Distance Trisector Curves in Regular Convex Distance Metrics

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Abstract

Given two points A and B in the plane, we are interested in separating them by two curves C_A and C_B such that C_A is equidistant from A and C_B, and C_B is equidistant from B and C_A. Such curves generalize the familiar notion of a bisector curve, and form the basis of a new kind of Voronoi diagram called a Zone diagram.

These curves, which are referred to as distance trisector curves, *have been studied in the Euclidean metric where they exist, are unique, and admit efficient approximations. Nevertheless, they have no known expression in terms of elementary functions and are conjectured to be non-algebraic.*

In this paper, we study distance trisector curves with respect to a parameterized family of distance metrics that provide arbitrarily close approximations to the Euclidean distance. The advantage of studying distance trisectors in this setting is that they have a simple piecewise-linear description and an efficient (exact) construction. We show that distance trisectors defined in this way provide a conceptually simple alternative proof of the existence and uniqueness of Euclidean trisector curves.

1. Introduction

Given two points A and B in the plane, we are interested in constructing two equally-spaced curves, C_A and C_B that trisect the plane with respect to A and B. Specifically, we want C_A and C_B to satisfy the following *equal-space property*:
(1) for any point p on the curve C_A, the point A and the curve C_B are at the same distance from p, that is, the distance from p to A is equal to that from p to any point on C_B that is closest to p, and
(2) for any point q on the curve C_B, the point B and the curve C_A are at the same distance.
We refer to such curves as *distance trisector curves* for A and B.

Trisector curves can be seen as a special case ($k = 2$) of a more general plane partitioning problem in which we ask for $k \geq 1$ equally-spaced curves that separate points A and B. In the case $k = 1$ this is just the bisector of A and B, which serves as the foundation of the classical notion of a Voronoi diagram. Asano *et al* [1] define a new Voronoi-like space partition, called a *Zone diagram* of which the trisector curves constitute an essential special case (with exactly two sites).

By the definition, for any point p on the trisector curve C_A (resp. C_B) there exists a point q that is closest to p on C_B (resp. C_A). Such a point q is called a *partner point* of p. If there exist distance trisector curves then any point on the curves has its partner point, although the partner point may not be unique, and some points on the curves do not serve as partner points.

When the distance is measured in the L_2 (Euclidean) metric, it is known that unique distance trisector curves exist for any two points in the plane [1, 2]. It is also known that such curves are convex, analytic (i.e. given by a convergent power series at a neighbourhood of each point), and computable to accuracy ϵ in time polynomial in $\log \frac{1}{\epsilon}$. However, no explicit polynomial equation determining the curves is known. Furthermore, the authors [1] conjecture that the curves are not expressible by elementary functions and, in particular, not algebraic.

Our goal in this paper is to study distance trisector curves in other distance metrics induced by a parameterized family of natural convex distance functions that approximate Euclidean distance. As we shall see, these trisector curves have structural and computational properties that make them interesting in their own right. However, our primary motivation is to shed further light on Euclidean trisectors. In particular, we outline a new and conceptually simpler proof of the existence and uniqueness of Euclidean trisectors, as

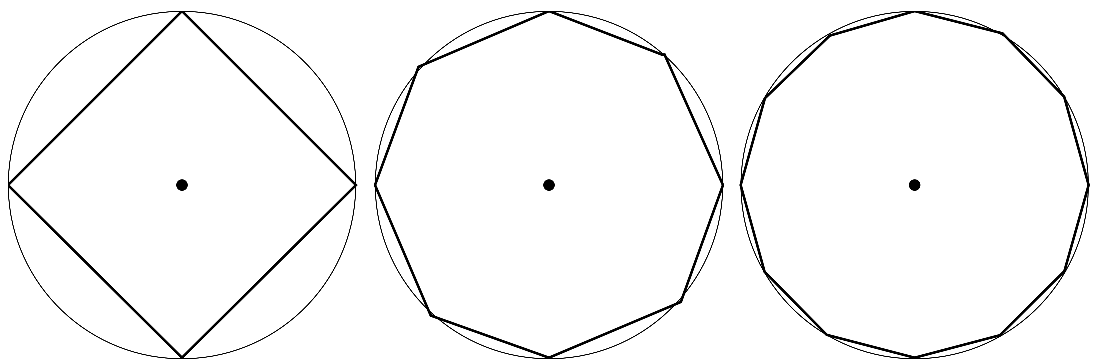


Figure 1. Regular convex $4m$-gons for distance measure D_m.

the limiting case of our convex distance trisectors.

