Voronoi Diagram with Respect to Criteria on Vision Information

Tetsuo Asano
School of Information Science,
JAIST, Japan
1-1 Asahidai, Nomi,
923-1292, Japan
t-asano@jaist.ac.jp

Naoki Katoh
Graduate School of Engineering,
Kyoto University, Japan
Kyoto University Katsura, Nishikyo-ku, Kyoto,
615-8540, Japan
naoki@archi.kyoto-u.ac.jp

Hisao Tamaki
Department of Computer Science,
Meiji University, Japan
Higashimita, Tama- Kawasaki,
214-8571, Japan
tamaki@cs.meiji.ac.jp

Takeshi Tokuyama
Graduate School of Information Sciences,
Tohoku University, Japan
Aramaki Aza-Aoba, Aoba, Sendai,
980-8579 Japan
tokuyama@dais.is.tohoku.ac.jp

Abstract

Voronoi diagram for a set of geometric objects is a partition of the plane (or space in higher dimensions) into disjoint regions each dominated by some given object under a predetermined criterion. In this paper we are interested in various measures associated with criteria on goodness of an input line segment with respect to each point in the plane as the "point of view". These measures basically show how the segment or information displayed on the segment can be seen from the point. Mathematically, the measures are defined in terms of the shape of the triangle determined by the point and the line segment. Given any such measure, we can define a Voronoi diagram for a set of line segments. In this paper we are interested in investigating their common combinatorial and structural properties. We investigate conditions for those measures to define regular Voronoi diagrams and also conditions that local optima on the measures lie only on Voronoi edges, not in the proper interior of Voronoi regions.

1. Introduction

Given a set of points in the plane, we can partition the plane into regions in such a way that any point in a region associated with some given point is closer to the point than to any other point in the set. The resulting partition of the plane is called a Voronoi diagram for the point set. Replacing the relation "closer to" with some other criteria we could define a number of variations of the diagram (see e.g. [3, 10]).

In this paper we propose a yet another abstraction of those Voronoi diagrams for a set of non-intersecting line segments possibly forming a polygon. We consider a measure associated with a criterion on how an input segment can be seen from a point in the plane as a point of view. There are several possible criteria. Naturally, the distances from the segments and the lengths of segments are important factors. If two segments are of the same length, one may conclude that the nearer one should be seen better; however, it is not always the case, since if we see a blackboard from the leftmost seat in the first row of a classroom, we have difficulty to read letters written on the blackboard. Also, liquid-crystal display of a laptop computer can be only seen from points in a limited region. Basically, the visual quality of information given on a line segment s seen from a point p depends on the shape of triangle $\triangle(p,s)$ defined by s and p such that $\triangle(p,s)$ has s as an edge and p as its opposite vertex. We consider $\mu(p,s)$ to measure how the visual information on s can be obtained at p, and naturally, $\mu(p,s)$ is defined as a function from the set of all triangles to the set of all nonnegative real numbers. Such a measure is not only useful in vision applications but also has potential applications to evaluation of quality of some geometric structures (e.g. polygonal meshes) related to input set of segments and also generation of a good geometric structure. One such example is the following. Suppose we have a point p and a line segment s. The internal angle $\theta_p(s)$ of the triangle $\triangle(p,s)$ at p is called a visual angle of s from p. Suppose the goodness $\mu(p,s)$ of the triangle is given by $\theta_p(s)$. Then, we can define a max-min type optimization problem using this measure. In this particular case, given a set S of line segments, we want to find a point that maximizes the minimum measure $\mu(p,s)$ over all $s \in S$. We can show that such an optimal point can be found on edges of the corresponding Voronoi diagram.

With a different criterion we can define a similar but different Voronoi diagram. Voronoi edges are characterized in a different manner. The purpose of this paper is to find combinatorial and structural properties common to those Voronoi diagrams associated with measures μ defined for a pair of point and line segment. We describe basic properties to be satisfied by the measures to possess those common properties. It is important for practical use. There may be a number of problems falling into a class which can be solved using our framework of Voronoi diagrams. Although it is impossible to enumerate all possible optimization criteria, it is possible to investigate basic conditions for those criteria to satisfy to have their corresponding Voronoi diagrams bear the same combinatorial properties. All the measures considered in this paper are associated with goodness of a triangle.

