Aspect-Ratio Voronoi Diagram with Applications

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Abstract

This paper considers a problem of finding an optimal point within a polygon P in the sense that when we connect the point to every vertex of P by straight line then the worst aspect ratio among all resulting triangles is optimized. This problem has an important application to triangular mesh improvement. We propose three different approaches toward this problem. The first one is based on some new Voronoi diagram defined by an aspect ratio, which is interesting in itself. The second approach is essentially a binary search defined by geometric intersection. The third one is grid-based heuristic, which might be practically best but has no theoretical guarantee on its performance.

1 Introduction

Voronoi diagrams have been applied in many different areas and different purposes[1, 6]. In this paper we define a new Voronoi diagram for a set of line segments in the plane, which is called an aspect-ratio Voronoi diagram. This Voronoi diagram is characterized as follows: given a set of line segments $s_1, \ldots, s_n$ and a point p in the plane, we can define a triangle $Tr(p, s_i)$ for each line segment s_i by drawing lines from p to two endpoints of s_i. Now, we define an aspect ratio $asp(p, s_i)$ of the triangle $Tr(p, s_i)$ by the ratio of the longest side over the shortest side. In our criterion the smaller the ratio the better the quality of the triangle. Given a set of line segments in the plane, the plane is divided into so-called Voronoi regions each associated with one of the line segments. A point belongs to a Voronoi region $V(s_i)$ for the line segment s_i if s_i gives the worst (largest) aspect ratio among given line segments. Such a Voronoi diagram is well defined and it is quite interesting it itself.

Once we construct such an aspect-ratio Voronoi diagram for a set of line segments, using the diagram we can find a point p^* that minimizes (optimizes) the largest (worst) aspect ratio. This is not the only way to find such an optimal point. We could have a more direct algorithm. Fix some constant $\lambda \geq 1$. Then, for each line segment s_i we can determine a region of points at which the aspect ratio is at most λ. If there is non-empty intersection of all those regions, then a point in the intersection achieves a better aspect ratio. This suggests a numerical binary search on the best possible aspect ratio. Unfortunately, the binary search is not done in strongly polynomial time. It is also not so easy to compute the intersection precisely because the boundary of the above-mentioned regions are characterized by plane curves consisting of six different circular arcs. Thus, we have to decide which should be more emphasized numerical preciseness or reasonable running time with some approximation.

This Voronoi diagram has interesting properties, which are quite different from ordinary ones. First of all it looks quite different from ordinary Voronoi diagrams for points. In our case Voronoi edges consist of plane curves of degree-3 polynomial in x and y. A Voronoi region associated with a line segment is not always connected. It may be divided into a number of connected regions or cells. This fact leads to high complexity of the diagram.

So far we have mentioned two approaches for finding an optimal point to minimize the worst aspect ratio for a set of line segments, one based on an aspect-ratio Voronoi diagram and the other using numerical binary search. Unfortunately, either approach has some disadvantages. If we adopted the numerical binary search, we would have to compute intersection of n polygons with circular arcs. If we used our Voronoi diagram instead, we would have to find intersection of degree-3 curves. So none of them may yield a practically easy solution. Thus, we come up with a third approach for practical application. Although we have no theoretical analysis or bound on its time complexity, it is easy to implement and fast enough for our application to mesh improvement.

2 Problem Definition

In this paper we consider a problem related to mesh improvement [2, 3, 5, 7, 8] That is, given a triangular mesh of a polygon with a number of internal points, we want to improve the quality of the mesh by appropriately moving internal points, while fixing all other points, since adding a new internal point usually brings no advantage to improving the worst aspect ratio. Among many different criteria for evaluating triangular mesh, we define our aspect ratio for a triangle by the ratio of the longest side over the shortest side. In this paper we consider a problem of moving one internal point to optimize triangles incident to it, that is, to minimize the highest aspect ratio of triangles incident to the point.

Given a triangular mesh and one internal point p, we first remove all the triangles incident to p, which results in a star-shaped polygon P. Then, we look for a point p^* to interconnect to every vertex of the polygon P that optimizes the worst aspect ratio of those triangles in P.

