(t)$ *and* $\psi(t')$ *for a child* t' *of* (t) *in* T_1.

If there is a tree embedding (resp. gap embedding) from T_1 to T_2, we denote $T_1 \leq_T T_2$ (resp. $T_1 \leq_G T_2$).



Tree embedding $\leq_T$



Gap embedding $\leq_G$

Theorem 3. *(***Kruskal's theorem***) If* $(Q, \leq)$ *is a WQO,* $(\mathcal{T}(Q), \leq_T)$ *is a WQO.*

Theorem 4. *(***Kruskal's theorem with gap condition***) For* $Q = \{0, 1, \cdots, n\}$*,* $(\mathcal{T}(Q), \leq_G)$ *is a WQO.*

Our TPP mark is a simpler version, *Higman's lemma with gap condition*, i.e., *Kruakal's theorem with gap condition* on finite words, instead of finite trees. Please try based on above library of formal proofs. The proof will be based on induction on n, and the inductive step will use Higman's lemma and MBS (minimal bad sequence) [1].

References

1. S.G. Simpson. *Nonprovability of certain combinatorial properties of finite trees.* 87-117, in *Harvey Friedman's Research on the Fondations of Mathematics, Studies in Logic, Vol.117, North-Holland*
2. C.R. Murthy, J.R. Russell. *A constructive proof of Higman's lemma.* 257-267, IEEE LICS 1990.
3. F.J. Martin-Mateos, J.L. Ruiz-Reina, J.A. Alonso, M.J. Hidalgo. *Proof pearl: A formal proof of Higman's lemma in ACL2.* TPHOLs 2005, LNCS 3603.
4. M. Wu. *A formally verified proof of Kruskal's tree theorem in Lean.* Master thesis, CMU, 2017.

[1] `https://mizar.org`
[2] `https://nuprl.org`
[3] `https://https://www.isa-afp.org/`
[4] `https://math-comp.github.io/`
[5] `https://leanprover.github.io/theorem_proving_in_lean/`