Angular Voronoi Diagram with Applications

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Abstract

Given a set of line segments in the plane, we define an angular Voronoi diagram as follows: a point belongs to a Voronoi region of a line segment if the visual angle of the line segment from the point is smallest among all line segments. The Voronoi diagram is interesting in itself and different from an ordinary Voronoi diagram for a point set. After introducing interesting properties, we present an efficient algorithms for finding a point to maximize the smallest visual angle. Some applications to mesh improvement are also mentioned.

1 Introduction

Voronoi diagram has been one of the most important notions in computational geometry[1, 6]. A number of variations of a Voronoi diagram defined for a set of points has been presented for different applications and purposes. In this paper we propose a yet another Voronoi diagram using a visual angle of a line segment from a point and investigate combinatorial and structural properties of the diagram.

An original motivation of this Voronoi diagram comes from an application to mesh improvement. Mesh generation/improvement [2, 3, 5, 7, 8] is an important process for many purposes including Finite Element Method. In a simple setting, a given simple polygon is partitioned into many small triangles after inserting an appropriate number of points in its interior as vertices of triangular meshes. Several different criteria have been considered to evaluate the quality of such a triangular mesh. One of them is to maximize the smallest internal angle (or to minimize the largest internal angle). Since polygon vertices are fixed, the only way to improve the quality of triangular mesh is either to move internal points while fixing all other points or to insert new internal points (or even delete existing internal points). This paper presents an efficient algorithm using the above-stated angular Voronoi diagram for finding where to move an internal point so that the smallest internal angle among those associated with the point is maximized.

Our angular Voronoi diagram is defined using a visual angle of a line segment from a point. Given a set of non-intersecting line segments in the plane, a point belongs to a Voronoi region associated with a line segment giving the minimum visaul angle among those line segments. Such a Voronoi diagram is well-defined and has different, but interesting properties from those of the existing Voronoi diagrams. For a set of n non-intersecting line segments in the plane, a Voronoi region associated with one of them may be disconnected. In fact it may consist of many connected regions. A general theory on terrains by Halperin and Sharir [4] yields an upper bound $O(n^{2+\varepsilon})$ on the complexity of the whole Voronoi diagram, where ε is a small constant. In other words, the Voronoi diagram has $O(n^{2+\varepsilon})$ Voronoi edges, and vertices. Despite the high complexity in the worst case, actual complexity seems to be low by our experiments for a number of star-shaped polygons.

In our important application to mesh improvement we look for an optimal point in the interior of a star-shaped polygon resulting after removing an internal point from a triangular mesh. Such an optimal point to maximize the minimum visual angle can be well characterized by our angular Voronoi diagram. Following a natural expectation such an optimal point should be found at some Voronoi vertex. Unfortunately, it is not the case. However, we can also show that it suffices to examine all Voronoi vertices and edges to find an optimal point. A key is an observation that our objective function is unimodal on each Voronoi edge. Thus, once an angular Voronoi diagram is constructed, we can find an optimal solution in time proportional to the number of Voronoi vertices and edges.

This paper is organized as follows. In Section 2 we define an angular Voronoi diagram and investigate its combinatorial and structural properties. Section 3 describes an application problem to maximize the minimum internal angle. We present two algorithms, one using our angular Voronoi diagram and the other directly computing an optimal point by a parametric search technique. Section 4 describes an application to mesh improvement. Finally, Section 5 gives some concluding remarks together with some open problems and future works.

2 Angular Voronoi diagram

Given a line segment s and a point p in the plane, the visual angle of s from p is the angle formed by two rays emanating from p through two endpoints of s, and it is denoted by $\theta_p(s)$. Since we do not consider the direction of the angle, it is always between 0 and π.