An original motivation of this Voronoi diagram comes from applications to mesh improvement and robotics. Mesh generation/improvement [4, 5, 9, 11, 12] is an important process for many purposes including Finite Element Method. In a simple setting, a given simple polygon is partitioned into many small triangles after inserting an appropriate number of points in its interior as vertices of triangular meshes. Several different criteria have been considered to evaluate the quality of such a triangular mesh. One of them is to maximize the smallest internal angle (or to minimize the largest internal angle). Since polygon vertices are fixed, the only way to improve the quality of triangular mesh is either to move internal points or to insert new internal points (or even delete existing internal points). In robotics, locating a robot in the amidst of many polygonal obstacles by computing its relative position to the most outstanding polygon or line segment in a criterion on visual information.

Let μ be a measure of max-min type on goodness of a triangle defined by a point and a line segment. That is, $\mu(p,s_i) < \mu(p,s_j)$ means that the triangle defined by (p,s_i) is worse in the measure μ than the triangle (p,s_j). Then, a point p belongs to a Voronoi region of a line segment s_i if $\mu(p,s_i)$ is worst among given line segments. A min-max type measure μ can be similarly treated by replacing $\mu(p,s_i)$ by $-\mu(p,s_i)$. A partition of the plane (or space) into such regions defines a Voronoi diagram associated with μ. The Voronoi diagram is also given as the lower envelope of terrains, where a terrain for a line segment s_i is defined using $\mu(p,s_i)$ as height at the point p. A general theory on terrains by Halperin and Sharir [6] yields an upper bound $O(n^{2+\varepsilon})$ on the complexity of the lower envelope of those terrains that is Voronoi diagram, where ε is an arbitrarily small positive constant. In other words, the Voronoi diagram has $O(n^{2+\varepsilon})$ Voronoi edges, and vertices. Despite the high complexity in the worst case, actual complexity seems to be low by our experiments for a number of polygons.

2. Voronoi diagrams for various criteria on triangles

A classic Voronoi diagram for a point set S is a partition of the plane into disjoint regions of points which are closer to one element of S than to any other element. We can replace this relation ”closer to” with some other relations defined by points and a set of given geometric objects. Geometric objects considered in this paper are non-intersecting line segments. Given an arbitrary point p in the plane and a line segment s_i in a given set S, the pair (p,s_i) determines a triangle $\triangle(p,s_i)$. Here we assume that line segments are transparent, that is, the triangle $\triangle(p,s_i)$ is defined even if there is another line segment $s_j \in S$ intersecting the triangle. Let $\mu(p,s_i)$ be any measure on ”goodness” of the triangle. Then, given n line segments, for each point p we can compute n values $\mu(p,s_1),\ldots,\mu(p,s_n)$. Among them we take the worst value as the value at the point p, and we say that the point is dominated by the element giving the worst value.

On a max-min type measure we are interested in a point that maximizes the minimum value of the measure on the criterion. A point p is dominated by a line segment s_i if

$$\min\{\mu(p,s_i) | s_i \in S\} = \mu(p,s_i)$$

and p belongs to a region dominated by s_i, which is a *Voronoi region* $V(s_i)$ of s_i. In other words, a Voronoi region $V(s_i)$ is defined by

$$V(s_i) = \{p \in \mathbb{R}^2 | \mu(p,s_i) < \mu(p,s_j) \text{ for any } j \neq i\}.$$

Voronoi edges are defined by curves which are dominated by exactly two elements of S. Formally, a Voronoi edge $E(s_i,s_j)$ is defined by

$$E(s_i,s_j) = \{p \in \mathbb{R}^2 | \mu(p,s_i) = \mu(p,s_j) < \mu(p,s_k) \text{ for any } k \neq i,j\}.$$

Two Voronoi edges may meet at one point, that is a Voronoi vertex. It is defined by

$$v(s_i,s_j,s_k) = \{p \in \mathbb{R}^2 | \mu(p,s_i) = \mu(p,s_j) = \mu(p,s_k) \leq \mu(p,s_l) \quad \text{for any } l \neq i,j,k\},$$

which is a set of points dominated by exactly three elements of S.

This is a basic definition of our Voronoi diagram on a measure μ. In practice, we need some minor modifications of the definition. We implicitly assume that the measure μ is described by a combination of a constant number of algebraic functions and $\mu(p,s_i)$ is continuous in the region $\mathbb{R}^2 - \{s_i\}$.. Some Voronoi edges may be described by a single algebraic function and others by more than one function. If $E(s_i,s_j)$ is decomposed into parts each described by a single algebraic function, then we cut it into those pieces as individual Voronoi edges and put Voronoi vertices at those endpoints. We say a Voronoi diagram on μ is *regular* if every Voronoi edge is a curve without any area.