In a more general setting, we could take a set of line segments which do not necessarily form a star-shaped polygon. They could be disjoint, but no proper intersection among line segments is allowed. Then, we want to find a point to optimize the worst aspect ratio. But, in this setting, if we connect the optimal point to all endpoints of the given line segments, the resulting triangles may have proper intersections. To prohibit intersection among those triangles, we must limit our search space to a region of points from which all the line segments are visible. If the set of line segments forms a star-shaped polygon, such a region must exist and it is called a kernel. If it forms a convex polygon, the whole interior part of the polygon is the kernel. So, in this paper we only consider a set of line segments forming a star-shaped polygon and search for a point in its kernel to optimize the worst aspect ratio.

Now, the goal in this paper is to solve the following problem:

Problem Given a star-shaped polygon $P = (s_0, s_1, \ldots, s_n = s_0)$ in the plane, find a point p that optimizes the worst value among $asp(p, s_0), asp(p, s_1), \ldots, asp(p, s_{n-1})$, where $asp(p, s_i)$ is an aspect ratio of the triangle defined by the point p and the line segment s_i.

3 Basic properties concerning aspect ratio

Given a line segment $s = \overline{p_1 p_2}$ and a point p in the plane, the aspect ratio $asp(p, s)$ of the triangle defined by connecting the two endpoints of s to p is defined by the ratio of the longest side over the shortest side of the triangle, i.e.,

$$asp(p,s) = \frac{\max\{|pp_1|, |pp_2|, |p_1 p_2|\}}{\min\{|pp_1|, |pp_2|, |p_1 p_2|\}}. \quad (1)$$

By the definition, an aspect ratio is at least 1 for any triangle and a triangle of an aspect ratio 1 is a regular triangle of three equal sides.

Given a line segment $s = \overline{p_1 p_2}$ in the plane, the plane is partitioned into regions by circles defined by the two endpoints and their perpendicular bisector, as shown in Figure 1. More precisely, we define two circles C_1 and C_2. The circle C_1 is centered at p_1 and passes through p_2. C_2 has its center at p_2 and p_1 on it. The perpendicular bisector of the two points p_1 and p_2 partitions the plane into two halfplanes. The one containing p_1 is denoted by $H(p_1, p_2)$ and the other by $H(p_2, p_1)$.

The two circles and the bisecting line creates 6 different regions. They are coded as R_{ijk} when $|s_i| \le |s_j| \le |s_k|$ for three sides s_i, s_j and s_k of a triangle. Here, $s_0 = s$, $s_1 = \overline{pp_1}$ and $s_2 = \overline{pp_2}$. The following is a list of those 6 regions with their associated aspect ratio.

R_{012}: $\overline{C_1} \cap \overline{C_2} \cap H(p_1, p_2)$. $asp(p,s) = d(p, p_2)/|s|$, where $|s|$ is the length of s.

R_{021}: $\overline{C_1} \cap \overline{C_2} \cap H(p_2, p_1)$. $asp(p,s) = d(p, p_1)/|s|$.

R_{102}: $C_1 \cap \overline{C_2} \cap H(p_1, p_2)$. $asp(p,s) = d(p, p_2)/d(p, p_2)$.

R_{201}: $\overline{C_1} \cap C_2 \cap H(p_2, p_1)$. $asp(p,s) = d(p, p_1)/d(p, p2)$.

R_{120}: $C_1 \cap C_2 \cap H(p_1, p_2)$. $asp(p,s) = |s|/d(p, p_1)$.

R_{210}: $C_1 \cap C_2 \cap H(p_2, p_1)$. $asp(p,s) = |s|/d(p, p_2)$.

In the following, by $R_{012}(p_1, p_2)$ we denote a region R_{012} for a line segment $s = \overline{p_1 p_2}$.

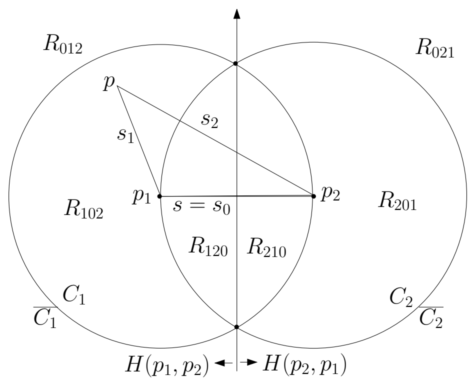

Figure 1. Partition of the plane with respect to an aspect ratio for a line segment in the plane.

