On-line uniformity of points

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A number of applications need uniformly distributed points over a specific domain. It is commonly known that randomly generated points are not always good enough, and therefore there are a number of studies in Discrepancy Theory for generation of uniformly distributed points over a specified region.

In this paper we first consider how to evaluate the uniformity of static points by introducing several different criteria for uniformity. Then, we extend it to evaluate uniformity to dynamic points. That is, the uniformity of a sequence of points is recursively defined by a uniformity measure of its subsequences.

Let S_0 be a set of 4 corner points of the unit square. Then, we insert n points $p_1, p_2, \ldots, p_n$ in order. Given a set of points $S_i = S_0 \cup \{p_1, \ldots, p_i\}$, its **static uniformity** is denoted by $\mu_s(p_1, \ldots, p_i)$, which measures the uniformity of a set of points $S_0 \cup \{p_1, \ldots, p_i\}$. Then, the **on-line uniformity** $\mu(p_1, \ldots, p_n)$ determined for a sequence of points $\langle p_1, \ldots, p_n \rangle$ is defined by

$$\mu(p_1, \ldots, p_i) = \min_{1 \le j \le i} \mu_s(p_1, \ldots, p_j) \tag{1}$$

Possible Uniformity Criteria

Here is a list of possible criteria for evaluating uniformity of a static point set S_i define above.

Measure by triangle areas Let $\mathcal{T}(S_i)$ be a family of all triangulations of S_i and let T be any such triangulation. Then, $\mu^{(1)}$ is defined by

$$\mu_s^{(1)}(p_1, \ldots, p_i; T) = \frac{\text{the area of the smallest triangle in } T}{\text{the area of the largest triangle in } T},$$
$$\mu^{(1)}(p_1, \ldots, p_i) = \min_{T \in \mathcal{T}(S_i)} \mu_s^{(1)}(p_1, \ldots, p_i; T).$$

Measure by triangle edge lengths The third measure $\mu^{(2)}$ is defined by

$$\mu_s^{(2)}(p_1, \ldots, p_i; T) = \frac{\text{the length of the shortest triangular edge in } T}{\text{the length of the longest triangular edge in } T},$$
$$\mu^{(2)}(p_1, \ldots, p_i) = \min_{T \in \mathcal{T}(S_i)} \mu_s^{(2)}(p_1, \ldots, p_i; T).$$

Measure by closest-point distances For each point $p \in S_i$, $d_{\min}(p, S_i)$ is the distance from p to its closest point in the set $S_i \backslash \{p\}$. Then, the first uniformity measure $\mu^{(3)}$ is defined by

$$\mu^{(3)}(p_1, \ldots, p_i) = \frac{\min_{p \in S_i} d_{\min}(p, S_i)}{\max_{p \in S_i} d_{\min}(p, S_i)}.$$

Measure by empty circles For a set S_i of points, a smallest empty circle is a circle of the smallest radius that does not contain any point of S_i in its proper interior but includes two points on the boundary. A largest empty circle is a circle of the largest radius with its center being in the convex hull of S_i that does not contain any point of S_i. Then, the fifth measure $\mu^{(4)}$ is defined by

$$\mu^{(4)}(p_1, \ldots, p_i) = \frac{\text{the diameter of a smallest empty circle for } S_i}{\text{the diameter of a largest empty circle for } S_i}. \tag{2}$$

Note that the diameter of a smallest empty circle coincides with the minimum pairwise distance among points in S_i.

Greedy algorithm

We can define a sequence of points $\langle p_1, \ldots p_n \rangle$ in such a way that p_i is a point that maximizes the measure $\mu_s(p_1, \ldots, p_i)$ when the $i-1$ points $p_1, \ldots, p_{i-1}$ are fixed. It is rather easy to implement this greedy algorithm, say in $O(n \log n)$ time. In each iteration we construct a Voronoi diagram for the current set of points. Then, we evaluate each Voronoi vertex within the unit square as the candidate for the next point. If a Voronoi edge intersects some square edge, then the intersection is also considered as a candidate. Among those candidates it chooses a point that achieves the best uniformity. The performance of the algorithm is not so bad. In fact, it is as good as that of an optimal algorithm for a large value of n.

Known Results

Some results are known for the problem. In one dimension, triangles cannot be defined. So, we use a notion of empty interval between two given points which contains no given point in its interior. Then, uniformity is defined by the ratio of the shortest interval over the longest interval. In that case, an exact bound on the uniformity is known [1]. That is, for any integer $n > 0$ there is a sequence of n points on a line with uniformity $2^{1/(\lfloor n/2 \rfloor + 1)}$ and also any sequence of n points has uniformity at least $2^{1/(\lfloor n/2 \rfloor + 1)}$. Such an optimal sequence can be computed in $O(n)$ time.

Comparison of Measures

Let us compare the measure listed above.

No explicit lower bound is known for the first measure $\mu^{(1)}$ on triangle areas, but we can easily achieve a ratio strictly better than that of the greedy algorithm. More precisely, we can apply the 1-d algorithm on a line to distribute points on one diagonal of a unit square. Note that triangulation for this point set is unique. This sequence achieves the uniformity $> 1/2$, but such a pattern does not look good.

The second measure $\mu^{(2)}$ on triangle edge lengths are not good either by a similar reason. If the first point is put on one of the square edges, then the ratio of the longest edge over the shortest edge in any triangulation must be at most $1/2$. So, to achieve the ratio strictly greater than $1/2$, the first point cannot be on any square edge. Similarly, the second point cannot lie on any square edge by the same reason. So, to achieve a better ratio, we cannot put points on any square edge. Thus, the square edges remain the longest edges in any triangulation. Thus, after some point we cannot avoid a short triangular edge $< 1/2$, which implies the uniformity $< 1/2$.

The third measure $\mu^{(3)}$ on pointwise distances and the fourth one $\mu^{(4)}$ on empty circles may be better. The greedy algorithm achieves the ratio $1/\sqrt{2}$ for the third measure and $1/2$ for the fourth measure. It is not trivial to find a sequence with strictly better ratio in either measure.

The greedy algorithm generates the same point sequence for all the measures listed above on the plane, but it is well characterized for the measure $\mu^{(4)}$ on empty circles since it repeatedly finds a largest empty circle and put its center as the next point.

An optimal point sequence for the measure is still not known. In [1] a reasonable sequence of length 50 is presented as an experimental result. We have invented a new heuristic algorithm and succeeded in improving the uniformity. The details will be presented in the symposia.

References

1. S. Teramoto, T. Asano, N. Katoh, and B. Doerr, " Inserting Points Uniformly at Eery Instance," IEICE Trans. INF. & SYST., Vol. E89-D, 8, pp.2348-2356, 2006.