We also put another implicit assumption on a set S of line segments. In our definition, a set of points dominated by exactly two elements of S can form curves but not regions with positive area. A set of points dominated by exactly three elements of S must be a set of individual points not forming curves or regions. Otherwise, we have degeneracy. If degeneracy cannot be broken by small perturbation of an input set S then we say that the measure causes degeneracy and we exclude the measure from our considerations.

Consider a max-min type measure $\mu(p,s_i)$ that is the distance from a point p to the line including a line segment s_i. If two line segments s_i and s_j lie on a line l, any point p has the same distance to s_i and s_j. This violates the condition (2) above. So, this is a degeneracy. But it can be resolved by rotating or translating one of them. Thus, this is not a degeneracy caused by the measure. Let us modify the measure slightly by introducing some upper bound on the measure. That is, $\mu(p,s_i)$ is the distance from p to the line containing s_i if it is at most the length of s_i. Otherwise, it is defined as $||s_i||$, the length of s_i. If the input set S consists of only two line segments s_1 and s_2 of the same length d, then any point which is away from both of them by at least d has the same value in the measure. This degeneracy cannot be resolved by changing placements of two line segments. So, this is a degeneracy caused by the measure.

3. Examples of measures

There are a number of ways of evaluating a triangle. Our purpose of this paper is to characterize topological and structural properties of Voronoi diagrams associated with measures on triangles and also characterize measures to define good Voronoi diagrams. Here is a list of criteria or measures on triangles determined by a pair of point p and a line segment s_i. There are basically two types of measures, max-min types and min-max types. In the max-min type (min-max type, respectively) measure we look for a point that maximizes the minimum value (minimizes the maximum value, resp.) of measures. Formally, an optimal point p^* is characterized by

$$p^* = \arg[\max_p \min\{\mu(p,s_j)|s_j \in S\}] \tag{1}$$

for a max-min type measure μ and

$$p^* = \arg[\min_p \max\{\mu(p,s_j)|s_j \in S\}] \tag{2}$$

for a min-max type measure μ.

3.1. List of possible measures

max-min visual angle Define $\mu_1(p,s_i) = \theta_p(s_i)$ that is the visual angle of s_i from p. See Fig. 1(a).

max-min height Define $\mu_2(p,s_i)$ as the minimum height of a triangle define by (p,s_i). See Fig. 1(b).

min-max circumcircle Define $\mu_3(p,s_i)$ as the radius of the circumcircle of a triangle define by (p,s_i). See Fig. 1(c).

max-min aspect-ratio Define $\mu_4(p,s_i) = h/L$ where h and L are the minimal height and the length of the longest side of a triangle define by (p,s_i), respectively. See Fig. 1(d).

min-max eccentricity Define $\mu_5(p,s_i)$ as follows: it is 0 if the center of the circumcircle of a triangle $\triangle(p,s_i)$ lies in the interior of the triangle. Otherwise, it is the distance from the center to the closest edge of the triangle. See Fig. 1(e).

3.2. Algebraic expressions

There may be a number of different measures on triangles. First of all we exclude all measures which cannot be computed by explicit algebraic expressions. More precisely, we need algebraic formulae defined by constants, polynomials in x and y, and r-th roots for positive integers r. Or we can relax the condition slightly. What we need is a function to compare $\mu(p,s_i)$ and $\mu(p,s_j)$ for two different segments s_i and s_j in a given set. Below are concrete algebraic expressions for the measures listed earlier.

(1) max-min visual angle

Instead of giving an expression to determine a value of $\theta_p(s_i)$ that is the visual angle of s_i from p, we can use its cosine value to compare two such values. Exactly, we have

$$\cos\theta_p(s_i) = \frac{||pa||^2 + ||pb||^2 - ||ab||^2}{2||pa|| \cdot ||pb||},$$