Given a line segment s in the plane, it is well known that points giving the same visual angle of s form circular arcs touching the two endpoints of the line segment s. If the visual angle is less than $\pi/2$ then the circular arcs are major circular arcs while they are minor arcs if it is greater than $\pi/2$, as is seen in Figure 1. By $C(s,\alpha)$ we denote a set of points the visual angle from which is α, that is,

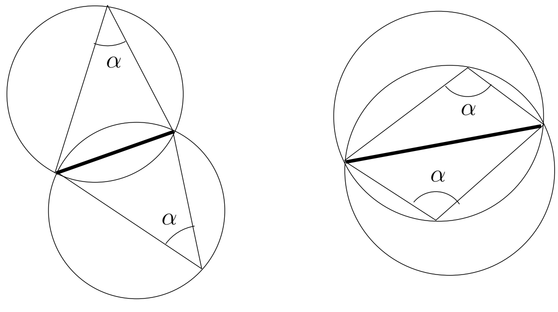
$$C(s,\alpha) = \{p \in \mathbb{R}^2 | \theta_p(s) = \alpha\}. \quad (1)$$

Given a set S of n line segments $s_1, s_2, \ldots, s_n$ and a point p in the plane, we define an angular Voronoi diagram $AVD(S)$ for S as follows:

Voronoi region: Each line segment s_i is associated with a region, called a Voronoi region $V(s_i)$, consisting of all point p such that the visual angle of s_i from p is smaller than that of any other line segment s_j. Formally, it is defined by

$$V(s_i) = \{p \in \mathbb{R}^2 | \theta_p(s_i) < \theta_p(s_j) \text{ for any } j \neq i\}.$$

Voronoi cell: A Voronoi region may not be connected. Connected parts of a Voronoi region are referred to as Voronoi cells to be distinguished from Voronoi regions.



(a) $\alpha < \pi/2$ (b) $\alpha \geq \pi/2$

Figure 1. Sets of points giving the same visual angle.

Voronoi edge: Voronoi edges form the boundary of Voronoi regions and thus they are defined for pairs of line segments:

$$E(s_i,s_j) = \{p \in \mathbb{R}^2 | \theta_p(s_i) = \theta_p(s_j) < \theta_p(s_k) \text{ for any } k \neq i,j\}.$$

Any point $p \in E(s_i,s_j)$ is given as intersection of two circles passing through the endpoints of s_i and s_j. Thus, a Voronoi edge is a planar curve.

Voronoi vertex: Voronoi vertices are points at which three or more Voronoi edges meet:

$$v(s_i,s_j,s_k) = \{p \in \mathbb{R}^2 | \theta_p(s_i) = \theta_p(s_j) = \theta_p(s_k) < \theta_p(s_l) \text{ for any } l \neq i,j,k\}.$$

A Voronoi vertex is given as intersection of three circles. Thus, it is a point unless there is any degeneracy.

An example of an angular Voronoi diagram is shown in Figure 2 for three line segments in the plane. Since for any point p on extension of a line segment s_i the visual angle of s_i from p is 0, i.e., $\theta_p(s_i) = 0$, extension of a line segment s_i is a part of the Voronoi cell for s_i. This observation makes it easier for readers to understand the figure.

One important difference from many other Voronoi diagrams is that Voronoi regions and Voronoi edges may consist of many disjoint connected parts. Figure 3 shows an example of an angular Voronoi diagram having quadratic complexity, which will be seen below.

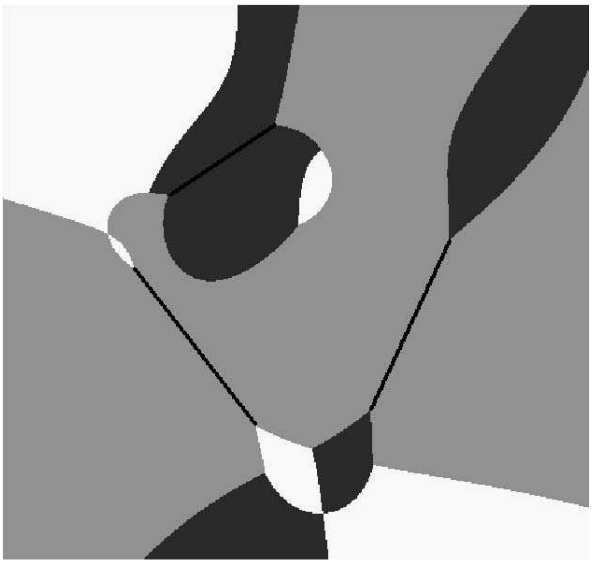


Figure 2. An angular Voronoi diagram $AVD(S)$ for a set S of three line segments (shown by black bold lines). A point belongs to a Voronoi region for a line segment giving the smallest visual angle.

