Distributing Distinct Integers Uniformly over a Square Matrix with Application to Digital Halftoning

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Abstract

This paper considers how to distribute n^2 integers between 0 and n^2-1 as uniformly as possible over an $n \times n$ square matrix. We introduce a discrepancy-based measure to evaluate the uniformity. More precisely, we take a sum of matrix elements over every $k \times k$ contiguous submatrix and define the discrepancy of the matrix as the largest difference among those sums. It is known that if n and k are both even integers then we can construct zero-discrepancy matrices. In this paper we present a scheme for achieving a new discrepancy bound $2n$ when n is odd and k is 2. This is an improvement from the previous bound $4n$. We borrow basic ideas behind orthogonal Latin squares and semi-magic squares. An n-ary number system also plays an important part.

This problem is closely related to digital halftoning. Low discrepancy matrices would improve the quality of commonly used Ordered Dither Algorithm.

1 Introduction

Digital Halftoning is an important technique for the rendition of continuous-tone pictures on displays that can only produce two levels. There have been a great number of methods for digital halftoning. One of the most popular methods is *Ordered Dithering* which determines an output level at each pixel by comparison with a threshold in a predetermined table called *Dither Matrix*. The performance of the algorithm heavily depends on the Dither matrix.

A Dither matrix is an $n \times n$ square matrix containing integers $0, \ldots, n^2-1$. It is good when those integers are *uniformly* distributed. To evaluate the uniformity we introduce a discrepancy-based measure. More precisely, we take a sum of matrix elements over every $k \times k$ contiguous submatrix (region) and define the discrepancy of the matrix as the largest difference among those sums. This measure reflects human eye perception usually modeled using weighted sum of intensity levels with Gaussian coefficients over square regions around each pixel [2]. It is known by experience that a matrix with low discrepancy frequently produces good-looking pictures. This is the reason why we are interested in finding a good matrix with low discrepancy.

The analogous geometric problem of distributing n points uniformly in a unit square has been studied extensively in the literature [6, 9]. Usually, a family of regions is introduced to evaluate the uniformity of a point distribution. If the points of an n-point set P are uniformly distributed, for any region R in the family the number of points in R should be close to $\frac{1}{n}$ area(R), where $\frac{1}{n}$ is the point density of P in the entire square. Thus, the *discrepancy of P in a region R* is defined as the difference between this value and the actual number of points of P in R. The *discrepancy of the point distribution P* with respect to the family of regions is defined by the maximum such difference, over all regions. The problem of establishing discrepancy bounds for various classes of regions has been studied extensively [7]. One of the simplest families is that of axis-parallel rectangles for which $\Theta(\log n)$ bound is known [6, 9].

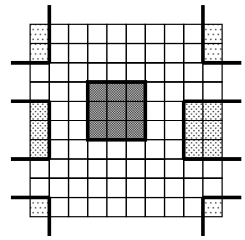

Figure 1: 3×3 contiguous submatrices (regions).

For the problem of establishing discrepancy bounds for families of regions (contiguous submatrices), some preliminary observations are obtained in [1]. One basic observation is that we can construct an $n \times n$ matrix of zero-discrepancy for a family of 2×2 regions if n is even. A space-efficient algorithm is also presented in [1] for constructing a $k^m \times k^m$ matrix of zero-discrepancy for a family of $k \times k$ regions. More precisely, given any matrix index (i, j) we can compute the corresponding matrix element of the index in constant time using only $O(k^2)$ working space instead of $O(k^{2m})$ required to store an output matrix. It is also shown in [1] that zero-discrepancy cannot be achieved if n is odd and k is even, and only trivial bound has been obtained for the discrepancy in that case. In this paper we present a new scheme for achieving a new discrepancy bound $2n$ when $n \geq 5$ is odd and k is 2. This is an improvement from the previous bound $4n$ [1]. Basic tools and theories are orthogonal Latin squares, semi-magic squares, and the n-ary number system.

2 Preliminary Definitions

For integers $n > 1$, let $\mathbb{Z}_n(n)$ be the class of all $n \times n$ integer matrices such that all the integers ranging from 0 to $n-1$ are included exactly n times and let $\mathbb{Z}_1(n)$ be that of all $n \times n$ matrices which contain every value $0, \ldots, n^2 - 1$ exactly once. In this paper we only deal with square matrices consisting of an odd number of rows (and columns) unless otherwise specified.