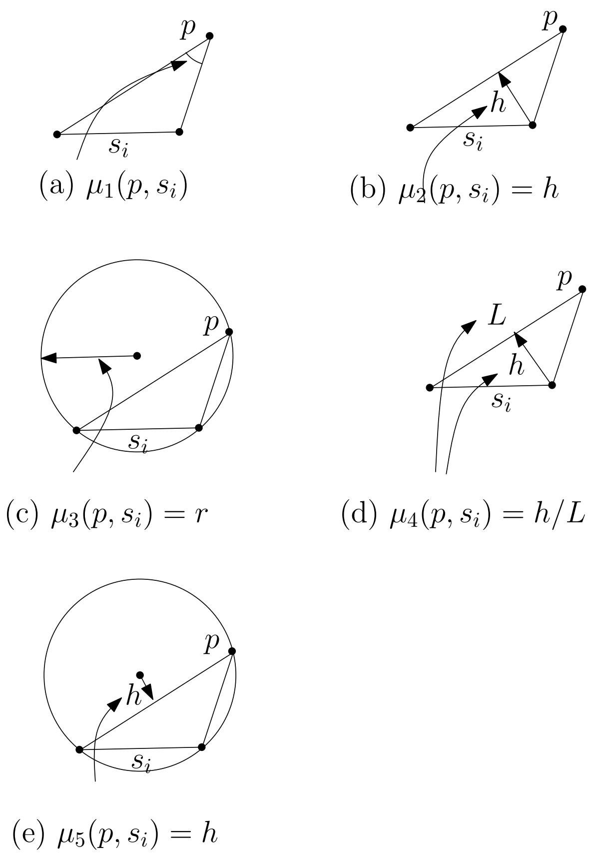

Figure 1. A list of measures on triangles. (a)max-min angle, (b) max-min height, (c) max-min aspect ratio, (d) min-max circumcircle, and (e) min-max eccentricity.

where a and b are two endpoints of s_i and $||pq||$ denotes the length of the segment $\overline{pq}$. Assuming $p = (x,y)$, $a = (a_x, a_y)$, and $b = (b_x, b_y)$, we have

$$\begin{aligned}
\cos\theta_p(s_i) &= \frac{T_1}{T_2}, \\
T_1 &= (x-a_x)^2 + (y-a_y)^2 + (x-b_x)^2 \\
&\quad + (y-b_y)^2 - (a_x-b_x)^2 - (a_y-b_y)^2, \\
T_2 &= 2\sqrt{(x-a_x)^2 + (y-a_y)^2} \\
&\quad \times \sqrt{(x-b_x)^2 + (y-b_y)^2}.
\end{aligned}$$

Using the cosine values we can easily compare $\theta_p(s_i)$ and $\theta_p(s_j)$. Especially, a Voronoi edge defined by $\theta_p(s_i) = \theta_p(s_j)$ leads to a polynomial equations of degree at most 8 without square roots. Of course, we have already known that Voronoi edges defined by the measure are polynomial curves of degree at most 3 through complex calculations [2]. The result above is short but enough to prove that Voronoi edges are characterized by polynomial curves of constant degrees.

(2) max-min height

Given a triangle, for each of three sides we can define a corresponding height. A minimal height of a triangle is one for the longest side. The height can be obtained using an area of a triangle, which can be calculated by coordinates of the three vertices of the triangle. So, if A and L are area and length of the longest side of the triangle then the minimal height h is given by $(2A)/L$. The expressions to give the minimal height of a triangle defined by a point p and a line segment $s_i = \overline{ab}$ are obtained in three different cases depending on which side is the longest side. For that we draw two circles centered at one of the endpoints of s_i and passing through the other endpoint. We also draw the perpendicular bisector of s_i. Now we have six regions bounded by those circles and line. If we take a point p in one of the regions, we can easily see which triangular edge is longest. In this way we have three different expressions to determine the minimal height as follows:

$$\mu_2(p, s_i) = \begin{cases}
\frac{|(a_y-b_y)x+(b_x-a_x)+a_xb_y-b_xa_y|}{\sqrt{(x-a_x)^2+(y-a_y)^2}} \\
\quad \text{if } \overline{pa} \text{ is the longest edge,} \\
\frac{|(a_y-b_y)x+(b_x-a_x)+a_xb_y-b_xa_y|}{\sqrt{(x-b_x)^2+(y-b_y)^2}} \\
\quad \text{if } \overline{pb} \text{ is the longest edge,} \\
\frac{|(a_y-b_y)x+(b_x-a_x)+a_xb_y-b_xa_y|}{\sqrt{(a_x-b_x)^2+(a_y-b_y)^2}} \\
\quad \text{if } \overline{ab} \text{ is the longest edge.}
\end{cases}$$

Thus, Voronoi edges are characterized by polynomial equations of degree at most 4 in x and y.