Section 2 introduces our family of convex distance metrics D_m and specifies their relationship with Euclidean distance. Section 3 discusses trisector curves with respect to D_m, with emphasis on the cases m equals 1 (which coincides with the Manhattan metric), 2 and 3. Section 4 summarizes some of the basic properties of general D_m-trisector curves. The proofs of these properties follow from the general construction described in Section 5. Section 6 sketches an application of our results to the existence, uniqueness and efficient construction of Euclidean trisector curves.

2. Regular convex distance metrics

We denote by d the Euclidean distance function. Points q satisfying $d(p,q) = 1$ describe a circle of radius one centred at p. We consider distance functions whose unit distance is defined by a regular convex polygon. Let $\mathcal{P}_m$ be a regular $4m$-gon as shown in Fig. 1. We could use a more general convex polygon to define a convex distance function, but our restriction guarantees that vertical and horizontal chords through the centre are included.

The regular $4m$-gon $\mathcal{P}_m$ has $4m$ vertices $v_0, v_1, \ldots, v_{4m-1}, v_{4m} = v_0$, as shown in Fig. 2. Each of $4m$ angular sectors is specified by angles associated with two vertices. We refer to the ray from the centre towards the i-th vertex v_i as r_i and denote by θ_i the angle formed by r_i with the positive x-axis. The sector bounded by r_{i-1} and r_i is referred to as sector i. Since there are $4m$ angular sectors, we have

$$\theta_i = \frac{\pi}{2m} i = \omega_m i,$$

where $\omega_m = \frac{\pi}{2m}$ is an angle of one sector.

It is easy to see that the line ℓ_i joining vertices v_{i-1} and v_i forms an angle $(2i-1)\omega_m/2$ with the y-axis, and the line normal to ray r_i forms an angle $i\omega_m$.

Given two points p and q, the distance $D_m(p,q)$ is defined by the factor by which $\mathcal{P}_m$ must be scaled so that q is located at its centre and p lies on the boundary. Of course, the distance $D_m(p,q)$ is always at least $d(p,q)$, and the difference approaches zero as m approaches ∞. In fact, the difference between $D_m(p,q)$ and $d(p,q)$ is maximized when q lies just in the middle of a polygon edge of $P_m(p,r)$, $r = D_m(p,q)$. The distance $d(p,q)$ is given by $r\cos(\pi/4m)$ in this case, and we have

$$\begin{aligned} d(p,q) &\leq D_m(p,q) \leq d(p,q)/\cos(\pi/4m) \\ &= d(p,q)(1 + O(1/m^2)). \end{aligned}$$

We first locate the centre of $\mathcal{P}_m$ at q and find an angular sector containing p. For simplicity we assume that $q = (0,0)$ and $p = (x_1, y_1)$ and the point p lies in the first quadrant, that is, $x_1 \geq 0$ and $y_1 \geq 0$. If p lies in the i-th angular sector (centred at q) we must have

$$\theta_i \leq \theta = \tan^{-1}\frac{x_1}{y_1} \leq \theta_{i+1}.$$

We define two associated angles α and β_i as shown in the right figure of Fig. 2.

$$\alpha = \frac{1}{2}(\pi - (\theta_{i+1} - \theta_i)) = \frac{1}{2}(\pi - \omega_m),$$

and

$$\beta_i = \alpha - \theta_i = \frac{1}{2}\pi - (i + \frac{1}{2})\omega_m.$$

The point p is projected to the ray $\overline{qv_i}$ by a line of angle $(\pi - \beta_i)$ to define its distance $D_m(p,q)$.

3. Distance D_m Trisector Curves

We now define distance trisector curves using the distance D_m defined above.

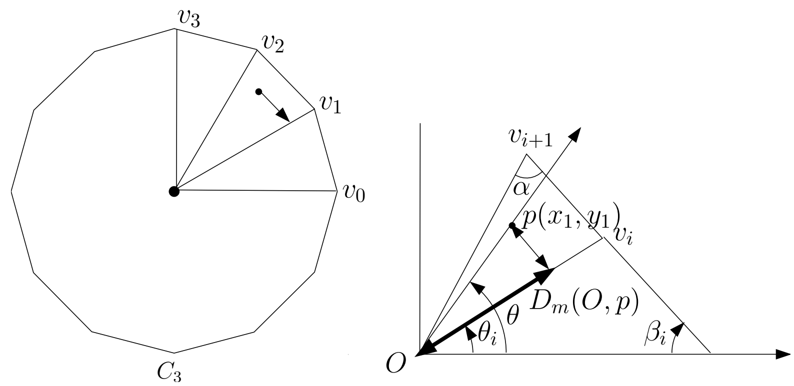


Figure 2. A regular polygon C_3 and a point p in an angular sector.