Two intersections of the two circles and the bisector are special points that achieve $asp(p,s) = 1$. Intuitively, as a point goes away from the points, its corresponding aspect ratio gets worse. Figure 2 shows contour lines of the same aspect ratio. The lower figure is a closer look at the same picture. In those figures, if the worst aspect ratio at a point is in some interval then it is painted in a color associated with the interval. Now define a region $R_{\geq\lambda}(s)$ for a line segment $s = \overline{p_1 p_2}$. A point $p = (x, y)$ belongs to the region $R_{\geq\lambda}(s)$ if and only if $asp(p, s) \geq \lambda$. Such a region exists for any value $\lambda \geq 1$. As is seen in the figures, such a region is not always connected. As a special case, $R_{\geq 1}(s)$ consists of two points which are the two points achieving $asp(p, s) = 1$ in both sides of s. As λ increases, the region $R_{\geq\lambda}(s)$ also grows, forming two connected regions. When λ reaches 2, the two regions are merged into one connected region with two holes around the two endpoints of the line segment s. When λ gets further larger, the two holes get shrunk toward the endpoints while the external boundary formed by two big circles centered at two endpoints grow toward infinity.

As is seen in the figures, the boundary of each such region (or contour line) consists of at most 6 circular arcs. In fact, a contour line giving the same aspect ratio is a circular arc in each region listed above. For example, if a point $p = (x, y)$ lies in R_{012}, a contour line for $asp(p, s = \overline{p_1 p_2}) = \lambda$ is characterized by an equation $\sqrt{(x - x_2)^2 + (y - y_2)^2}/d_{12} = \lambda$, where (x_2, y_2) is the coordinate of an endpoint p_2 and d_{12} is the length of the line segment $s = \overline{p_1 p_2}$. Since d_{12} and λ are constants, it defines a circle. Similarly , if p lies in R_{102}, we have $\sqrt{(x - x_2)^2 + (y - y_2)^2}/\sqrt{(x - x_1)^2 + (y - y_1)^2} = \lambda$, which is again an equation of a circle if it is defined since λ is a constant.

Figure 3 illustrates how the plane is subdivided by those regions associated with two line segments. Consider the intersection between $R_{210}(p_1, p_2)$ and $R_{120}(p_3, p_4)$ as shown in the figure. Take any point $p = (x, y)$ in the intersection. The point gives the same aspect ratio to these two line segments if and only if the following equation has a solution:

$$\frac{d(p, p_2)}{d_{12}} = \frac{d(p, p_3)}{d_{34}}, \tag{2}$$

where d_{12} and d_{34} are the lengths of the segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, respectively, which are constants. With $p_2 = (x_2, y_2)$ and $p_3 = (x_3, y_3)$, we have

$$\frac{\sqrt{(x - x_2)^2 + (y - y_2)^2}}{d_{12}} = \frac{\sqrt{(x - x_3)^2 + (y - y_3)^2}}{d_{34}}, \tag{3}$$

which represents a circle if the constant part is positive and $d_{12} \neq d_{34}$, and a line if it is positive and $d_{12} = d_{34}$. In a degenerate case where $d_{12} = d_{34}$ it represents a line since the quadratic terms disappear.

On the other hand, any point in the intersection $R_{201}(p_1, p_2) \cap R_{102}(p_3, p_4)$ yields an equation

$$\frac{\sqrt{(x - x_2)^2 + (y - y_2)^2}}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} = \frac{\sqrt{(x - x_3)^2 + (y - y_3)^2}}{\sqrt{(x - x_4)^2 + (y - y_4)^2}}, \tag{4}$$