(3) max-min aspect-ratio

Recall that the measure on the minimal height h is defined by the longest side L and area A of a triangle, that is, $h = 2A/L$. The aspect ratio of a triangle $\triangle(p, s_i)$ is the ratio h/L, which is equal to $2A/L^2$. Thus, we have

$$\mu_4(p,s) = \frac{2 \times |\triangle(p,s_i)|}{L^2},$$

where $|\triangle(p,s_i)|$ is the area of the triangle and L is the length of the longest side, that is,

$$2|\triangle(p,s_i)| = |(a_y - b_y)x + (b_x - a_x)y + a_x b_y - a_y b_x|,$$
$$L^2 = \max\{(x-a_x)^2 + (y-a_y)^2, (x-b_x)^2 + (y-b_y)^2, (a_x-b_x)^2 + (a_y-b_y)^2\}.$$

(4) min-max circumcircle

It is known that the radius r of a triangle pab with area A is given by

$$\mu_4(p,s_i) = r = \frac{||pa|| \cdot ||pb|| \cdot ||ab||}{4A}$$
$$||pa|| = \sqrt{(x-a_x)^2 + (y-a_y)^2},$$
$$||pb|| = \sqrt{(x-b_x)^2 + (y-b_y)^2},$$
$$||ab|| = \sqrt{(a_x-b_x)^2 + (a_y-b_y)^2},$$
$$4A = 2|(a_y - b_y)x + (b_x - a_x) + a_x b_y - b_x a_y|.$$

(5) min-max eccentricity

Let r be the radius of the circumcircle of a triangle pab. If the center of the circumcircle lies in the interior of the triangle, $\mu_5(p,s_i) = 0$. Otherwise, pa or pb is closest to the center. If pa is closest, the distance from the center to the edge pa gives the measure $\mu_5(p,s_i)$. Since it is complicated, we omit the exact expressions.

3.3. Voronoi diagrams for various measures

Given a measure on triangles, we can define a Voronoi diagram. If it is regular on the measure, Voronoi vertices are individual points and Voronoi edges are curves without areas. Using the measures listed earlier except the last one on min-max eccentricity, we have a regular Voronoi diagram. An example of a Voronoi diagram is shown in Fig. 2 for three line segments in the plane with the measure on max-min visual angle. That is, a point belongs to a Voronoi region dominated by a line segment giving the smallest visual angle among given line segments.

We have assumed that our measures are defined by algebraic expressions of constant degrees. If we define a terrain for a line segment s_i by determining the height at a point p by the value $\mu(p,s_i)$, the terrain is represented by a constant number of algebraic functions, or more precisely the lower

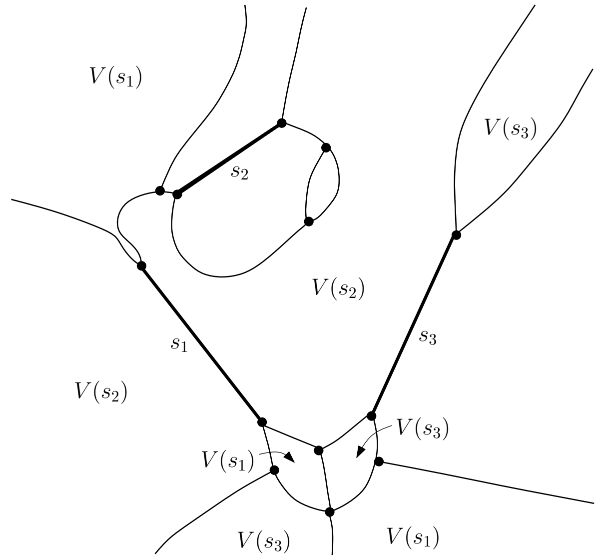

Figure 2. A Voronoi diagram for a set S of three line segments (shown by bold lines) under the measure of max-min visual angle. A point belongs to a Voronoi region dominated by a line segment giving the smallest visual angle.

envelope (or upper envelope, resp.) of constant number of algebraic functions for max-min type measures (min-max type measures, resp.). The Voronoi diagram is then defined by the lower envelope (or upper envelope, resp.) of n such terrains. The complexity of the envelope is known to be $O(n^{2+\varepsilon})$ by the analysis by Halperin and Sharir [6]. Thus, we have a theorem.