Given two points A and B in the plane, draw 2 equally-spaced curves, C_A^m and C_B^m between them so that
(1) for any point p on the curve C_A^m we have
$D_m(p, A) = D_m(p, C_B^m)$, and
(2) for any point q on the curve C_B^m we have
$D_m(q, B) = D_m(q, C_A^m)$.

The distance trisector curves start from those points at which the line segment $\overline{AB}$ is partitioned into three parts of equal lengths. One of the main advantages of using a distance measure characterized by a convex $4m$-gon $\mathcal{P}_m$ is that the distance trisector curves are polygonal lines, and thus we can compute them exactly in time polynomial in m.

We can describe (and simultaneously demonstrate the existence of) the polygonal line C_A^m by specifying how the centre of a convex polygon $\mathcal{P}_m$ for distance function D_m slides along the curve C_A^m while the polygon itself touches the point A and the other curve C_B^m by appropriate scaling. First, we consider explicitly the cases where m is 1, 2 and 3 since they are of interest in their own right and they provide useful intuition for the general case.

3.1. D_1-trisectors

In the case $m = 1$, the convex polygon $\mathcal{P}_m$ has a diamond shape. The associated distance metric is the familiar L_1 metric.

The first striking difference between D_m and Euclidean trisectors is that the former are not invariant under rotation. As we illustrate in detail for the case $m = 1$, D_m-trisectors appear very different depending on relative positions of two points.

Fig. 3 shows four different cases for relative positions of points when one point lies to the upper left of the other point. They are classified by an aspect ratio defined by the ratio between vertical length and horizontal length of the axis-parallel rectangle defined by the two points. A similar classification holds when one point lies to the lower left of the other point.

We only consider the cases (a) and (b) (skinny and fat, respectively) in Fig. 3 since other cases are treated symmetrically.

Fat case

When the bounding rectangle is *fat*, that is, its aspect ratio defined by W/H using its width W and height H is between $1/2$ and 2, the trisector curves are easily defined and computed. Fig. 4 shows the trisector curves in this case, which consist of 45 degree lines in the bounding rectangle and horizontal and vertical rays to the infinity which are perpendicular to the rectangle sides.

Denoting the L_1 distance between two points p and q by $D_1(p, q)$, the 45 degree lines are designed so that the distances $D_1(A, A_2)$, $D_1(A_2, B_2)$, and $D_1(B_2, B)$ are all equal. If we denote by d the L_1 distance from A to A_2, for any point p on the line segment A_2A_3 there is a point q (in fact, an interval of such points) on the line segment B_2B_3 such that $D_1(p, q) = d$. This guarantees that the two line segments A_2A_3 and B_2B_3 are really parts of the distance trisector curves.

Next, take any point p on the line segment, say A_1A_2. Then, the L_1 distance between p and A is given as the sum of $D_1(A, A_2)$, which is d, and $D_1(p, A_2)$. Since $D_1(A_2, B_2)$ is equal to d and $D_1(p, B_2)$ is given as the sum of $D_1(p, A_2)$ and $D_1(A_2, B_2)$, the line segment $\overline{A_1A_2}$ is also a part of the distance trisector curves. Proofs for other line segments are similar.

Skinny case

The distance trisector curves in the *skinny* case, when the aspect ratio W/H is greater than 2, look more complicated, but they are polygonal chains. Within the bounding rectangle, they consist of two 45 degree lines as shown in Fig. 5. Those 45 degree lines are placed in a similar way to the fat

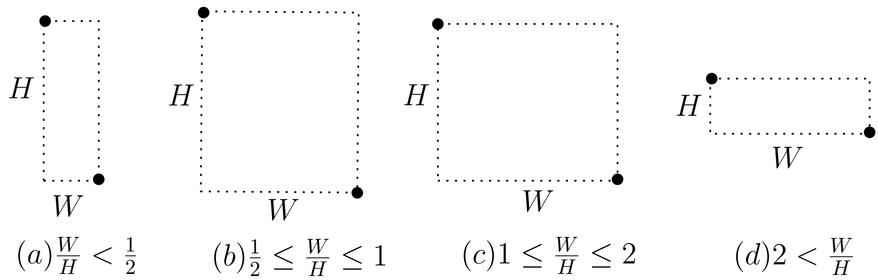


Figure 3. Several cases for relative positions of two points.