Given an $n \times n$ matrix $P = (p_{i,j})$ and a region $R \subseteq \{0, \ldots, n-1\}^2$, $P(R)$ denotes the sum of the elements of P in locations given by R, i.e., $P(R) = \sum_{(i,j)\in R} p_{i,j}$. A *contiguous* $k \times k$ *region* $R^{(k)}_{i,j} \subseteq \{0, \ldots, n-1\}^2$ with its upper left corner at (i, j) is defined by $R^{(k)}_{i,j} = \{(i', j') \mid i' = i, \ldots, i+k-1 \text{ and } j' = j, \ldots, j+k-1\}$, where indices are calculated modulo n. [1] The $k \times k$-*discrepancy* of an $n \times n$ matrix P for the family $\mathcal{F}_{k,n}$ of all $k \times k$ regions is defined as

$$\max_{R\in\mathcal{F}_{k,n}} P(R) - \min_{R'\in\mathcal{F}_{k,n}} P(R').$$

Let $\mathbb{N}(k, n)$ be the set of all such zero-$k \times k$-discrepancy matrices of order (k, n).

Theorem 1 *[1] The set $\mathbb{N}(k, n)$ of zero-$k \times k$-discrepancy matrices of order (k, n) has the following properties:*

(a) $\mathbb{N}(k, n)$ is non-empty if k and n are both even.

(b) $\mathbb{N}(k, n)$ is empty if k and n are relatively prime.

(c) $\mathbb{N}(k, n)$ is empty if k is odd and n is even.

(d) $\mathbb{N}(k, k^m)$ is non-empty for any integers k and m, $k \geq 2, m \geq 2$.

[1] Throughout this paper, index arithmetic is performed modulo matrix size n unless otherwise noted.

It follows from the theorem that zero-discrepancy cannot be achieved in a basic case of n odd and k even. In this paper we consider how much we can reduce the discrepancy of such a matrix in this basic case. One simple question is whether we can achieve a $\Theta(\log n)$ bound as in geometric discrepancy problems.

3 Basic Construction Schemes

A goal here is to design a low discrepancy matrix. We begin with some basic schemes for constructing matrices having some nice properties.

3.1 n-Ary Number System

¿From now on, we assume that n is an odd number not less than 3, and our target $n \times n$ matrix $C \in \mathbb{Z}_1(n)$. Since all such integers ranging from 0 and $n^2 - 1$ can be represented by two digits in the n-ary number system, we can associate to each $C \in \mathbb{Z}_1(n)$ two square matrices A and B representing upper and lower digits. That is, each element $c_{i,j}$ of the matrix C is given by

$$c_{i,j} = n \times a_{i,j} + b_{i,j}, \ \ 0 \le a_{i,j}, b_{i,j} < n, \ \ i,j = 0, 1, \ldots, n-1.$$

Observe that $C \in \mathbb{Z}_1(n)$ if and only if the two matrices A and B are in the class $\mathbb{Z}_n(n)$ and are mutually orthogonal, that is, no ordered pair $(a_{i,j}, b_{i,j})$ occurs more than once. So, we need schemes for generating two mutually orthogonal matrices in the class $\mathbb{Z}_n(n)$.

In the following, we first propose two schemes called *Alternating Diagonal Sequencing* and *Diagonal Repeating*, and then propose another scheme obtained based on the first two scheme.

3.2 Alternating Diagonal Sequencing

Alternating Diagonal Sequencing is a scheme for generating a matrix $A_n = (a_{i,j})$ in the class $\mathbb{Z}_n(n)$ as follows.

$$a_{i,j} = \begin{cases} i & \text{if } i+j \text{ is odd,} \\ n-1-i, & \text{otherwise.} \end{cases}$$

Recall that $R^{(2)}_{i,j} = \{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\}$ is the 2×2 contiguous region on an $n \times n$ matrix, and $A_n(R^{(2)}_{i,j})$ is the sum of elements of A_n in the region $R^{(2)}_{i,j}$, that is,

$$A_n(R^{(2)}_{i,j}) = a_{i,j} + a_{i+1,j} + a_{i,j+1} + a_{i+1,j+1}.$$

By definition and the fact that n is odd, we have

$$a_{i,j} + a_{i,j+1} = \begin{cases} 2i & \text{if } i \text{ is odd and } j = n-1, \\ 2n-2-2i & \text{if } i \text{ is even and } j = n-1, \\ n-1 & \text{otherwise.} \end{cases}$$

Hence, we have

$$A_n(R^{(2)}_{i,j}) = \begin{cases} 2n & \text{if } i < n-1 \text{ is even and } j = n-1, \\ 2n-4 & \text{if } i \text{ is odd and } j = n-1, \\ 2n-2 & \text{otherwise.} \end{cases} \tag{1}$$