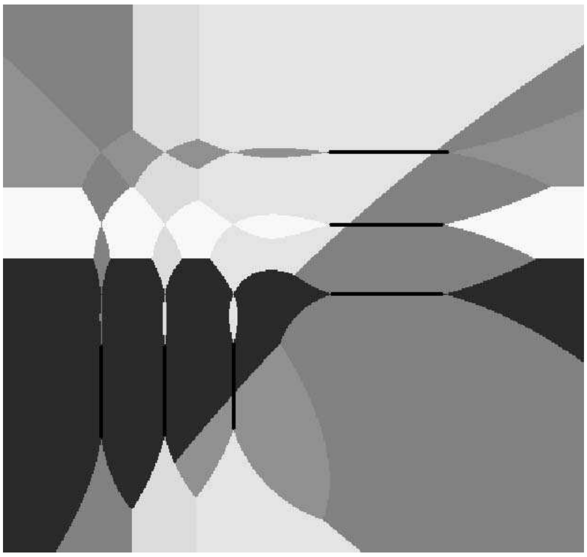


Figure 3. Worst case Voronoi diagram $AVD(S)$ for a set S of horizontal and vertical line segments (shown by black bold lines) having quadratic number of Voronoi edges and vertices.

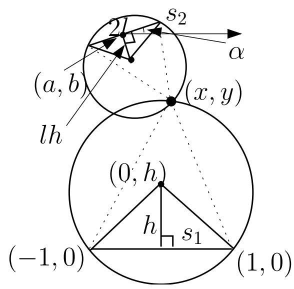


Figure 4. Intersection of two circles of the same visual angle for two line segments.

Equation for Voronoi edges

An angular Voronoi diagram AVD looks quite different from the existing ones for a point set or a set of line segments. Then, how is a Voronoi edge described?

Lemma 1 *Edges of an angular Voronoi diagram are described by polynomial curves of degree at most three.*

Proof: Each Voronoi edge is defined by a pair of lines segments. Let us consider one for a pair (s_1, s_2) of line segments. For simplicity we assume that s_1 is fixed on the x-axis between two points $(-1,0)$ and $(1,0)$ and s_2 of length $2l$ is specified by the coordinate (a,b) of its midpoint and a slope α as is illustrated in Figure 4.

Suppose we have two circles C_1 and C_2 giving the same visual angle for the line segments s_1 and s_2, respectively, and that they intersect each other at (x,y). The center of C_1 is on the vertical line $x = 0$. So, we assume that the center of C_1 is at $(0,h)$. Then, we have

$$x^2 + (y-h)^2 = 1 + h^2, \tag{2}$$

from which we have

$$h = \frac{x^2 + y^2 - 1}{2y}. \tag{3}$$

Because of similarity of the two triangles, it turns out that the center of the circle C_2 is given by $(a + lh\sin\alpha, b - lh\cos\alpha)$ and the radius by $l\sqrt{1+h^2}$. Therefore, we have

$$(x - a - lh\sin\alpha)^2 + (y - b + lh\cos\alpha)^2 = l^2(1+h^2),$$

which is simplified into

$$\begin{aligned}&(x-a)^2 + (y-b)^2 - l^2 \\ = \;& 2l\{(x-a)\sin\alpha - (y-b)\cos\alpha\}h. \end{aligned} \tag{4}$$

Substituting Eq.(3) into Eq.(4), we have

$$\begin{aligned}&y\{(x-a)^2 + (y-b)^2) - l^2\} \\ &-l\{(x-a)\sin\alpha - (y-b)\cos\alpha\}(x^2+y^2-1) \\ &= 0. \end{aligned} \tag{5}$$

This completes the proof of the lemma. □

Combinatorial properties

An angular Voronoi diagram $AVD(S)$ for a set S of n line segments has the following properties.