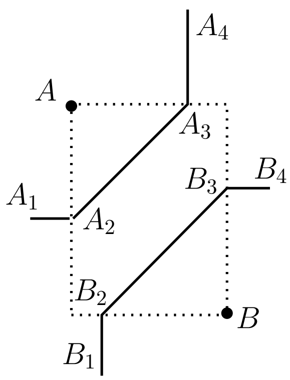


Figure 4. Fat case

case. That is, we walk along the left and bottom sides starting at A, we encounter the endpoints A_3 and B_3 and finally B in the same distance:

$D_1(A, A_3) = D_1(A_3, B_3) = D_1(B_3, B)$.

When they leave from the bounding rectangle, they are extended in the directions roughly perpendicular to the rectangle sides as shown in the figure. More precisely, they lie between two lines starting at the endpoint. For example, at the endpoint A_3 we draw two rays from A_3, one horizontal (perpendicular to the left side) and the other -45 degree which is a part of the equidistant diamond centered at A in the distance AA_3. The line $\overline{A_3A_2}$ is defined so that the vertical distance from $\overline{A_3A_2}$ to the -45 degree line is twice larger than that from $\overline{A_3A_2}$ to the horizontal line. The line segment $\overline{A_4A_5}$ is defined in the same way. It is located between the line perpendicular to the rectangle side (horizontal line) and 45 degree line from A_4 that is a part of the equidistant diamond centred at A in the distance $D_1(A, A_4)$, and the vertical distance from $\overline{A_4A_5}$ to the 45 degree line is twice larger than that from $\overline{A_4A_5}$ to the horizontal line. The endpoint A_5 is determined as the intersection between the ray from A_4 to A_5 and the horizontal line passing through A. The curve for the point B is also defined similarly. The endpoint B_2 is determined by the y-coordinate of the point B.

The point A_2 is determined by the x-coordinate of B_2 while B_5 is determined by the x-coordinate of A_5.

An easy observation here is that any point p on the polygonal line from A_2 to A_5 has its closest point on the polygonal line from B_2 to B_5 right below it (on the vertical line through p).

Degenerate case

A extremal *degenerate* case happens when two points are aligned on a vertical or horizontal line. In this case, the distance trisector curves look quite different from others; furthermore, they are not even uniquely defined. Fig. 6 shows various forms of distance trisector curves for the case. A basic form of the curves shown in Fig. 6 (a) consists of line segments of slope ± 3 followed by horizontal rays. Horizontal rays can be replaced with unbounded rectangular regions shown in Fig. 6 (b). Although two rays on the A side are replaced with rectangular regions in the figure, any horizontal ray can be replaced provided that only one of the rays is replaced in each of the left and right sides. Horizontal rays can be even replaced with non-horizontal rays as shown in Fig. 6 (c) if angles from the horizontal rays are at most 45 degrees.

Lemma 3.1 *Given two points $A(t, 0)$ and $B(-t, 0)$ in the plane, a pair of polygonal lines defined by $C_A = ((+\infty, 3t), (t, 3t), (t/3, 0), (t, -3t), (+\infty, -3t))$ and its symmetric polygonal line defined by $C_B = ((-\infty, 3t), (-t, 3t), (-t/3, 0), (-t, -3t), (-\infty, -3t))$ are distance trisector curves between A and B.*

Proof: If we take any point p on the edge $\overline{A_2A_3}$ between $A_2(t, 3t)$ and $A_3(t/3, 0)$. Then, the point q on C_B that is closest to p is the point on C_B directly left of p (at the same y-coordinate as p). The point p lies on the line $y = 3x + t$. Thus, the L_1 distance between $p = (x, 3x + t)$ and $A(0, t)$ is given by