Let $\hat{A}_n = (\hat{a}_{i,j})$ be an $n \times n$ matrix defined by

$$\hat{a}_{i,j} = a_{j,n-1-i}, \ \text{for each } i,j = 0, \ldots, n-1.$$

Roughly speaking, $\hat{A}_n$ can be obtained by a clockwise 90 degree rotation of A_n. Thus, we have

$$\hat{A}_n(R^{(2)}_{i,j}) = \begin{cases} 2n-4 & \text{if } i=n-1 \text{ and } j<n-1 \text{ is even,} \\ 2n & \text{if } i=n-1 \text{ and } j \text{ is odd,} \\ 2n-2 & \text{otherwise.} \end{cases} \tag{2}$$

Consider the $n \times n$ matrix $nA_n + \hat{A}_n$. For example, when $n=5$, we have

$$5A_5 + \hat{A}_5 = 5 \times \begin{pmatrix} 4 & 0 & 4 & 0 & 4 \\ 1 & 3 & 1 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 1 & 3 \\ 0 & 4 & 0 & 4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 & 2 & 1 & 4 \\ 4 & 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 & 4 \\ 4 & 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 20 & 3 & 22 & 1 & 24 \\ 9 & 16 & 7 & 18 & 5 \\ 10 & 13 & 12 & 11 & 14 \\ 19 & 6 & 17 & 8 & 15 \\ 0 & 23 & 2 & 21 & 4 \end{pmatrix}.$$

Lemma 2 *The matrix $nA_n + \hat{A}_n$ belongs to $\mathbb{Z}_1(n)$.*

Proof: First, it is obvious that all elements in $nA_n + \hat{A}_n$ are integers ranging from 0 to n^2-1. Then, to complete the proof, each integer ranging from 0 to n^2-1 appears exactly once in $nA_n + \hat{A}_n$.

Suppose there are two locations $(i,j),(i',j') \in \{0,1,\ldots,n-1\}^2$ such that $na_{i,j} + \hat{a}_{i,j} = na_{i',j'} + \hat{a}_{i',j'}$. It follows that $a_{i,j} = a_{i',j'}$ and $\hat{a}_{i,j} = \hat{a}_{i',j'}$. By definition, each $a_{i,j}$ is either i or $n-1-i$. From $a_{i,j} = a_{i',j'}$, we have $i' = i$ if $i+j$ and $i'+j'$ have the same parity, and $i' = n-1-i$ otherwise. On the other hand, we have $a_{j,n-1-i} = a_{j',n-1-i'}$ from $\hat{a}_{i,j} = \hat{a}_{i',j'}$. Then, we have $j' = j$ if $i+j$ and $i'+j'$ have the same parity, and $j' = n-1-j$ otherwise.

As a consequence, $i+j$ and $i'+j'$ have the same parity. Otherwise, $i+j+i'+j'$ must be an odd, however, $i+j+i'+j' = i+j+(n-1-i)+(n-1-j) = 2n-2$. Hence, $i'=i$ and $j'=j$. Therefore, we can conclude that each integer ranging from 0 to n^2-1 appears exactly once in $nA_n + \hat{A}_n$, i.e., $nA_n + \hat{A}_n \in \mathbb{Z}_1(n)$. □

Lemma 3 *The 2×2 discrepancy of $nA_n + \hat{A}_n$ is $4n$.*

Proof: Observe from (1) and (2) that there is no pair (i,j) such that $A_n(R^{(2)}_{i,j})$ and $\hat{A}_n(R^{(2)}_{i,j})$ are both $2n$ or both $2n-4$. Hence, the 2×2 sums of $nA_n+\hat{A}_n$ is at most $2n\times n+2n-2 = 2n^2+2n-2$ and are at least $(2n-4)\times n + 2n - 2 = 2n^2 - 2n - 2$. Indeed, $nA_n(R^{(2)}_{0,n-1}) + \hat{A}_n(R^{(2)}_{0,n-1}) = 2n^2+2n-2$ and $nA_n(R^{(2)}_{1,n-1}) + \hat{A}_n(R^{(2)}_{1,n-1}) = 2n^2-2n-2$. Therefore, the 2×2 discrepancy of $nA_n + \hat{A}_n$ is $4n$. □

Observe that, when n is even, the 2×2 discrepancy of $nA_n+\hat{A}_n$ becomes 0, because we have $A_n(R^{(2)}_{i,j}) = \hat{A}_n(R^{(2)}_{i,j}) = 2n-2$ for every (i,j) when n is even. Notice that the non-emptyness of $\mathbb{N}(2,n)$ for even n is already shown in (a) of Theorem 1.

