# Aspect-Ratio Voronoi Diagram and Its Complexity Bounds

Tetsuo Asano
School of Information Science,
JAIST (Japan Advanced Institute of Science and Technology)
1-1 Asahidai, Nomi, Ishikawa, 923-1292, Japan
t-asano@jaist.ac.jp.

**Abstract**

This letter first defines an aspect ratio of a triangle by the ratio of the longest side over the minimal height. Given a set of line segments, any point $p$ in the plane is associated with the worst aspect ratio for all the triangles defined by the point and the line segments. When a line segment $s_i$ gives the worst ratio, we say that $p$ is dominated by $s_i$. Now, an aspect-ratio Voronoi diagram for a set of line segments is a partition of the plane by this dominance relation. We first give a formal definition of the Voronoi diagram and give $O(n^{2+\varepsilon})$ upper bound and $\Omega(n^2)$ lower bound on the complexity, where $\varepsilon$ is any small positive number. The Voronoi diagram is interesting in itself, and it also has an application to a problem of finding an optimal point to insert into a simple polygon to have a triangulation that is optimal in the sense of the aspect ratio.

**keywords:** algorithms, combinatorial problems, computational geometry.

## 1 Introduction

Voronoi diagrams have been applied in many different areas and different purposes [1, 3]. In this letter we define a new Voronoi diagram for a set of line segments in the plane, which is called an aspect-ratio Voronoi diagram. This Voronoi diagram is characterized as follows: given a set of line segments $s_1, \ldots, s_n$ and a point $p$ in the plane, we can define a triangle $Tr(p, s_i)$ for each line segment $s_i$ by drawing lines from $p$ to two endpoints of $s_i$. Now, we define an aspect ratio $asp(p, s_i)$ of the triangle $Tr(p, s_i)$ by the ratio of the longest side over the minimal height. In our criterion the smaller the ratio the better the quality of the triangle. Given a set of line segments in the plane, the plane is divided into so-called Voronoi regions each associated with one of the line segments. A point belongs to a Voronoi region $V(s_i)$ for a line segment $s_i$ if $s_i$ gives the worst (largest) aspect ratio among given line segments. Such a Voronoi diagram is well defined and it is quite interesting in itself. We investigate the complexity of the Voronoi diagram, that is the number of Voronoi vertices, edges, and cells. We prove $O(n^{2+\varepsilon})$ upper bound and $\Omega(n^2)$ lower bound for a set of $n$ line segments.

Once we construct such an aspect-ratio Voronoi diagram for a set of line segments, using the diagram we can find a point $p^*$ that minimizes (optimizes) the largest (worst) aspect ratio. We are especially interested in the case where a set of line segments form a polygon.

This Voronoi diagram has interesting properties, which are quite different from ordinary ones. First of all it looks quite different from ordinary Voronoi diagrams for points. In our case Voronoi edges consist of plane curves that are polynomials of degree at most 3 in $x$ and $y$. A Voronoi region associated with a line segment is not always connected. It may be divided into a number of connected regions or cells. This fact leads to high complexity for the diagram.

This Voronoi diagram can be applied to another geometric optimization problem; Given a simple polygon $P$, we want to find an optimal Steiner point $p^*$ such that the worst aspect ratio of the best triangulation of $P \cup \{p^*\}$ is optimized. Difficulty is how to determine topology of triangulation (as a graph), but once topology is fixed, an optimal Steiner point can be found on edges of the aspect-ratio Voronoi diagram for $P$.

## 2 Definitions of an aspect ratio

Given a line segment $s = \overline{p_1 p_2}$ and a point $p$ in the plane, the aspect ratio $asp(p, s)$ of the triangle defined by connecting the two endpoints of $s$ to $p$ is defined by the ratio of the longest side $L$ over the minimal height $h$ of the triangle, i.e.,

$$asp(p, s) = \frac{L}{h}. \tag{1}$$

By the definition, the aspect ratio is at least $2/\sqrt{3}$ for any triangle and a triangle of aspect ratio $2/\sqrt{3}$ is a regular triangle of three equal sides. There are some other definitions for aspect ratio. We could use the shortest side instead of the minimal height. Or we could also use the circumcircle radius instead of the longest side. Any such definition leads to an aspect-ratio Voronoi diagram. Reasons why we are interested in our definition using the longest side and minimal height are that (1) it is common in Finite Element Methods, (2) it is easy to extend to higher dimensions, and (3) the resulting Voronoi diagram is mathematically or combinatorially simplest among them.