Theorem 1 *A regular Voronoi diagram associated with a measure μ on triangles for a set of n line segments consists of $O(n+2+\varepsilon)$ cells, edges, and vertices, where ε is an arbitrarily small positive constant.*

3.4. Level Region

Given a max-min type measure μ, we define a level region by

$$R_{>t}(s_i) \quad = \quad \{p \in \mathbb{R}^2 | \mu(p,s_i) > t\}.$$

That is, it consists of all points at which the measure is greater than some given value t. $R_{\geq t}(s_i)$ is similarly defined. Fig. 3 shows level regions for the first four measures listed above (although there is another region symmetric with respect to a line segment, only one of them is shown). For the measure μ_1 on max-min visual angle, the level region

$R_{\geq\theta}(s_i)$ is the interior of a circle on which the circular angle is exactly θ, as shown in Fig. 3(a). For the measure μ_2 on max-min height, $R_{\geq h}(s_i)$ above the line segment s_i is characterized by two lines each passing through an endpoint of the line segment and the line parallel to s_i separated by h from s_i. Thus, the region is an infinite region bounded by two rays and one line segment (which may be degenerated to a point). The measure μ_3 on max-min aspect ratio has the level region $R_{\geq\alpha}(s_i)$ bounded by two circular arcs and one line segment parallel to s_i. The gap between two parallel lines is $\alpha \times ||s_i||$ where $||s_i||$ is the length of s_i. The two circles determining the circular arcs have their center on lines perpendicular to s_i and passing through the two endpoints of s_i. The level region for the measure on min-max circumcircle is not convex. It is bounded by two circular arcs of the same radius and both passing through the two endpoints of s_i. As is easily seen, whenever a point p lies on the boundary, the circumcircle of the triangle $\triangle(p, s_i)$ is given by the circle shown in the figure.

For a line segment s_i and a real value $t > 0$, the level region $R_{\geq t}(s_i)$ appears in both sides of the line segment. In Fig. 3 we only illustrate one of the two regions since they are symmetric. Every boundary curve of the level region is described by a polynomial equation of constant degrees in x and y. We assume that a level region in one side is convex if it is defined. Otherwise we exclude such a measure from our considerations. Referring to Fig. 3, the first three measures give convex level regions while the level region is not convex for the fourth measure on min-max circumcircle. This is one reason why we do not include the measure. There is another reason to exclude the measure μ_4, which will be described later on.

3.5. Feasibility

We say a value t is *feasible* for a measure μ if there is a point p such that $\mu(p, s_i)$ is at least t for every element s_i of a given set S. In other words, if the following intersection is not empty:

$$\bigcap_{s_i \in S} R_{\geq t}(s_i) \neq \emptyset. \tag{3}$$

Now, consider the following problem:

Problem Given a set S of line segments and a measure μ, find a maximum feasible value t^* and a point p^* that achieves the value t^*, that is,

$$p^* \in \bigcap_{s_i \in S} R_{\geq t^*}(s_i). \tag{4}$$

What we are interested in here is whether an optimal point for a given measure μ lies on some edge of a Voronoi diagram associated with the measure μ. To investigate properties for this to hold, we introduce a few properties to be satisfied by measures.