Lemma 2 *Each Voronoi region is partitioned into at least two connected parts if $n \geq 2$.*

Proof: We have $\theta_p(s_i) = 0$ for any point p on an extension of a line segment s_i. It implies that the Voronoi region for s_i contains the extensions of s_i in two opposite directions. Suppose we have only two line segments s_i and s_j, and assume that they are not parallel to each other. Then, the Voronoi region of s_i is partitioned into at least two parts by extensions of s_j. When there are more line segments, we can always find line segments playing the same role as s_j. □

For any point p in the plane, we define a function $\theta_{\min}(p,S)$, or simply $\theta_{\min}(p)$ if no confusion, by

$$\theta_{\min}(p) = \min\{\theta_p(s_1), \theta_p(s_2), \ldots, \theta_p(s_n)\} \qquad (6)$$

Each visual angle $\theta_p(s_i)$ is computed neglecting other line segments, that is, assuming that other line segments are transparent. A point p is called a *peak* if $\theta_{\min}(p)$ is largest at p, in other words, $\theta_{\min}(p) > \theta_{\min}(p')$ for any $p' \in \mathbb{R}^2, p' \neq p$.

Lemma 3 *Given a set S of line segments in the plane, a peak lies on some Voronoi edge of an angular Voronoi diagram for S.*

Proof: Suppose that a peak p lies in some Voronoi region $V(s_i)$ for contradiction. By the definition of a Voronoi region, we have

$$\theta_{\min}(p) = \theta_p(s_i) < \theta_p(s_j) \text{ for any } s_j \in S\backslash\{s_i\}. \qquad (7)$$

Since the Voronoi region is open by the definition and the function $\theta_p()$ is unimodal, there is a point p' in the neighborhood of p within the same cell $V(s_i)$ such that

$$\theta_p(s_i) < \theta_{p'}(s_i) < \theta_{p'}(s_j) \text{ for any } s_j \in S\backslash\{s_i\}.$$

This implies $\theta_{\min}(p') > \theta_{\min}(p)$, a contradiction. □

Lemma 4 *The function $\theta_{\min}(p)$ is convex in each Voronoi edge.*

Proof: Referring to Figures 1 and 4, a Voronoi edge $E(s_i,s_j)$ is defined by intersections of those two circles passing through the two endpoints of s_i and s_j which give the same visual angle for s_i and s_j. The visual angle is maximized when these two circles are tangent to each other, that is, meet at one tangential point. Otherwise, they intersect at two points, which naturally define an interval on the edge $E(s_i,s_j)$. Here, note that any point in the interior of the circle has a larger visual angle than a point on the circular boundary. This property guarantees that the function $\theta_{\min}(p)$ is convex on the edge. □

Theorem 5 *An angular Voronoi diagram for a set of n line segments in the plane consists of $O(n^{2+\varepsilon})$ cells, edges, and vertices, where ε is a small constant.*

Proof: Given a line segment in the plane, we can uniquely determine the visual angle of the line segment at each point. Using this angle as height at the point, we define a terrain. Given n line segments, they define n terrains and its associated Voronoi diagram is given as the lower envelope since each point belongs to a territory of a line segment giving the smallest visual angle. The theorem follows from the result by Halperin and Sharir [4]. □

It is not so hard to give a set of n points for which an angular Voronoi diagram has quadratic complexity, that is, $O(n^2)$ vertices, edges, and cells. In fact, if we arrange $n/2$ horizontal line segments and $n/2$ vertical line segments as shown in Figure quadratic, each Voronoi region for a horizontal line segment is divided into $O(n)$ pieces by extensions of the $n/2$ vertical line segments, thus yielding $O(n^2)$ Voronoi cells in total. We haven't obtained an example achieving the bound $O(n^{2+\varepsilon})$ in the theorem. However, if we restrict ourselves to a set of line segments forming a star-shaped polygon and also to its interior part, our experience tells us linear complexity. But there is no proof yet.

So far we have been interested in a point maximizing the smallest visual angle for a set of line segments. It is natural to define a symmetric notion of a point minimizing the largest visual angle. We could define a similar angular Voronoi diagram using this symmetric notion. If we have only two line segments, a point maximizing the smallest visual angle also minimizes the largest visual angle. But they are different in general for three or more line segments.

3 Finding a peak

Now we have a naive algorithm for finding a peak for a given set of line segments in the plane.

Naive Algorithm

(1) Construct an angular Voronoi diagram for a set of line segments.

(2) For each Voronoi edge, find its peak.

(3) Choose a point among those peaks and Voronoi vertices that gives the largest $\theta_{\min}()$ values as a solution.