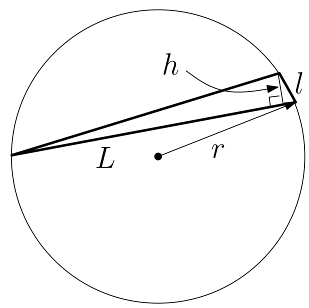


Figure 1: Three different definitions of aspect ratio. (1) $L/h$, the longest side $L$ over the minimal height $h$, (2) $L/l$, the longest side over the shortest side $l$, and $L/r$ the longest side over the circumcircle radius $r$.

Figure 1 illustrates the three different definitions of aspect ratio of a triangle characterized by the longest side $L$, the minimal height $h$, the shortest side $l$, and the circumcircle radius $r$. Our definition is $L/h$, while the other two are $L/l$ and $L/r$. In our definition the ratio $L/h$ goes to infinity if a triangle is almost flat. On the other hand, for a flat triangle consisting of three sides of lengths $2, 1+\varepsilon$, and $1+\varepsilon$ for a small positive constant $\varepsilon$, the ratio $L/l$ is very close to 2, but $L/h$ is roughly proportional to $\sqrt{2/\varepsilon}$ (since $\varepsilon^2 \ll 2\varepsilon$), which can be arbitrarily large.

## 3 Fundamental Properties

In this section we investigate fundamental properties related to aspect ratio. Suppose we are given a line segment $s$ connecting two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$. An arbitrary point $p(x, y)$ in the plane defines a triangle $Tr(s, p)$ by connecting $p$ to the two endpoints of $s$. Let $L$ be the length of the longest side of $Tr(s, p)$ and $h$ be the corresponding height, which is also a minimal height of the triangle. Recall that the area $S$ of $Tr(s, p)$ is given by

$$\begin{aligned} 2S &= |x(y_1 - y_2) + x_1(y_2 - y) + x_2(y - y_1)| \\ &= |(y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1|, \end{aligned} \tag{2}$$

which is linear in $x$ and $y$. Since $2S = Lh$, we have

$$asp(p, s) = \frac{L}{h} = \frac{L^2}{2S}. \tag{3}$$

It is very important to note that the longest side length is squared in the ratio since otherwise we have to deal with a square root. The numerator, $L^2$, is either quadratic in $x$ and $y$ or a constant which is the squared length of the line segment $s$. The denominator, $2S$, is linear in $x$ and $y$. These facts are useful to simplify mathematical treatment.

There are two cases to consider depending on whether $s$ is the longest side of the triangle $Tr(s,p)$ or not.

Case 1: $pp_i$ is the longest side ($i = 1, 2$)

$L^2$ is given by $(x - x_i)^2 + (y - y_i)^2$, and thus we have

$$asp(p, s) = \frac{(x - x_i)^2 + (y - y_i)^2}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|}. \tag{4}$$

Case 2: $s = p_1p_2$ is the longest side

$L^2$ is given by $(x_1 - x_2)^2 + (y_1 - y_2)^2$, which is a constant, say $C$, and thus we have

$$asp(p, s) = \frac{C}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|}. \tag{5}$$

Consider the level curve of aspect ratio of a triangle $Tr(p,s)$ for a value $\lambda$, denoted by $L_s(\lambda)$, that is,

$$L_s(\lambda) = \{p \mid asp(p, s) = \lambda\}. \tag{6}$$

From the above observations it turns out that the level curve consists of a line segment parallel to $s$ and two circular arcs $C_{p_1}$ and $C_{p_2}$ passing through the endpoints $p_1$ and $p_2$, respectively. The curve symmetric with respect to the supporting line of $s$ is also the level curve for the same value $\lambda$ although it is not illustrated in Fig. 2 to save space.