$$D_1(p, A) = 2x.$$

The L_1 distance between $p = (x, 3x + t)$ and $q = (-x, 3x + t)$ is

$$D_1(p, q) = 2x.$$

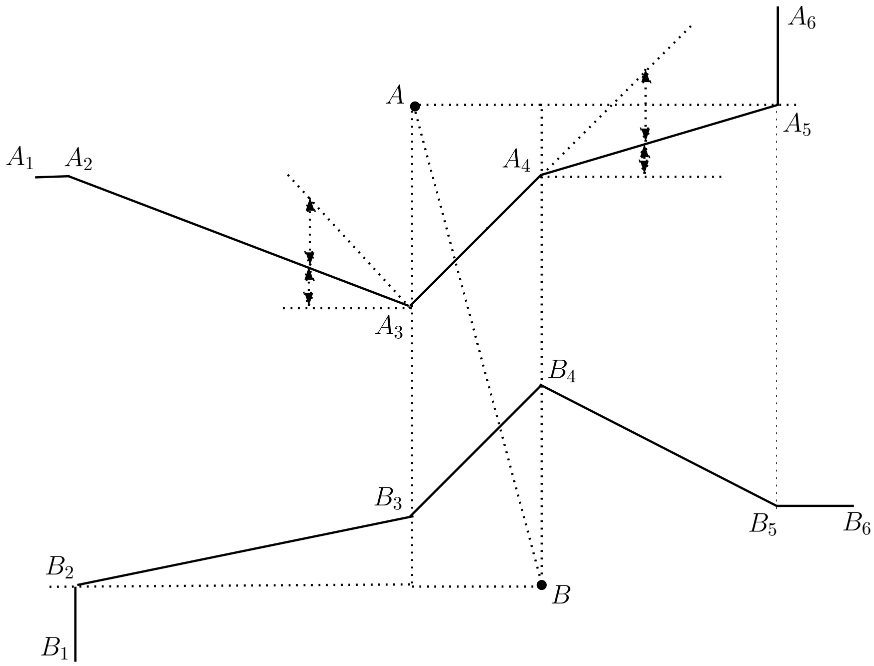
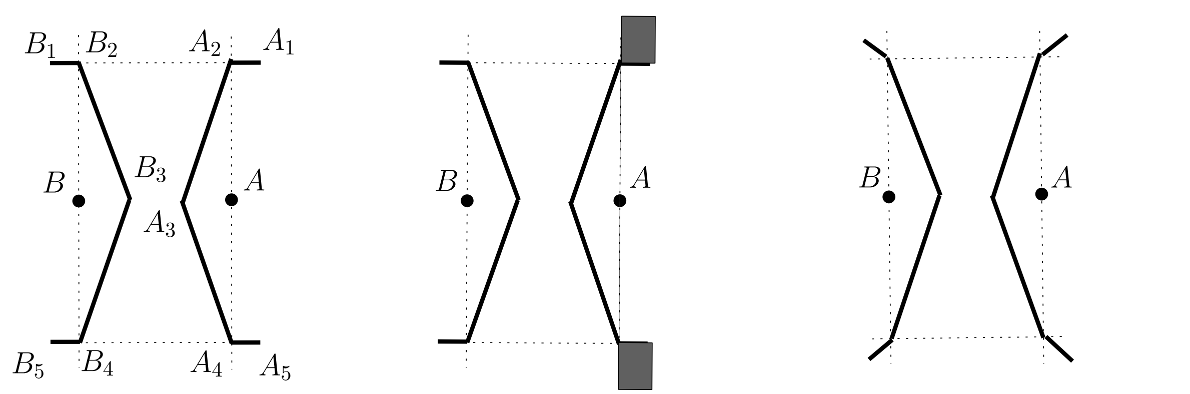


Figure 5. Skinny case.



(a) Basic form (b) curves with regions (c) curves with non-horizontal rays

Figure 6. Degenerate case where two points have the same x coordinates.

Thus, we have

$$D_1(p, A) = D_1(p, q) = D_1(p, C_B).$$

If we take a point p on the horizontal ray emanating from $A_2(t, 3t)$. Then, a point on C_B that is closest to p is $B_2(-t, 3t)$. Since the length of the vertical line segment between A and A_2 is equal to that of the horizontal line segment between A_2 and B_2, we have

$$\begin{aligned} D_1(p, A) &= D_1(p, A_2) + D_1(A_2, A) \\ &= D_1(p, A_2) + D_1(A_2, B_2) \\ &= D_1(p, B_2) \\ &= D_1(p, C_B). \end{aligned}$$

The rest of the proof is symmetric to the above. □

When the upper right horizontal ray is replaced with a region as shown in Fig. 6(b), for any point p in the region its closest point on the curve C_B is the point B_2. Here, the L_1 distance $D_1(p, A)$ between p and A is given as the sum of $D_1(p, A_2)$ and $D_1(A_2, A)$. It is also true that $D_1(p, B_2)$ is given as the sum of $D_1(p, A_2)$ and $D_1(A_2, B_2)$. Thus, in fact we have $D_1(p, A) = D_1(p, B_2) = D_1(p, C_B)$.