3.3 Diagonal Repeating

We now define another scheme called *Diagonal Repeating* for generating low discrepancy matrices. We partition the n^2 locations of an $n\times n$ matrix into n disjoint regions $L^{(+)}_0,\ldots,L^{(+)}_{n-1}$ along 45 degree lines as follows:

$$L_s = \{(i,j) \mid (i+j) \bmod n = s\}, \text{ for } s = 0,\ldots,n-1.$$

We define an $n\times n$ matrix $D_n = (d_{i,j})$ as follows:

$$d_{i,j} = \begin{cases} s & \text{if } (i,j) \in L_s \text{ with } s \text{ is even,} \\ n-1-s & \text{if } (i,j) \in L_s \text{ with } s \text{ is odd.} \end{cases}$$

For example, when $n = 5$, we have

$$D_5 = \begin{pmatrix} 0 & 3 & 2 & 1 & 4 \\ 3 & 2 & 1 & 4 & 0 \\ 2 & 1 & 4 & 0 & 3 \\ 1 & 4 & 0 & 3 & 2 \\ 4 & 0 & 3 & 2 & 1 \end{pmatrix}.$$

Observe that the same number is repeated along 45 degree lines. So, it is called *Diagonal Repeating*.

By definition, we have

$$d_{i,j} + d_{i,j+1} = \begin{cases} n-2 & \text{if } (i,j) \in L_s \text{ with } s < n-1 \text{ is even,} \\ n & \text{if } (i,j) \in L_s \text{ with } s \text{ is odd,} \\ n-1 & \text{otherwise.} \end{cases}$$

Hence, we have

$$D_n(R^{(2)}_{i,j}) = \begin{cases} 2n-1 & \text{if } (i,j) \in L_{n-2}, \\ 2n-3 & \text{if } (i,j) \in L_{n-1}, \\ 2n-2 & \text{otherwise.} \end{cases} \tag{3}$$

Let $\hat{D}_n = (\hat{d}_{i,j})$ be an $n \times n$ matrix defined by

$$\hat{d}_{i,j} = d_{j,n-1-i}, \text{ for each } i, j = 0, \ldots, n-1.$$

Here, $\hat{D}_n$ can be obtained by a clockwise 90 degree rotation of D_n. Analogous to L_ss, we define sets $L'_0, \ldots, L'_{n-1}$ along -45 degree lines:

$$L'_s = \{(i,j) \mid (j-i) \bmod n = s\}, \text{ for } s = 0, \ldots, n-1.$$

Then, $\hat{D}_n = (\hat{d}_{i,j})$ can be defined in terms of L'_ss as follows.

$$\hat{d}_{i,j} = \begin{cases} n-1-s & \text{if } (i,j) \in L'_s \text{ with } s \text{ is even,} \\ s & \text{if } (i,j) \in L'_s \text{ with } s \text{ is odd.} \end{cases}$$

Moreover, we have

$$\hat{D}_n(R^{(2)}_{i,j}) = \begin{cases} 2n-1 & \text{if } (i,j) \in L'_0, \\ 2n-3 & \text{if } (i,j) \in L'_{n-1}, \\ 2n-2 & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 4 *The matrix $nD_n + \hat{D}_n$ belongs to $\mathbb{Z}_1(n)$.*

Proof: Again, it is obvious that all elements in $nD_n + \hat{D}_n$ are integers ranging from 0 to $n^2 - 1$. Then, to complete the proof, each integer ranging from 0 to $n^2 - 1$ appears exactly once in $nD_n + \hat{D}_n$.

Recall that n is odd. Then, for every two pairs (i,j) and (i',j'), $d_{i,j} = d_{i',j'}$ if and only if $(i+j) \bmod n = (i'+j') \bmod n$, and $\hat{d}_{i,j} = \hat{d}_{i',j'}$ if and only if $(j-i) \bmod n = (j'-i') \bmod n$. Moreover, the two sets L_s and L'_t intersect at a single location for every $s, t = 0, \ldots, n-1$. Hence, for every two pairs (i,j) and (i',j'), $nd_{i,j} + \hat{d}_{i,j} = nd_{i',j'} + \hat{d}_{i',j'}$ if and only if $i = i'$ and $j = j'$. Therefore, we can conclude that each integer ranging from 0 to $n^2 - 1$ appears exactly once in $nD_n + \hat{D}_n$, i.e., $nD_n + \hat{D}_n \in \mathbb{Z}_1(n)$. $\square$

The following lemma shows that Diagonal Repeating can produce lower discrepancy matrices than Alternating Diagonal Sequencing.