Generally speaking, efficient construction of an angular Voronoi diagram is hopeless due to its high complexity. It consists of $O(n^{2+\varepsilon})$ disjoint Voronoi cells in the worst case for a set of n line segments. Fortunately, in our application to mesh improvement we have a set of line segments forming a star-shaped polygon instead of arbitrary set of line segments. Following our experience based on experiments, angular Voronoi diagrams seem to have linear complexity in the kernel of the starshaped polygon. This observation based on our experience is just an observation which remains unproved.

As was remarked earlier in this paper, Voronoi edges are characterized by plane curves of degree at most three. So, we need to solve a system of equation of two such degree-3 polynomial equations in x and y. If we can solve such a system of equations in time T and the angular Voronoi diagram consists of E edges, then the part of the angular Voronoi diagram within the kernel of an input star-shaped polygon of n edges can be constructed in $O(ET\log n)$ time. This also means that a peak can be found in $O(ET\log n)$ time.

Parametric search

There is another algorithm for finding a peak that maximizes the smallest visual angle based on parametric search technique. Given a line segment s in the plane, the visual angle of s from a point p is at least α if p lies in the region $C(s,\alpha) = \{p \in \mathbb{R}^2 | \theta_p(s) = \alpha\}$ defined by two circles passing through the two endpoints of s. It is bounded by major arcs of the circles if $\alpha < \pi/2$ and minor arcs otherwise. So, if we define a region $R_{\geq\alpha}(s)$ by

$$R_{\geq\alpha}(s) = \{p \in \mathbb{R}^2 | \theta_p(s) \geq \alpha\}, \tag{8}$$

it is bounded by the two circles and for any point p in the region the visual angle $\theta_p(s)$ is at least α.

Given a set of line segments $s_1, s_2, \ldots, s_n$, the value $\theta_{\min}(p)$ for a peak p is at least α if all of the regions $R_{\geq\alpha}(s_1), R_{\geq\alpha}(s_2), \ldots, R_{\geq\alpha}(s_n)$ have non-empty intersection. Unfortunately, $R_{\geq\alpha}(s_i)$ is not convex if $\alpha < \pi/2$. So, it is not so efficient to compute those intersections. A good news here is that if we are interested in the interior of a star-shaped polygon to find a peak then we can represent each region $R_{\geq\alpha}(s_i)$ by a disk instead of a region bounded by two circles since the part of the region in the kernel of the polygon is characterized by a single circle passing through the two endpoints of s_i. Then, the intersection of all of the regions $R_{\geq\alpha}(s_1), R_{\geq\alpha}(s_2), \ldots, R_{\geq\alpha}(s_n)$ is convex in the kernel and thus it is computed efficiently. In fact, we can find a peak by parametric search based on the following parallel algorithm for finding the convex intersection.

Parallel algorithm for finding circle intersection
Input: a set S of n circles $C_1, C_2, \ldots, C_n$.
Output: Intersection $meet(S)$ of all the disks, which is maintained by an alternating list of vertices and circular edges and all the vertices are sorted in the increasing x order. Note that the intersection is convex.
Divide-and-conquer algorithm
1. Divide a set S into two disjoint subsets S_1 and S_2 of almost equal sizes.
2. Compute $meet(S_1)$ and $meet(S_2)$ recursively.
3. Compute $meet(S) = meet(S_1) \cap meet(S_2)$.

The third step for merge is done in parallel as follows:
3.1 Apply a parallel merge algorithm for vertex lists of $meet(S_1)$ and $meet(S_2)$.
3.2 At each vertex find at most four circular arcs from $meet(S_1)$ and $meet(S_2)$ that intersect the vertical line passing through the vertex using a binary search.
3.3 For each interval between two adjacent vertices in the lists, compute an arrangement of those four arcs to find the corresponding intersection by finding new vertices.
3.4 Combine those pieces.

Lemma 6 *Given a set of n circles, the above parallel algorithm for finding their intersection is implemented in* $O(\log^2 n)$ *time using n processors.*

Theorem 7 *Given a star-shaped polygon P of n edges* $s_0, s_1, \ldots, s_{n-1}$, *a point (peak) in its kernel that maximizes the smallest visual angle* $\theta_{\min}(p) = \min\{\theta_p(s_0), \theta_p(s_1), \ldots, \theta_p(s_{n-1})\}$ *can be found in* $O(n\log^2 n)$ *time.*

Proof: Apply a parametric search algorithm based on the parallel algorithm above. □