The same argument holds when a horizontal ray is replaced with a non-horizontal ray as far as the angle with the opposite horizontal ray is at most 45 degrees. This is because the corner points other than those on the line AB remain the closest points for those points in non-horizontal rays unless the angles exceed 45 degrees.

Construction of the D_1-trisectors

We now outline a procedure for constructing C_A^1 and C_B^1 in the case where points A and B both lie on the x-axis. This will serve as a template for our more general ($m > 1$) construction procedure.

When we construct C_A^1, the centre of $\mathcal{P}_1$ starts at the trisector point A_0 and $\mathcal{P}_1$ touches the point A at vertex v_0 (the right vertex of $\mathcal{P}_1$) and the curve C_B^1 at the other trisector point B_0 by the vertex v_2 (the left vertex of $\mathcal{P}_1$). Then, the centre of $\mathcal{P}_1$ slides along the edge $\overline{A_0A_1}$ of C_A^1, while $\mathcal{P}_1$ touches A by the polygon edge $\overline{v_0v_1}$ and the edge $\overline{B_0B_1}$ of C_B^1 at the vertex v_2, until it reaches A_1 at which it touches A at the vertex v_1 and C_B^1 at B_1 by the vertex v_2. As we have noted, at this point there is some ambiguity on how the curves are extended. We could extend them horizontally or in some angle at most 45 degrees from the horizontal lines. In all cases the side $\overline{v_1v_2}$ of $\mathcal{P}_1$ remains in contact with both B_1 and A. Fig. 7 illustrates slides of the convex polygon $\mathcal{P}_1$ as the trisector curve is extended at 45 degrees.

We can calculate the coordinates of the points A_0, A_1, B_0 and B_1. For simplicity, let $A = (3, 0)$ and $B = (-3, 0)$. We have $A_0 = (1, 0)$ and $B_0 = (-1, 0)$. Since A, B, B_1, A_1 form a square, we have $A_1 = (3, -3)$ and $B_1 = (-3, -3)$.

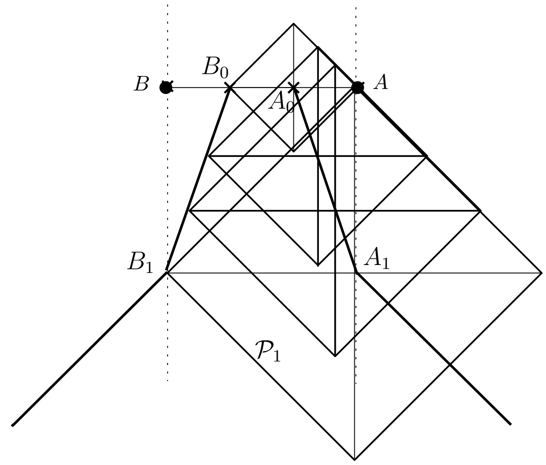


Figure 7. The distance trisector curves for the case $m = 1$.

3.2. D_2-trisectors

The convex polygon $\mathcal{P}_2$ we use for the case $m = 2$ is an octagon. The polygon $\mathcal{P}_2$ is initially centred at the trisector point A_0 where it touches the point A at it vertex v_0 and the curve C_B^m at B_0 by v_4.

After this, the centre of $\mathcal{P}_2$ slides along the edge $\overline{A_0A_1}$ of C_A^2, while $\mathcal{P}_2$ touches A by the edge $\overline{v_0v_1}$ and the edge $\overline{B_0B_1}$ of C_B^2 at v_4, until the centre of $\mathcal{P}_2$ reaches A_1 at which point $\mathcal{P}_2$ touches A at the vertex v_1 and C_B^2 at B_1 by the vertex v_4. The polygon $\mathcal{P}_2$ further slides along $\overline{A_1A_2}$ while touching A by $\overline{v_1v_2}$ and $\overline{B_1B_2} \in C_B^2$ at $\overline{v_3v_4}$ until the centre reaches A_2 at which point $\mathcal{P}_2$ touches A at v_2 and $\overline{B_1B_2} \in C_B^2$ at $\overline{v_3v_4}$. Now, $\overline{B_1B_2}$ coincides with $\overline{v_3v_4}$. At this moment, the centre of the polygon located at A_2 lies just below the point A and just to the right of B_2; we can easily calculate the coordinates of A_2 and B_2, that is, $A_2 = (3, -3)$ and $B_2 = (-3, -3)$.