Lemma 5 *The 2×2 discrepancy of $nD_n + \hat{D}_n$ is $2n + 2$.*

Proof: From (3) and (4), the maximum and minimum 2×2 sums of $D_n(R^{(2)}_{i,j})$s are respectively $D_n(R^{(2)}_{n-1,n-1}) = (2n-1)\times n + 2n - 1 = 2n^2 + n - 1$ and $D_n(R^{(2)}_{0,n-1}) = (2n-3)\times n + 2n - 3 = 2n^2 - n - 3$. $\square$

Fig. 2 shows a 9×9 matrix generated by the method.

$$\begin{bmatrix} 0 & 7 & 2 & 5 & 4 & 3 & 6 & 1 & 8 \\ 7 & 2 & 5 & 4 & 3 & 6 & 1 & 8 & 0 \\ 2 & 5 & 4 & 3 & 6 & 1 & 8 & 0 & 7 \\ 5 & 4 & 3 & 6 & 1 & 8 & 0 & 7 & 2 \\ 4 & 3 & 6 & 1 & 8 & 0 & 7 & 2 & 5 \\ 3 & 6 & 1 & 8 & 0 & 7 & 2 & 5 & 4 \\ 6 & 1 & 8 & 0 & 7 & 2 & 5 & 4 & 3 \\ 1 & 8 & 0 & 7 & 2 & 5 & 4 & 3 & 6 \\ 8 & 0 & 7 & 2 & 5 & 4 & 3 & 6 & 1 \end{bmatrix} \times 9 + \begin{bmatrix} 8 & 1 & 6 & 3 & 4 & 5 & 2 & 7 & 0 \\ 0 & 8 & 1 & 6 & 3 & 4 & 5 & 2 & 7 \\ 7 & 0 & 8 & 1 & 6 & 3 & 4 & 5 & 2 \\ 2 & 7 & 0 & 8 & 1 & 6 & 3 & 4 & 5 \\ 5 & 2 & 7 & 0 & 8 & 1 & 6 & 3 & 4 \\ 4 & 5 & 2 & 7 & 0 & 8 & 1 & 6 & 3 \\ 3 & 4 & 5 & 2 & 7 & 0 & 8 & 1 & 6 \\ 6 & 3 & 4 & 5 & 2 & 7 & 0 & 8 & 1 \\ 1 & 6 & 3 & 4 & 5 & 2 & 7 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 64 & 24 & 48 & 40 & 32 & 56 & 16 & 72 \\ 63 & 26 & 46 & 42 & 30 & 58 & 14 & 74 & 7 \\ 25 & 45 & 44 & 28 & 60 & 12 & 76 & 5 & 65 \\ 47 & 43 & 27 & 62 & 10 & 78 & 3 & 67 & 23 \\ 41 & 29 & 61 & 9 & 80 & 1 & 69 & 21 & 49 \\ 31 & 59 & 11 & 79 & 0 & 71 & 19 & 51 & 39 \\ 57 & 13 & 77 & 2 & 70 & 18 & 53 & 37 & 33 \\ 15 & 75 & 4 & 68 & 20 & 52 & 36 & 35 & 55 \\ 73 & 6 & 66 & 22 & 50 & 38 & 34 & 54 & 17 \end{bmatrix}$$

Figure 2: Low discrepancy matrix of size 9×9 (discrepancy = 20).

The discrepancy bound $2n+2$ achieved by Diagonal Repeating is much better than the bound $4n$ done by Alternating Diagonal Sequencing. Unfortunately, this bound is not optimal. In fact, we reduce the discrepancy bound further to $2n$ in the next section. Nevertheless, this bound looks near optimal because of the following theorem establishing a lower bound 2 on the 2×2 discrepancy for matrices in the class $\mathbb{Z}_n(n)$. In fact, the theorem shown a stronger result.

Theorem 6 *Let k be any integer such that $2 \le k < n$. If n and k are relatively prime, then the $k \times k$ discrepancy of each $A \in \mathbb{Z}_n(n)$ is at least 2.*

Proof: Let $A \in \mathbb{Z}_n(n)$. Observe that each matrix element is included in exactly k^2 different $k \times k$ regions, and each integer between 0 and $n-1$ appears exactly n times in A. Thus,

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} A(R^{(2)}_{i,j}) = k^2(0+1+\cdots+n-1)\times n = k^2 n^2 \frac{(n-1)}{2}.$$

The lemma is proven by contradiction.

Suppose the $k \times k$ discrepancy of A is 0 for some $A \in \mathbb{Z}_n(n)$. It follows that that every $k \times k$ region has the same sum $k^2(n-1)/2$. Define a row sum $r_{i,j}$ by $r_{i,j} = a_{i,j} + \cdots + a_{i,j+k-1}$. In terms of $r_{i,j}$s, we have $r_{i,j} + r_{i+1,j} + \cdots + r_{i+k-1,j} = k^2(n-1)/2$ for $i, j = 0, 1, \ldots, n-1$, which implies $r_{i,j} = r_{i+k,j}$. Since n and k are relatively prime, we have $r_{i,0} = r_{i,1} = \ldots = r_{i,n-1}$ for each $i = 0, 1, \ldots, n$, and thus, $r_{i,j} = k(n-1)/2$ for each $i, j = 0, 1, \ldots, n-1$. By applying the same arguments to elements of A, it follows that all the elements of the matrix A are $(n-1)/2$, which contradicts to the assumption $A \in \mathbb{Z}_n(n)$.