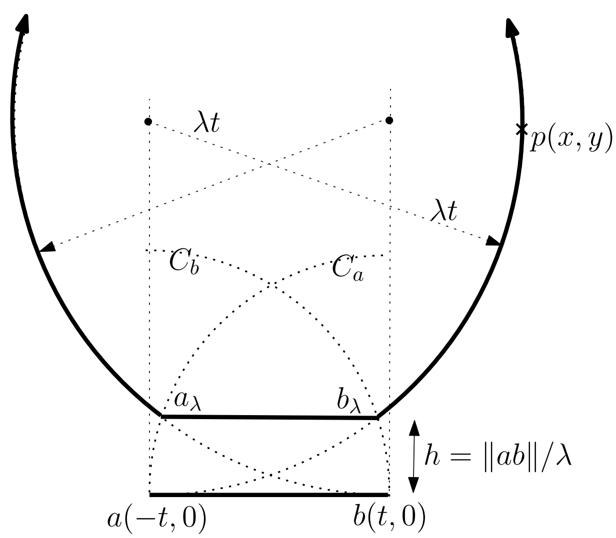


Figure 2: A level curve of aspect ratio for the value of $\lambda$. It consists of circular arcs passing through the endpoints $a$ and $b$ of a given line segment $ab$ and a line segment $a_\lambda b_\lambda$ parallel to the given one spaced by the length of the segment divided by $\lambda$. Those endpoints $a_\lambda$ and $b_\lambda$ are determined so that $\|ab_\lambda\| = \|ba_\lambda\| = \|ab\|$.

Figure 2 illustrates a level curve of aspect ratio for a value $\lambda$. As is easily seen, given an aspect ratio, points of the same aspect ratio are either on circular arcs or on a straight line segment. In fact, for any constant $\lambda$

$$asp(p, s) = \frac{(x - x_1)^2 + (y - y_1)^2}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|} = \lambda \tag{7}$$

gives a circle and

$$asp(p, s) = \frac{C}{|(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1|} = \lambda \tag{8}$$

gives a line.

Now we can characterize Voronoi edges at which the worst aspect ratio is given by two different line segments.

**Lemma 1** *For two non-intersecting line segments in the plane, the trace of points giving the same aspect ratio for them is either a line or a degree-3 curve.*

**Proof:** The trace of those points is characterized by one of the following equations

$$\frac{(x-x_1)^2+(y-y_1)^2}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} = \frac{(x-x_3)^2+(y-y_3)^2}{|(y_3-y_4)x+(x_4-x_3)y+x_3y_4-x_4y_3|},$$
$$\frac{(x-x_1)^2+(y-y_1)^2}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} = \frac{C}{|(y_3-y_4)x+(x_4-x_3)y+x_3y_4-x_4y_3|},$$
$$\frac{C}{|(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1|} = \frac{C}{|(y_3-y_4)x+(x_4-x_3)y+x_3y_4-x_4y_3|}.$$

The first and second equations give degree-3 curves and the third one gives a line. □

# 4 Aspect Center of a Triangle

A great number of centers are known for triangles. We can define a yet another triangle center using the aspect ratio. Let $P$ be a point in the interior of a triangle $ABC$. Then, it is called an *aspect center* of the triangle if the three triangles $PAB, PBC$, and $PCA$ have the same aspect ratio.

**Theorem 2** *Any triangle has a unique aspect center.*

**Proof:** Let $\triangle ABC$ be a triangle with three sides $|AB| \geq |BC| \geq |CA|$. Consider a set $S_A$ of points $P$ in the interior of $\triangle ABC$ that satisfy $asp(PAB) = asp(PAC)$. The vertex $A$ belongs to $S_A$. If we move a point $P$ within $\triangle ABC$ while keeping the condition $asp(PAB) = asp(PAC)$, their aspect ratios must monotonically change. The reason is as follows: consider a level curve $L_{XY}(\alpha)$ which consists of points $P$ such that $asp(PXY) = \alpha$. Since $AB$ is the longest side of $\triangle ABC$, its level curve $L_{AB}(\alpha)$ always consists of a line parallel to $AB$. On the other hand, $L_{AC}(\alpha)$ is either a line segment parallel to $AC$ (if $AC$ is a longest side of $\triangle PAC$) or a circular arc (otherwise) passing through $A$. The level curve for $AC$ can be a circular arc passing through $C$, but we can neglect the possibility because any point in $S_A$ must be above the line bisecting the angle at $A$. Another important observation here is that the aspect ratio decreases if a point moves upward from somewhere on the level curve $L_{AB}(\alpha)$. It is also true that the aspect ratio decreases if a point moves to the right of the level curve $L_{AC}(\alpha)$. To keep the condition $asp(PAB) = asp(PAC)$, the point $P$ must be between those two level curves $L_{AB}(\alpha)$ and $L_{AC}(\alpha)$. This means that point $P$ must move in a direction between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In other words, the set $S_A$ forms a simple curve partitioning the triangle $\triangle ABC$ into two disjoint parts, depending on which of $asp(PAB)$ and $asp(PAC)$ is larger and the curve is monotone in both of the directions $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