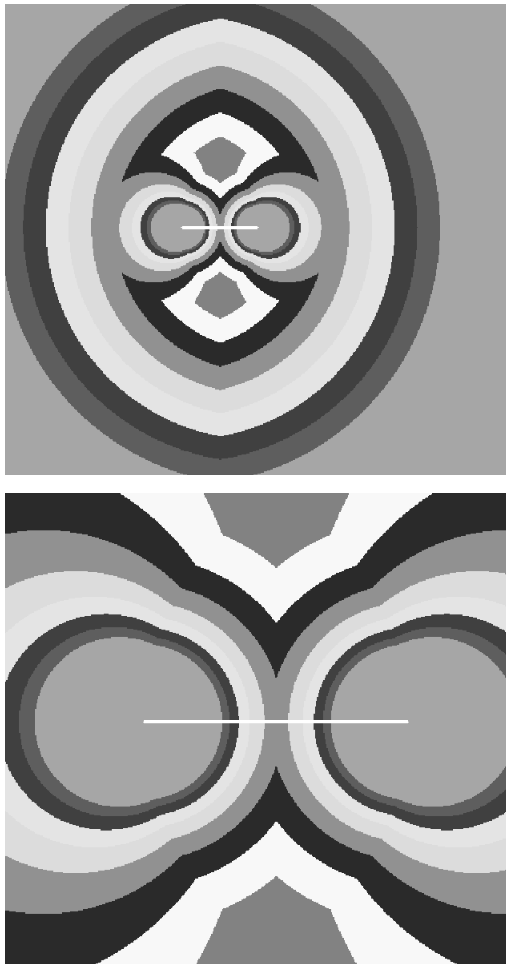

Figure 2. Contour lines defined by aspect ratios with different scales. Colors correspond to some interval of aspect ratios. For low aspect ratio, the corresponding region consists of two separate regions, each defined by six circular arcs. For high aspect ratio ≥ 1, the region is merged into one with two holes around two endpoints of the given line segment.

which is a curve of degree 3 since the terms x^4, x^2y^2 and y^4 disappear. In a degenerate case where $x_2 - x_1 = x_4 - x_3$ and $y_2 - y_1 = y_4 - y_3$ hold, it becomes a degree-2 curve, that is a hyperbola. By a careful analysis we have the following conclusions.

Lemma 1 *For two nonintersecting line segments in the plane, a trace of points giving the same aspect ratio for them is either a circular arc or a degree-3 curve (a line or a hyperbola in degenerate cases, respectively).*

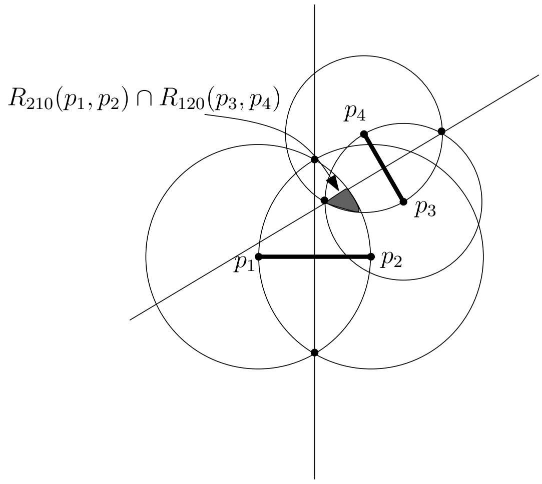

Figure 3. Subdivision of the plane by regions associated with two line segments by four circles and two bisecting lines determined by the two line segments.

Figure 4 illustrates how the entire plane is partitioned into regions. Two different colors are assigned to two line segments. When one of them gives worse (larger) aspect ratio for a point, the point is painted in a color assigned to the line segment. The boundary between different regions are circular arcs or plane curves of degree 3 (if no degeneracy) characterized by Eq.(4).

4 Aspect-ratio Voronoi diagram

Now we are ready to define an aspect-ratio Voronoi diagram for a given set S of line segments $s_1, s_2, \ldots, s_n$ in the plane. We define an aspect-ratio Voronoi diagram in such a way that a point p belongs to a Voronoi region associated with a line segment s_i if and only if s_i gives the worst (largest) aspect ratio, that is,

$$asp(p, s_i) \geq asp(p, s_j) \text{ for any } j \neq i. \tag{5}$$

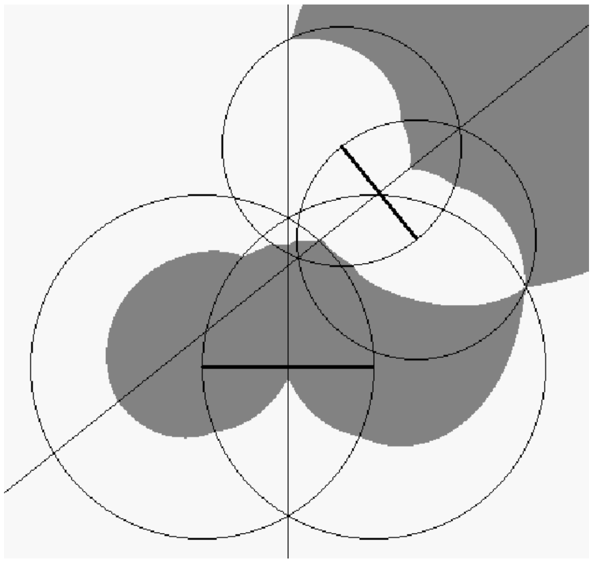

Figure 4. Subdivision of the plane by aspect ratios, which are defined by two circles and one bisecting line for each line segment. Each point is painted in a color assigned to a line segment which gives a larger aspect ratio.