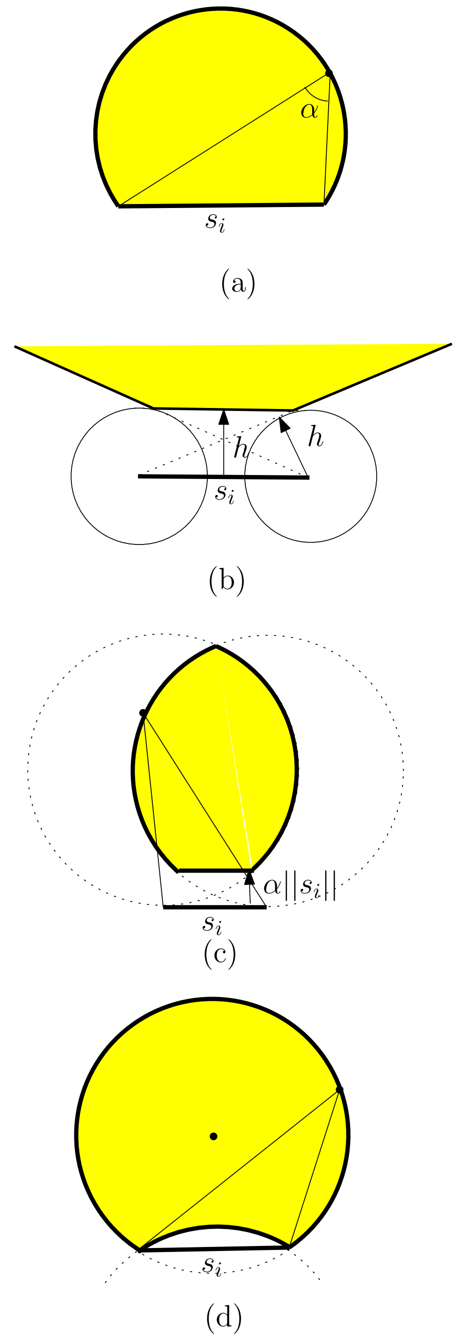

Figure 3. Level regions $R_{\geq t}(s_i)$ for four different measures: (a) max-min visual angle, (b) max-min height, (c) max-min aspect-ratio, and (d) min-max circumcircle.

3.6. Properties of measures

Strict Monotonicity If p is not a *local maximum* (or *peak*) on the measure μ, then there is a positive real number ε such that for any point p' in the neighborhood of p $\mu(p', s_i) > \mu(p, s_i)$ holds for any $s_i \in S$.

Peak Sharpness For any $s_i \in S$, a set of peaks (local maxima) does not form a region with positive area.

Single Peak Value All $s_i \in S$ have the same peak value. That is, $\max\{\mu(p, s_i) | p \in R^2\}$ is just the same independently of s_i.

Infiniteness of Peak Position For any $s_i \in S$, its peaks are located infinitely far away from s_i.

A point p is called a *peak* for a line segment s_i and a measure μ if there is a small real number $\varepsilon > 0$ such that there is no point p' in the ε-neighborhood of p that is strictly better than p on the measure μ.

Theorem 2 *If a measure* μ *on triangles satisfies*
(1) strict monotonicity,
(2) peak sharpness, and
(3) single peak value or (3') infiniteness of peak position,
then an optimal point on the measure μ *lies on an edge of a Voronoi diagram associated with* μ.

Proof: Suppose the property (3) is satisfied. For contradiction, suppose a peak p is contained in a Voronoi region $V(s_i)$. For any other s_j, we have $\mu(p, s_i) < \mu(p, s_j)$ by the definition of $V(s_i)$ if μ is of max-min type. However, since p is a peak and all peaks have a single peak value, say t^*, we have $t^* = \mu(p, s_i)$ and $t^* \geq \mu(p, s_j)$. Therefore, it contradicts to $\mu(p, s_i) < \mu(p, s_j)$. The proof is symmetric for a measure of min-max type.

If the property (3') is satisfied, then no Voronoi region within a given polygon cannot have a peak in its proper interior since the polygon is finite.

If we combine the property proved above and the conditions (1) and (2) in the theorem, we can prove that an optimal point must lie on Voronoi edges. □

Let us consider whether the measures listed above satisfy the conditions of the theorem. The first measure on max-min visual angle and the third one on max-min aspect-ratio satisfy the conditions (1), (2) and (3). The measure on max-min height satisfies (1), (2) and (3'). However, the fourth measure on min-max circumcircle does not satisfy (3) or (3'). The peak value for s_i is just the half of the length of the line segment since the circle with s_i as its diameter is the smallest circle passing through the two endpoints of s_i. Thus, two line segments having different lengths have different peak values. Thus, the condition (3) is not satisfied. Also, peaks are located on the smallest circles. Therefore, the condition (3') is not satisfied. In fact, in the Voronoi diagram associated with the measure shown in Fig. 4 an optimal point is not located on Voronoi edges.