This is also true for a set $S_B$ of points $P$ with $asp(PBA) = asp(PBC)$. It forms a simple curve monotone in both of the directions $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Therefore, the two curves for $S_A$ and $S_B$ must meet at a single point, which is a unique aspect center. □

# 5 Aspect-ratio Voronoi diagram

Now we are ready to define an aspect-ratio Voronoi diagram for a given set $S$ of line segments $s_1, s_2, \ldots, s_n$ in the plane. We define the aspect-ratio Voronoi diagram in such a way that a point $p$ belongs to a Voronoi region associated with a line segment $s_i$ if and only if $s_i$ gives the worst (largest) aspect ratio, that is,

$$asp(p, s_i) \geq asp(p, s_j) \text{ for any } j \neq i. \tag{9}$$

This also implies that a Voronoi region $V(s_i)$ for $s_i$ is defined by

$$V(s_i) = \{p \in \mathbb{R}^2 | asp(p, s_i) \geq asp(p, s_j) \text{ for any } j \neq i\}. \tag{10}$$

Each Voronoi region is bounded by curves at which two line segments give the same aspect ratio, which are either straight lines or degree-3 curves defined earlier (hyperbolas if any degeneracy). Endpoints or intersections of those curves (referred to as primitive curves, hereafter) are Voronoi vertices and those primitive curves joining two such vertices are Voronoi edges. A minimal region bounded by Voronoi edges is called a Voronoi cell. Every Voronoi cell is associated with a line segment, but the reverse is not always true. That is, a Voronoi region for a line segment may be divided into many Voronoi cells.

The partition of the plane into Voronoi cells is called an aspect-ratio Voronoi diagram for the set of line segments. Two such Voronoi diagrams are shown in Figure 3, one for a triangle and the other for a convex polygon. In each case Voronoi regions are distinguished by colors each defined for a line segment. In the right Voronoi diagram the Voronoi region for the top short edge is disconnected.

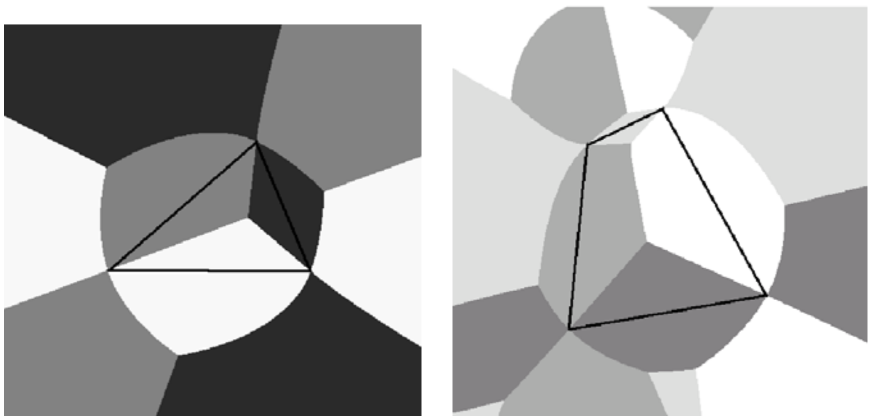

Figure 3: Two aspect-ratio Voronoi diagrams.

## 6 Complexity Bounds

Now, let us analyze the combinatorial complexity of our Voronoi diagram for $n$ line segments. First recall that an aspect ratio $asp(p)$ at a point $p$ is defined as the largest value among aspect ratios for all line segments, that is,

$$asp(p) = \max\{asp(p, s_1), asp(p, s_2), \ldots, asp(p, s_n)\}. \tag{11}$$

Note that each line segment is transparent, that is, any triangle $Tr(p, s_i)$ can intersect any other line segment $s_j$.