This also implies that a Voronoi region $V(s_i)$ for s_i is defined by

$$V(s_i) = \{p \in \mathbb{R}^2 | asp(p, s_i) \geq asp(p, s_j) \text{ for any } j \neq i\}. \tag{6}$$

Each Voronoi region is bounded by curves at which two line segments give the same aspect ratio, which are either circular arcs or degree-3 curves defined earlier (lines and hyperbolas if any degenracy). Endpoints or intersection of those curves (referred to as primitive curves, hereafter) are Voronoi vertices and those primitive curves joining two such vertices are Voronoi edges. A minimal region bounded by Voronoi edges is called a Voronoi cell. Every Voronoi cell is associated with a line segment, but the reverse is not always true. That is, a Voronoi region for a line segment may be divided into many Voronoi cells.

The partition of the plane into Voronoi cells is called an aspect-ratio Voronoi diagram for the set of line segments. Two such Voronoi diagrams are shown in Figure 5, one for a triangle and the other for a convex polygon. In each case Voronoi regions are distinguished by colors each defined for a line segment. The lower Voronoi diagram contains disconnected Voronoi regions.

Now, let us analyze the combinatorial complexity of our Voronoi diagram for n line segments. First recall that an aspect ratio $asp(p)$ at a point p is defined as the largest value among aspect ratios for all line segments, that is,

$$asp(p) = \max\{asp(p,s_1), asp(p,s_2), \ldots, asp(p,s_n)\}. \quad (7)$$

Since $asp(p,s_i)$ is defined at any point p, if we regard the value as the height at the point, the function $asp(p,s_i)$ is a terrain. We have n terrains and an aspect ratio $asp(p)$ at a point p is given by the largest value among them. Thus, our aspect-ratio Voronoi diagram is an upper envelope of those n terrains. The result by Halperin and Sharir [4] gives a bound $O(n^{2+\varepsilon})$ for any small positive constant ε. Thus, the complexity of our aspect-ratio Voronoi diagram is $O(n^{2+\varepsilon})$. Fortunately, we can improve the bound to $O(n^2)$ in the following theorem.

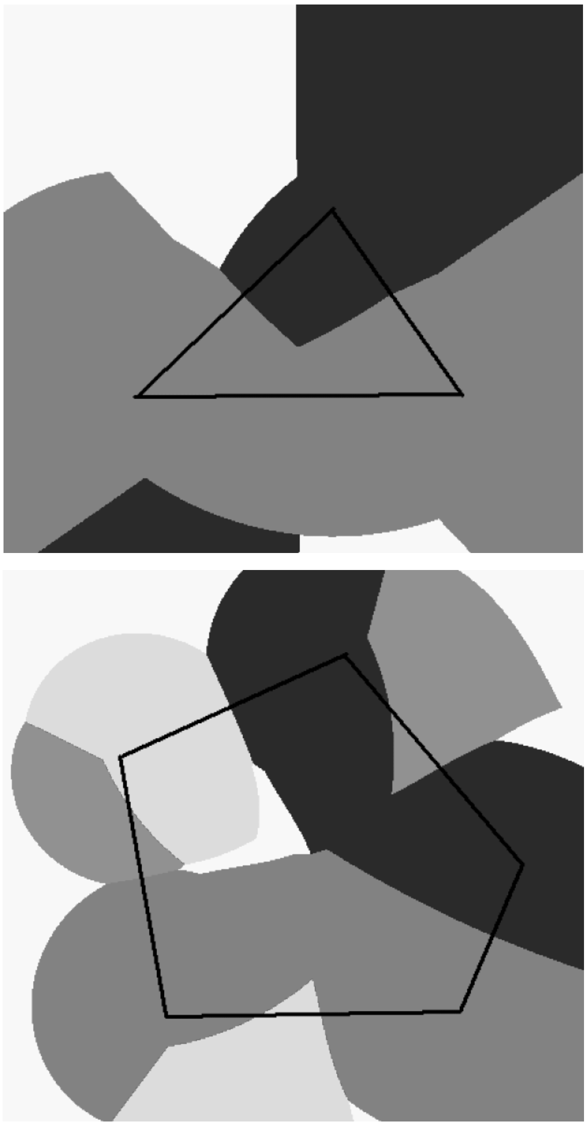

Figure 5. Two aspect-ratio Voronoi diagrams. In general, the shortest side has the largest Voronoi region.