Next, suppose the $k \times k$ discrepancy of A is 1 for some $A \in \mathbb{Z}_n(n)$. Then, there are only two different $k \times k$ sums S_0 and S_1 with $S_1 = S_0 + 1$. For $k = 0, 1$, we denote by R_k the number of $k \times k$ regions whose sums are S_k. Then, we have $R_0 + R_1 = n^2$ and $0 < R_0, R_1 < n^2$. Then, we have $R_0 S_0 + R_1 S_1 = k^2 n^2 (n-1)/2$. From $R_0 + R_1 = n^2$ and $S_1 = S_0 + 1$, we have

$$R_1 = k^2 n^2 \frac{(n-1)}{2} - n^2 S_0 = n^2 \left(k^2 \frac{(n-1)}{2} - S_0 \right).$$

Since n is odd and S_0 is an integer, R_1 is a multiple of n^2, which contradicts $0 < R_1 < n^2$. $\Box$

Observe from (3) that D_n has 2×2 discrepancy 2, and according the theorem, D_n has the lowest 2×2 discrepancy among all matrices in $\mathbb{Z}_n(n)$.

4 Combined Strategy for Improving Discrepancy

We have presented two schema for generating low discrepancy matrices, i.e., Alternating Diagonal Sequencing and Diagonal Repeating. The discrepancy bound achieved is $2n+2$ for those matrices in the class $\mathbb{Z}_1(n)$ with n odd. In this section we propose yet another scheme, called *Modified Alternating Diagonal Sequencing*, based on these two strategies to achieve a better bound, $2n$.

Here we take D_n as the matrix for upper digits, which has the lowest 2×2 discrepancy among all matrices in $\mathbb{Z}_n(n)$. Recall that all the elements of D_n in the location given by L_s have the same value (for each $(i,j) \in L_s$, $d_{i,j} = s$ if s is even, and otherwise $d_{i,j} = n - 1 - s$).

For lower digits, we take the matrices $M_n = (m_{i,j})$ defined as follows:

$$m_{i,j} = \begin{cases} i & \text{if } (i,j) \in L_1, \\ i & \text{if } (i,j) \in L_s \text{ with } s \geq 2 \text{ and } s \text{ is even}, \\ n-1-i & \text{otherwise}, \end{cases}$$

For example, when $n = 5$,

$$M_5 = \begin{pmatrix} 4 & 0 & 0 & 4 & 0 \\ 1 & 1 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 1 & 3 & 3 \\ 4 & 0 & 4 & 4 & 0 \end{pmatrix}.$$

By the definition of M_n, for each $s = 0, \ldots, n-1$, all elements of the matrix M_n in the locations given by L_s have different values, i.e., on matrix M_n, each of $0, \ldots, n-1$ appears exactly once in the locations given by L_s. This proof the following lemma.

Lemma 7 *The matrices $nD_n + M_n$ belongs to $\mathbb{Z}_1(n)$.*

Now we estimate the 2×2 discrepancy of $nD_n + M_n$. Recall that we have

$$D_n(R^{(2)}_{i,j}) = \begin{cases} 2n-1 & \text{if } (i,j) \in L_{n-2}, \\ 2n-3 & \text{if } (i,j) \in L_{n-1}, \\ 2n-2 & \text{otherwise}, \end{cases}$$

Moreover, by definition, we have

$$m_{i,j} + m_{i,j+1} = \begin{cases} 2i & \text{if } (i,j) \in L_1, \\ n-1 & \text{otherwise}, \end{cases}$$

and thus,

$$M_n(R^{(2)}_{i,j}) = \begin{cases} n+2i+1 & \text{if } (i,j) \in L_0 \text{ and } i < n-1, \\ n-1 & \text{if } (i,j) \in L_0 \text{ and } i = n-1, \\ n+2i-1 & \text{if } (i,j) \in L_1, \\ 2n-2 & \text{otherwise (i.e, if } (i,j) \in L_s \text{ with } 2 \le s \le n-1). \end{cases}$$

Then, we obtain the following theorem.