4 Application to mesh improvement

The algorithms and notions developed in the paper can be applied to mesh generation, or more exactly mesh improvement. Figure 5(a) shows a triangulated mesh of a polygon using some internal points. Although external points are fixed and cannot be moved, internal points are usually free to move for better triangulation. Suppose the interior of a polygon of n vertices is partitioned into triangles using m internal points. Then, we have $O(m+n)$ internal angles. For each internal point p_i we associate the smallest internal angle incident to the point, which is denoted by $\alpha(p_i)$. Then, we evaluate the quality of the triangular mesh by

$$< \alpha(p_1), \alpha(p_2), \ldots, \alpha(p_n) > \tag{9}$$

in their dictionary order.

A greedy heuristic proceeds as follows;
(1) Let $< \alpha(p_1), \alpha(p_2), \ldots, \alpha(p_n) >$ be a list of smallest

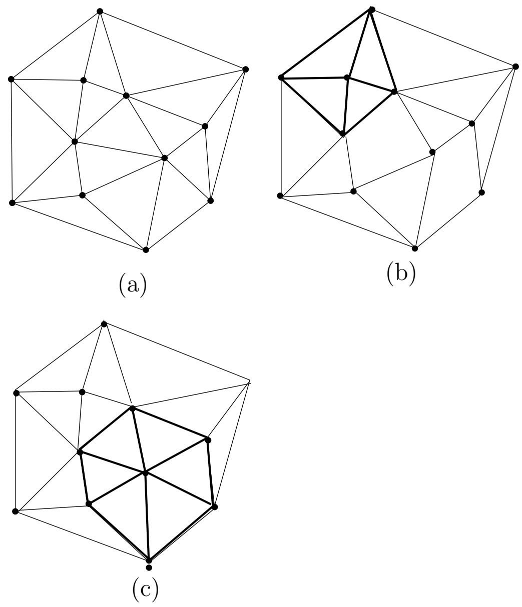


Figure 5. Improving triangular mesh by moving internal points.

angles associated with internal points. (2) Choose an internal point p_i with the least $\alpha(p_i)$.
(3) Remove all (internal) edges incident to p to form a star-shaped polygon P with p in its interior,
(4) Find the best location of p_i to maximize the minimum visual angle from p_i.

Figure 5(b) is a result after moving upper left internal point and Figure 5(c) is one after moving lower right internal point. Of course, we could extend this greedy algorithm so that it optimizes the second or third smallest angles if the smallest angle cannot be improved.

So far we have been interested in maximizing the smallest internal angle around an internal point. However, moving the point may cause a smaller internal angle along the boundary of the star-shaped polygon. Suppose a star-shaped polygon Phas a small internal angle at a vertex v_i. Then, we must be careful whether the smallest internal angle around the peak is not greater than the smaller internal angle at v_i resulting after connecting the peak to v_i. If this is the case, we should either make a structural change by drawing a chord connecting the two adjacent vertices (see Figure 6) or find a peak on the angular bisection through the vertex v_i.

5 Concluding Remarks

In this paper we have presented a new Voronoi diagram based on visual angle of a line segment from a point. Unfortunately, our complexity result of $O(n^{2+\varepsilon})$ is not encour-

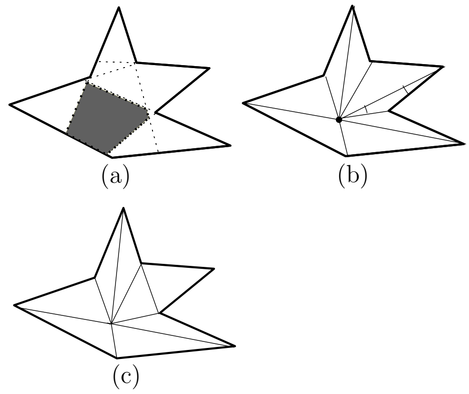


Figure 6. Triangulation of a star-shaped polygon with one internal point. (a) A kernel of the polygon from which any point in the polygon is visible. (b) Triangulation by connecting the internal point to all the vertices. (c) Another triangulation by removing an ear.

aging practical application, but this is just an upper bound on the worst case complexity. Since the worst case is not known, it may be possible to lower the complexity. We have also described an important application of our Voronoi diagram to mesh improvement. More experimental works are required to judge whether this idea is useful for practical use.

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