Theorem 2 *An aspect-ratio Voronoi diagram for a set of n lines in the plane consists of $O(n^2)$ Voronoi vertices, edges, and cells.*

Proof: As stated above, an aspect ratio is given as an upper envelope of n terrains. We further decompose each terrain. Recall that an aspect ratio for a point p and a line segment $s = (p_1, p_2)$ is defined by the ratio of the longest side over the shortest side. In other words, it is the maximum among six ratios between two sides of a triangle $Tr(p,s)$. Therefore, we have

$$\begin{aligned} asp(p,s) \quad = \quad & \max\{\frac{d(p,p_1)}{|s|}, \frac{d(p,p_2)}{|s|}, \frac{|s|}{d(p,p_1)}, \frac{|s|}{d(p,p_2)}, \\ & \frac{d(p,p_2)}{d(p,p_1)}, \frac{d(p,p_1)}{d(p,p_2)}\}, \end{aligned}$$

where $|s|$ is the length of s and $d(p,q)$ is the distance between two points p and q. If we consider the above six terms separately, two of them are convex cylinders with the bottom at endpoints of s, other two of them are concave cylinders with the top again at endpoints of s, and the the remaining two are not cylinders but concave with infinite height at endpoints of s. So, we have $2n$ convex terrains and $4n$ concave terrains and the upper envelope of those terrains gives us an aspect-ratio Voronoi diagram. The complexity of the upper envelope of $4n$ concave terrains is $O(n^2)$ and so is that of $2n$ convex terrains. Thus, the combined upper envelope also has the complexity $O(n^2)$. □

Lemma 3 *Given an aspect-ratio Voronoi diagram for a set of n line segments in the plane, an optimal point to optimize the worst aspect ratio is found either on Voronoi edges or at Voronoi vertices. There are only a constant number of peaks on each Voronoi edge.*

Proof: A Voronoi edge is a primitive curve defined by two line segments. Due to the shape of contour lines consisting of 6 circular arcs, there are only a constant number of peaks on the edge. If a point lies in the interior of a Voronoi cell, we can always find a better point in its neighborhood. □

Given a set of n line segments in the plane, we can construct an aspect-ratio Voronoi diagram in $O(n^3)$ time. It looks quite slow and there may be a more efficient algorithm, but it also looks hopeless to have an algorithm running in subquadratic time in the worst case. In our case our interest is only on a part of the Voronoi diagram in the interior of a given star-shaped polygon. It is not known whether the diagram is simpler in the interior. A number of experimental results indicate linear complexity. So, a naive algorithm may be good enough for practical application (but no theoretical guarantee at present).

5 Direct approach for finding an optimal point

Our goal here was to find a point p^* with the minimum $asp(p)$ value. We can find such an optimal point in a more direct fashion. It is basically a binary search on the optimal value $asp(p^*)$. Fix a constant $\lambda \geq 1$. Then, for each line segment we can calculate a region $R_{\geq\lambda}$ of points p such that $asp(p) \geq \lambda$. When we restrict our search space to the interior of a given star-shaped polygon P, for any constituent edge s_i the intersection of $R_{\geq\lambda}(s_i)$ with the interior of P always consists of one connected region. If all those regions have non-empty intersection, that is, if

$$R_{\geq\lambda}(s_0) \cap \cdots \cap R_{\geq\lambda}(s_{n-1}) \neq \emptyset \tag{8}$$

then we can conclude that the optimal value $asp(p^*)$ is not greater than λ. Otherwise, that is, if the intersection is empty, it is greater than λ. This enables a binary search. Although the binary search does not give us an algorithm running in strongly polynomial time, parametric search leads to a strongly polynomial time. Since the detail of an implementation of a parallel algorithm is so complicated due to non-convexity of a region $R_{\geq\lambda}(s_i)$, we just suggest possibility of a polynomial-time algorithm.