After this point we can extend the curves C_A^2 and C_B^2 at 45 degrees from the horizontal lines while touching A by $\overline{v_2v_3}$ and B_1 at v_3. Here, note that B_1 always coincides with v_3, and thus the centre of $\mathcal{P}_2$ moves along the chord incident to v_3 which forms -45 degrees to the horizontal line. Fig. 8 illustrates how the convex polygon $\mathcal{P}_2$ slides along the curves.

As before, it is straightforward to confirm that the curves defined above are the distance trisector curves for two points A and B. For any point p on the first edge $\overline{A_0A_1}$, we have

$$D_2(p, A) = D_2(p, \overline{v_0v_1}) = D_2(p, v_0),$$
$$D_2(p, C_B^2) = D_2(p, \overline{B_0B_1}) = D_2(p, v_4).$$

They are equal to each other since they are chords of $\mathcal{P}_2$ when its centre is located at p.

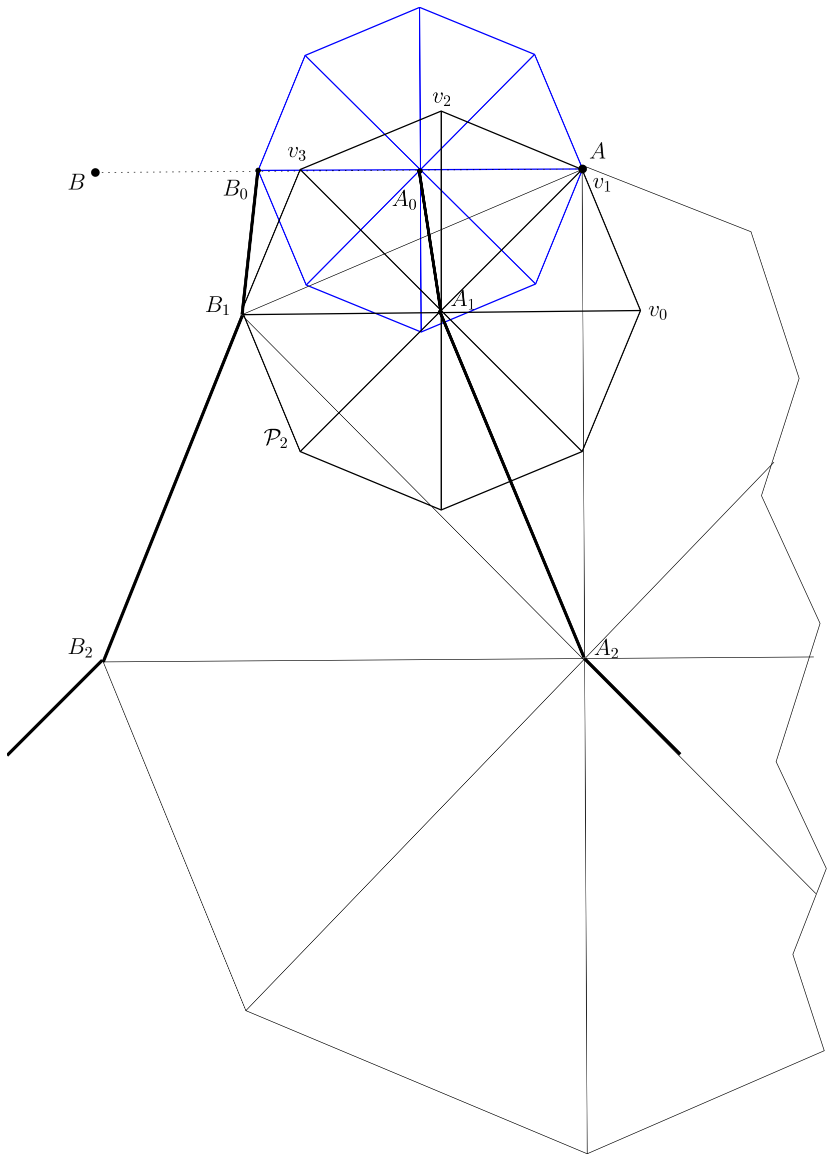


Figure 8. The distance trisector curves for the case $m = 2$.

For any point p on the second edge $\overline{A_1A_2}$, we have

$$D_2(p, A) = D_2(p, \overline{v_1v_2}) = D_2(p, v_1),$$
$$D_2(p, C_B^2) = D_2(p, \overline{B_1B_2}) = D_2(p, v_3).$$

They are equal to each other for the same reason as above.

For any point p on the third infinite edge $\overline{A_2A_3}$, we have

$$D_2(p, A) = D_2(p, \overline{v_2v_3}) = D_2(p, v_2),$$
$$D_2(p, C_B^2) = D_2(p, \overline{B_1B_2}) = D_2(p, v_3).$$

Again, they are equal to each other for the same reason as above.