Theorem 8 *The 2×2 discrepancy of the matrix $nD_n + M_n$ is $2n$ when $n > 3$.*

Proof: Assume that $n > 3$. Then, from $n - 2 > 1$,

$$(nD_n + M_n)(R^{(2)}_{i,j}) = \begin{cases} 2n^2-n+2i+1 & \text{if } (i,j) \in L_0 \text{ and } i < n-1, \\ 2n^2-n-1 & \text{if } (i,j) \in L_0 \text{ and } i = n-1, \\ 2n^2-n+2i-1 & \text{if } (i,j) \in L_1, \\ 2n^2+n-2 & \text{if } (i,j) \in L_{n-2}, \\ 2n^2-n-2 & \text{if } (i,j) \in L_{n-1}. \\ 2n^2-2 & \text{otherwise (i.e., if } (i,j) \in L_s \text{ with } 2 \le s \le n-3). \end{cases}$$

Hence, the largest and the smallest 2×2 sums in $nD_n + M_n$ are respectively $2n^2 + n - 2$ (on L_{n-2}) and $2n^2 - n - 2$ (on L_{n-1}), and thus, the 2×2 discrepancy is $2n$. $\square$

In the following, the matrix $nD_n + M_n$ with $n = 9$ is shown.

$$\begin{bmatrix} 0 & 7 & 2 & 5 & 4 & 3 & 6 & 1 & 8 \\ 7 & 2 & 5 & 4 & 3 & 6 & 1 & 8 & 0 \\ 2 & 5 & 4 & 3 & 6 & 1 & 8 & 0 & 7 \\ 5 & 4 & 3 & 6 & 1 & 8 & 0 & 7 & 2 \\ 4 & 3 & 6 & 1 & 8 & 0 & 7 & 2 & 5 \\ 3 & 6 & 1 & 8 & 0 & 7 & 2 & 5 & 4 \\ 6 & 1 & 8 & 0 & 7 & 2 & 5 & 4 & 3 \\ 1 & 8 & 0 & 7 & 2 & 5 & 4 & 3 & 6 \\ 8 & 0 & 7 & 2 & 5 & 4 & 3 & 6 & 1 \end{bmatrix} \times 9 + \begin{bmatrix} 8 & 0 & 0 & 8 & 0 & 8 & 0 & 8 & 0 \\ 1 & 1 & 7 & 1 & 7 & 1 & 7 & 1 & 7 \\ 2 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 2 \\ 5 & 3 & 5 & 3 & 5 & 3 & 5 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 5 & 3 & 5 \\ 6 & 2 & 6 & 2 & 6 & 6 & 2 & 6 & 2 \\ 1 & 7 & 1 & 7 & 7 & 1 & 7 & 1 & 7 \\ 8 & 0 & 8 & 8 & 0 & 8 & 0 & 8 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 63 & 18 & 53 & 36 & 35 & 54 & 17 & 72 \\ 64 & 19 & 52 & 37 & 34 & 55 & 16 & 73 & 7 \\ 20 & 51 & 38 & 33 & 56 & 15 & 74 & 6 & 65 \\ 50 & 39 & 32 & 57 & 14 & 75 & 5 & 66 & 21 \\ 40 & 31 & 58 & 13 & 76 & 4 & 67 & 22 & 49 \\ 30 & 59 & 12 & 77 & 3 & 68 & 23 & 48 & 41 \\ 60 & 11 & 78 & 2 & 69 & 24 & 47 & 42 & 29 \\ 10 & 79 & 1 & 70 & 25 & 46 & 43 & 28 & 61 \\ 80 & 0 & 71 & 26 & 45 & 44 & 27 & 62 & 9 \end{bmatrix}$$

5 Conclusions

In this paper we have introduced a discrepancy-based measure to evaluate a class of integer-valued matrices for application to digital halftoning. In the context of digital halftoning, a problem is defined on a discrete plane instead of a continuous plane used in the traditional discrepancy theory. We are also required to fill in lattice points by distinct integers instead of just placing points on the continuous plane. We have observed that discrepancy is affected by integral properties between a matrix size n and a region size k.

We have devoted to minimizing the discrepancy against a family of 2×2 regions. The region size may be too small for practical application to digital halftoning. We could define a combined discrepancy measure, that is, sum of discrepancy bounds for several families of regions such as $\mathcal{D}_{2,n}(P)+\mathcal{D}_{3,n}(P)+\cdots+\mathcal{D}_{k,n}(P)$. What is know so far is that there is no matrix P for which the combined discrepancy bound $\mathcal{D}_{2,n}(P)+\mathcal{D}_{3,n}(P)$ is zero.

One technical open problem is to prove optimality of the discrepancy bound $2n$ we have established in this paper. It is not so easy even for a small value of n. For example, when n is five, we have 24! different matrices to check.

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Appendix: Dither Matrix

A motivation of this study comes from digital halftoning, which is a technique to convert a multiple-level image A given as a matrix into a binary black-white image specified as a binary matrix B to print it out in a printer. Although this is a quite practical topic, several theoretical studies have been presented in relation to discrete geometry and combinatorial optimization [3, 2, 4]. A number of algorithms have been proposed so far. The following two are the simplest ones among them.