6 Practically reasonable algorithm

So far we have proposed two different approaches to find an optimal point to minimize the worst aspect ratio. Unfortunately, however, none of the two algorithms seems to be practical. In many applications we do not insist on an exact solution. In such a case we could use a heuristic or approximation algorithm. One way is to distribute many points in the kernel of a given star-shaped polygon and then evaluate those points p by their $asp(p)$ values. Choose the k best points among them (k should be say something like one fourth of the number of those points). Construct the convex hull of those points and limit our search only to the interior of the convex hull. We iterate this search some certain number of times until the search space becomes sufficiently small. Then, take the best point as a solution. This is a heuristic algorithm.

7 Another Definition of Aspect Ratio

There is another way of defining an aspect ratio, which might be more popular than the one above. A common equation for an area of a triangle is to multiply the base times the one-half the height. If we choose the longest side as the base, the aspect ratio is defined by the ratio of the longest side over the corresponding height, that is,

$$asp(T) = \frac{\text{longest side}}{\text{corresponding height}} \tag{9}$$

Suppose we are given a line segment s between two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$. An arbitrary point $p(x, y)$ in the plane defines a triangle $Tr(s, p)$ by connecting p to the two endpoints of s. Let L be the length of the longest side of $Tr(s, p)$ and h be the corresponding height. Recall that the area S of $Tr(s, p)$ is given by

$$\begin{aligned} 2S &= |x(y_1 - y_2) + x_1(y_2 - y) + x_2(y - y_1)| \\ &= |(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|, \end{aligned} \tag{10}$$

which is linear in x and y. Since $2S = Lh$, we have

$$asp(s, p) = \frac{L}{h} = \frac{L^2}{2S}. \tag{11}$$

Thus, we have three cases:

Case 1: pp_1 is the longest side

L^2 is given by $(x - x_1)^2 + (y - y_1)^2$, and thus we have

$$asp(s, p) = \frac{(x - x_1)^2 + (y - y_1)^2}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|}. \tag{12}$$

Case 2: pp_2 is the longest side

L^2 is given by $(x - x_2)^2 + (y - y_2)^2$, and thus we have

$$asp(s, p) = \frac{(x - x_2)^2 + (y - y_2)^2}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|}. \tag{13}$$

Case 3: p_1p_2 is the longest side

L^2 is given by $(x_1 - x_2)^2 + (y_1 - y_2)^2$, which is a constant, say C, and thus we have

$$asp(s, p) = \frac{C}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|}. \tag{14}$$

From the above observations it turns out that a contour line of the same aspect ratio is a circle in the cases 1 and 2 and a line in the case 3.

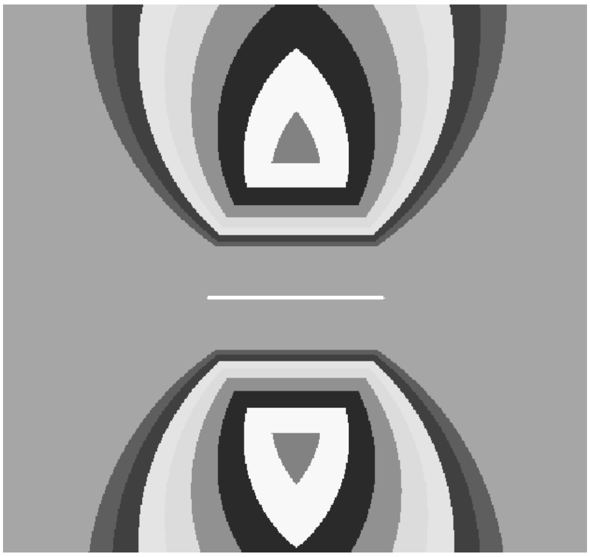

Figure 6. Contour lines defined by aspect ratios.

Figure 6 illustrates contour lines of different aspect ratios. As is easily seen, fixed an aspect ratio, points of the aspect ratio are either on circular arcs or on a straight line segment. In fact, for any constant λ

$$asp(s,p) = \frac{(x-x_1)^2+(y-y_1)^2}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} = \lambda \quad (15)$$

gives a circle and

$$asp(s,p) = \frac{C}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} \quad (16)$$

gives a line.