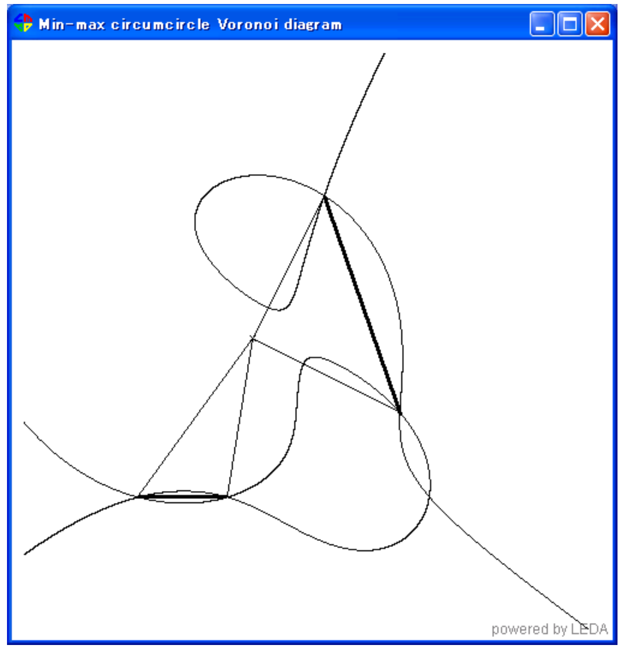

Figure 4. A Voronoi diagram associated with a measure on min-max circumcircle for two line segments (drawn by bold lines). An optimal point that minimizes the radius of the maximum circumcircles is depicted by a cross and connected with four endpoints of the two line segments. The optimal point does not lie on a Voronoi edge.

4. Concluding Remarks

In this paper we have presented a new family of Voronoi diagrams for a set of line segments or a polygon based on various measures on goodness of triangles. We have succeeded in characterizing their common combinatorial and structural properties. Unfortunately, our complexity result of $O(n^{2+\varepsilon})$ is not encouraging for practical applications, but this is just an upper bound on the worst case complexity. Since the worst case is not known, it may be possible to lower the complexity. More experimental works are required to judge whether this idea is useful for practical use, which is left for future work.

It is known in [8] that the problem of finding a point in a star-shaped polygon that maximizes the minimum visual angle when the point is connected to all the vertices of the polygon by straight edges is formulated as an LP-type problem and thus it can be solved by implementing $O(n)$ basic operations in the framework. So, it is more efficient than our approach based on the Voronoi diagram associated with the measure μ_1 on max-min visual angle. Although it is hard to describe in limited space, there is an application in which we are required to find a point that maximizes the smallest visual angle in a star-shaped polygon in some region bounded by some planar curves such as circular arcs. In such cases the diagram may be useful and expected to be efficient because we do not need the whole diagram but just a part of it in the restricted area.

Acknowledgment

The part of this research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research on Priority Areas. Work of T.A. was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (B).

References

[1] T. Asano, ”Aspect-ratio Voronoi Diagram with Applications,” Proc. International Symposium on Voronoi Diagram in Science and Engineering, Banff, Canada, pp.217-223, 2006

[2] T. Asano, N. Katoh, H. Tamaki and T. Tokuyama, “ Angular Voronoi Diagram with Applications, ” ibid, pp.32-39, 2006.

[3] F. Aurenhammer, ”Voronoi Diagrams – A Survey of a Fundamental Geometric Data Structure,” ACM Computing Surveys, 23, 3, pp.345-405, 1991.

[4] H. Edelsbrunner and N. Shah, ”Incremental topological flipping works for regular triangulations, ”Proc. of the 8th ACM Symp. on Computational Geometry 1992, pp.43–52, 1992.

[5] L. A. Freitag and P. Plassmann, ”Local optimization-based simplicial mesh untangling and improvement,” Int. J. for Numerical Methods in Engineering, 49, pp.109–125, 2000.

[6] D. Halperin and M. Sharir, ”Almost tight upper bounds for the single cell and zone problems in three dimensions,” Proc. Symp. on Comput. Geom., pp.11-20, 1994.

[7] ”LEDA: Library of Efficient Data Types and Algorithms,” Algorithmic Solutions Software, GmbH, Germany.

[8] J. Matoušek, M. Sharir, and E. Welzl, ”A Subexponential Bound for Linear Programming,” Algorithmica, 16, pp.498-516, 1996.

[9] S. A.Mitchell and S. A. Vavasis, ”Quality Mesh Generation in Three Dimensions,” Proc. of the Eighth Annual Symp. on Computational Geometry, pp212–221, 1992.

[10] A. Okabe, B. Boots, and K. Sugihara, ”Spatial Tessellations, Concepts and Applications of Voronoi Diagrams,” John Wiley & Sons, New York, 1992.

[11] J. R. Shewchuk, ”Delaunay refinement algorithms for triangular mesh generation,” Comput. Geom., Theory and Appl., 22, 1-3, pp.21-74, 2002.

[12] K. Shimada and D. C. Gossard, ”Bubble mesh: automated triangular meshing of non-manifold geometry by sphere packing,” Proc. of the Third Symposium on Solid Modeling and Applications, pp.409-419, 1995.