Simple Thresholding

Given an $n \times n$ array A of real numbers between 0 and 1 representing intensity levels of pixels, we want to obtain a binary array B of the same size which looks similar to A. The most naive method for obtaining B is simply to binarize each input entry by a fixed threshold, say 0.5. It is simplest, but the quality of the output image is worst since any uniform gray region could become totally white or totally black depending on the intensity levels.

Ordered Dither

Instead of using a fixed threshold over an entire image, we could use different thresholds. A simple way of implementing this idea is as follows: We prepare an $M \times M$ matrix of integers from 0 to $M^2 - 1$. This matrix *(dither matrix)* is tiled periodically to cover the image. Each pixel in the image is compared with the corresponding threshold from the dither array to decide which color should be put at that location. That is, for each entry (i, j) we have an input value $A(i, j)$ of a given image matrix A and an integer $D(i \bmod M, j \bmod M)$ in the matrix. If $A(i, j) > D(i \bmod M, j \bmod M)/M^2$ then the corresponding output value is determined to be 1, and otherwise it is 0.

The performance of the ordered dither algorithm heavily depends on a dither matrix used. Then, how can we define an optimal dither matrix? Imagine an artificial image of gradually increasing intensity. During the transition from dark to bright, the number of white dots should gradually increase. This means that for any number i between 0 and $2^{2k} - 1$ those entries having numbers greater than i must be as uniformly distributed as possible in the dither matrix. The uniformity can be measured in several different ways. One measure is based on the ratio of the minimum pairwise distance against the diameter of the maximum empty circle.

One of most commonly used dither matrix is known as Bayer's matrix given by Bayer in 1973 [5], which is shown in Fig. 4. Fig. 5 shows an output image using this dither matrix. Artifacts caused by this dither matrix are clearly visible.

0	32	8	40	2	34	10	42
48	16	56	24	50	18	58	26
12	44	4	36	14	46	6	38
60	28	52	20	62	30	54	22
3	35	11	43	1	33	9	41
51	19	59	27	49	17	57	25
15	47	7	39	13	45	5	37
63	31	55	23	61	29	53	21

Figure 3: 8×8 dither matrix by Bayer.

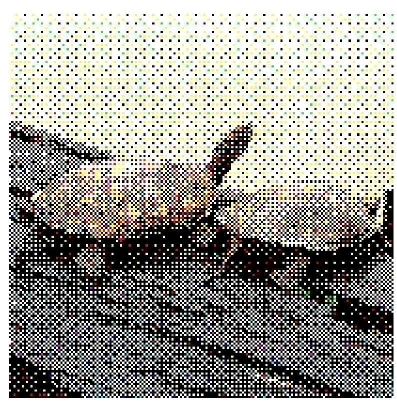

Figure 4: An output image by Ordered Dithering.

This dither matrix is defined as follows [8]. First, starting from a 1×1 matrix $D_0 = [0]$, we recursively define $D_k (k = 1, 2, \ldots)$ as follows:

$$D_k = \left[\begin{array}{ll} 4D_{k-1} & 4D_{k-1} + 2U_{k-1} \\ 4D_{k-1} + 3U_{k-1} & 4D_{k-1} + U_{k-1} \end{array} \right]. \tag{5}$$

Here, U_k is a $2^k \times 2^k$ square matrix consisting of all 1s.

The above regular grid-like construction of the dither matrix is good enough in the measure based on pairwise distance since it is constructed under the notion of *incremental Voronoi insertion.* However, this matrix is very bad in the discrepancy measure. Consider a 2×2 region R. It is not so hard to see that the sum in R is smallest when R is the upper left corner and largest when it is the lower left corner. More concretely, their difference is $4^k - 4$ for a $2^k \times 2^k$ matrix. This is almost equivalent to the total number of entries of the matrix. Thus, the discrepancy of the dither matrix is $\Theta(n^2)$. Since the upper bound on the 2×2 discrepancy of any matrix containing $0, 1, \ldots, n^2 - 1$ is $O(n^2)$, the Bayer's dither matrix is the worst matrix in the sense of discrepancy. In the next section we propose two schema for achieving low discrepancy for 2×2 regions.

Lemma 9 *Any $n \times n$ square matrix P defined by Eq.(5) is worst in the discrepancy measure, that is,*

$$\mathcal{D}_{2,n}(P) = \Theta(n^2), \tag{6}$$

for the family of all 2×2 regions, and

$$\mathcal{D}_{2,n}(P') = O(n^2), \tag{7}$$

for any square matrix P' in the class $\mathbb{Z}_1(n)$.