Using the aspect ratio, we can also define a Voronoi diagram as before. Then, how can we characterize Voronoi edges? We have seen that Voronoi edges for aspect ratios defined by longest and shortest sides are given by circular arcs and degree-3 curves. The circular arcs may be replaced with lines and the degree-3 curves may be with hyperbolas if any degeneracy. For the aspect ratio using longest side and height Voronoi edges are either lines or degree-3 curves and there are no degenerate cases.

Lemma 4 *For two nonintersecting line segments in the plane, a trace of points giving the same aspect ratio for them is either a line or a degree-3 curve.*

Proof: Voronoi edges are characterized by equations

$$\begin{aligned}
&\frac{(x-x_1)^2+(y-y_1)^2}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} \\
&= \frac{(x-x_3)^2+(y-y_3)^2}{|(y_3-y_4)x+(x_4-x_3)y+x_3y_4-x_4y_3|}, \\
&\frac{(x-x_1)^2+(y-y_1)^2}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} \\
&= \frac{C}{|(y_3-y_4)x+(x_4-x_3)y+x_3y_4-x_4y_3|}, \\
&\frac{C}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} \\
&= \frac{C}{|(y_3-y_4)x+(x_4-x_3)y+x_3y_4-x_4y_3|}.
\end{aligned}$$

The first and second equations give degree-3 curves and the third one gives a line. □

Figure 7 shows two Voronoi diagrams using the new aspect ratio.

8 Concluding Remarks

We have presented three different approaches to find a point that optimizes the worst aspect ratio of triangles resulting by inserting a point. The first two algorithms run in polynomial time and find an optimal point, but the last one does not. Nevertheless, the last one seems to be the best among three for practical use. So, one important future work is to establish a theoretical guarantee on the performance of the third algorithm. It is also interesting to improve the performance of the first two algorithms. It is also interesting to use a different definition for aspect ratio.

Acknowledgment

The part of this research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research on Priority Areas. The author would like to thank Kiyoshi Kumahata of JAIST for his valuable suggestions. The author also would like to thank Oswin Aichholzer, Franz Aurenhammer, and Thomas Hackl for their discussions.

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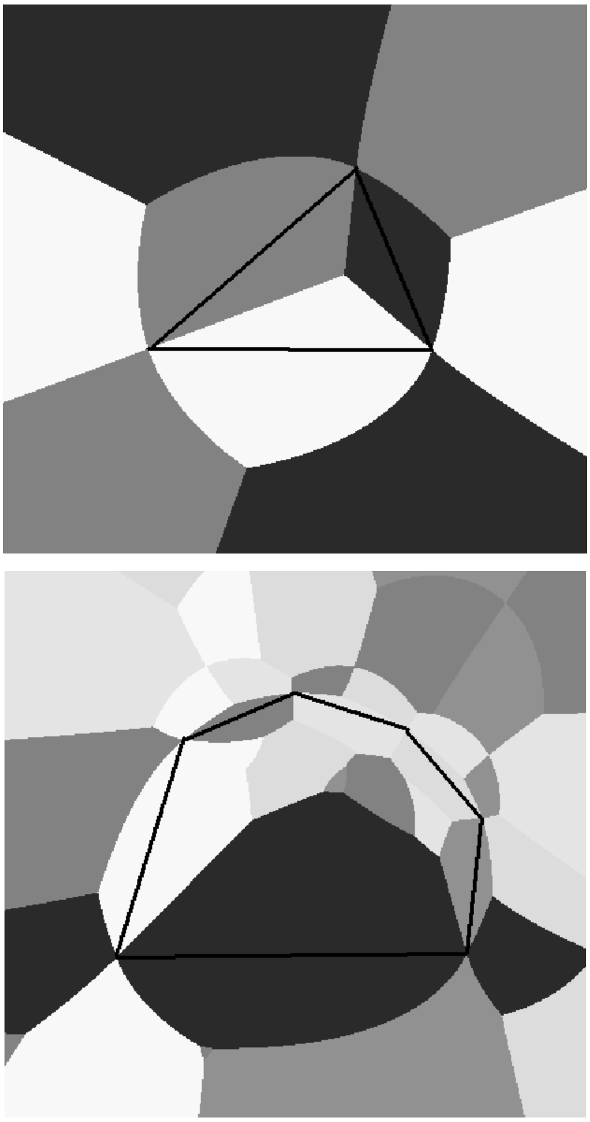

Figure 7. Two aspect-ratio Voronoi diagrams.

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