**Theorem 3** *An aspect-ratio Voronoi diagram for a set of $n$ lines in the plane consists of $O(n^{2+\varepsilon})$ Voronoi vertices, edges, and cells, where $\varepsilon$ is an arbitrarily small positive constant.*

**Proof:** The aspect ratio $asp(p, s_i)$ is defined at any point $p$ in the plane. It is natural to define a terrain by regarding the value as the height at the point. We have $n$ terrains and an aspect ratio $asp(p)$ at a point $p$ is given by the largest value among them. Thus, our aspect-ratio Voronoi diagram is an upper envelope of those $n$ terrains.

We further decompose each terrain. Recall that an aspect ratio for a point $p$ and a line segment $s = (p_1, p_2)$ is defined by the ratio $L^2/(2S)$ where $L$ and $S$ are the longest side length and the area of the triangle, respectively. For simplicity we assume that the area is given by

$$2S = ax + by + c,$$

using appropriate constants $a, b$ and $c$. Then, the aspect ratio at $p(x,y)$ for $s$ is the maximum among three ratios:

$$asp(p,s) = \max\{\frac{(x-x_1)^2+(y-y_1)^2}{ax+by+c}, \frac{(x-x_2)^2+(y-y_2)^2}{ax+by+c}, \frac{\|s\|^2}{ax+by+c}\},$$

where $\|s\|$ is the length of $s$.

Define three terrains:

$$z = \frac{(x-x_1)^2+(y-y_1)^2}{ax+by+c}, z = \frac{(x-x_2)^2+(y-y_2)^2}{ax+by+c}, \text{ and } z = \frac{\|s\|^2}{ax+by+c}.$$

Then, the aspect ratio $asp(p,s)$ is given as the point of the upper envelope of these three terrains just above at $p(x,y)$. These terrains are infinite at all points $p$ lying on the supporting line of $s$. As a point $p$ moves away from the supporting line of $s$, the corresponding $z$ value monotonically decreases. So, we have $3n$ such terrains and the upper envelope of those terrains gives us an aspect-ratio Voronoi diagram. Halperin and Sharir [2] proved $O(n^{2+\varepsilon})$ complexity of a single cell in such an arrangement for any small positive constant $\varepsilon$. The upper envelope mentioned above is a cell of the arrangement. Thus, their theorem applies to our case. □

Although we do not have an example that matches the upper bound, the quadratic lower bound is established:

**Theorem 4** *There is a star-shaped polygon with $n$ vertices whose aspect-ratio Voronoi diagram contains $\Omega(n^2)$ Voronoi vertices.*

**Proof:** Consider the following star-shaped polygon $P$ and a small square $R$ in it. It has $k = (n-6)/4$ convex corners in each of the upper and lower sides. Each corner in the upper (resp., lower) side has an edge whose extension hits the upper (resp., lower) right corner of $R$. There are $\Theta(k^2) = \Theta(n^2)$ intersections among those extensions. Consider any two edges $s_1$ and $s_2$ whose supporting lines intersect at a point $p$ in the interior of the polygon. Then, if we walk away from $p$ on the supporting line of $s_1$, we enter the Voronoi cell for $s_1$. Same for the edge $s_2$. So, each such intersection is a Voronoi vertex. □

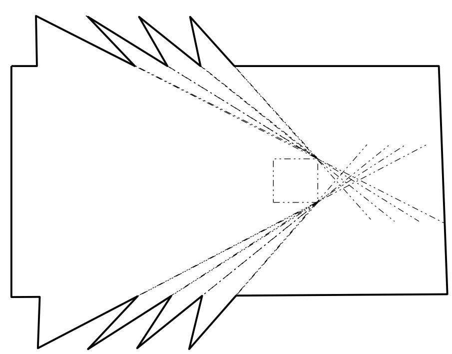

Figure 4: Star-shaped polygon with an aspect-ratio Voronoi diagram of $\Omega(n^2)$ Voronoi vertices.

## 7 Conclusions and Future Works

In this letter we have proposed a new Voronoi diagram using the notion of aspect ratio of a triangle, which is commonly used in Finite Element Methods. We have also revealed the complexity bounds of the aspect-ratio Voronoi diagram. A small gap still remains between its upper bound $O(n^{2+\varepsilon})$ and its lower bound $\Omega(n^2)$. The author conjectures that it is $\Theta(n^2)$. A related open problem is to know whether we can triangulate a set of points in the plane in polynomial time so that the worst aspect ratio is optimized.

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