3.3. D_3-trisectors

A similar construction can be used for the D_3-trisectors. Refer to Fig. 9 for an illustration:
(0) We first centre the polygon $\mathcal{P}_3$ at A_0 to touch A at v_0 and C_B^3 at B_0 by the vertex v_7.
(1) Then, the centre of $\mathcal{P}_3$ slides along the edge $\overline{A_0A_1}$ as before until it reaches A_1 at which point $\mathcal{P}_3$ touches A at v_1 and B_1 at v_6.
(2) $\mathcal{P}_3$ slides along the edge $\overline{A_1A_2}$ while touching A at $\overline{v_1v_2}$ and $\overline{B_1B_2}$ at $\overline{v_5v_6}$ until its centre reaches A_2 at which point $\mathcal{P}_3$ touches A at v_2 and $\overline{B_1B_2}$ at $\overline{v_5v_6}$. Now, $\overline{B_1B_2}$ coincides with $\overline{v_5v_6}$.
(3) $\mathcal{P}_3$ slides along the edge $\overline{A_2A_3}$ while touching A at $\overline{v_2v_3}$ and $\overline{B_1B_2}$ at $\overline{v_5v_6}$ until its centre reaches A_3 at which point $\mathcal{P}_3$ touches A at v_3 and B_2 at v_5.
(4) $\mathcal{P}_3$ slides along the edge $\overline{A_3A_4}$ while touching A at $\overline{v_3v_4}$ and $\overline{B_2B_3}$ at v_5 until its centre reaches A_4 at which point $\mathcal{P}_3$ touches B_3 at vertex v_5. At this point the trisector curve exhibits its first non-convexity.

4. Basic properties of D_m-trisector curves

We summarize below some of the important properties of D_m-trisector curves. The proof of these properties follows from an analysis of the general construction procedure presented in the next section.

Polygonal Chains: C_A^m and C_B^m are both polygonal chains consisting of $O(m)$ line segments.

Symmetricity: C_A^m and C_B^m are symmetric with respect to the x- and y-axes.

Monotonicity: The lower parts of C_A^m and C_B^m are both x- and y-monotone.

Distance Monotonicity: As a point walks along the lower part of the curve C_A^m (resp. C_B^m), the distance from the point A (resp. B) monotonically increases.

Star-shapedness: The curve C_A^m (resp. C_B^m) is star-shaped with respect to point A (resp. B).

Co-star-shapedness: The curve C_A^m (resp. C_B^m) is star-shaped with respect to its closest points on the curve C_B^m (resp. C_A^m).

Non-convexity In general, the curves C_A^m and C_B^m, unlike the Euclidean trisector curves, are not convex.

Partner Point: If p is any point on one of the curves C_A^m, $t(p)$ is its partner point and $t(t(p))$ is its partner's partner point, then $|y(p)| \geq |y(t(p))|$. Furthermore, $\angle(A, B, t(p)) + \angle(B, A, t(t(p)) \leq 4/3\angle(B, A, p)$.

5. Construction of D_m-trisectors in the general case

One of the main advantages of using the distance functions D_m is that distance trisector curves are given as polygonal lines that can be easily calculated and consist of $O(m)$ line segments, as we will show later.

Just as before, the distance trisector curves start from those points at which the line segment $\overline{AB}$ is partitioned into three parts of equal lengths. We represent the lower halves of the distance trisector curves (C_A^m, C_B^m) for two points A and B aligned on a horizontal line (we can assume that they are located at $(3, 0)$ and $(-3, 0)$, respectively, without any loss of generality) using two polygonal lines: $C_A^m = (A_0, A_1, A_2, \cdots)$ and $C_B^m = (B_0, B_1, B_2, \cdots)$, where A_0 and B_0 are trisector points of the horizontal line $\overline{AB}$. Because of the symmetry in the definition, A_i is symmetric to B_i with respect to the y-axis for each $i = 0, 1, \ldots$, that is, if we denote the coordinates of A_i by (x_i, y_i) then the coordinates of B_i must be $(-x_i, y_i)$.

We can describe the polygonal line C_A^m by specifying how a convex polygon $\mathcal{P}_m$ for distance function slides along the curve C_A^m while touching the point A and the other curve C_B^m by appropriate scaling of the polygon $\mathcal{P}_m$. Recall the notation of section 2, describing the structure of $\mathcal{P}_m$.

5.1. The first segments of the curve C_A^m and C_B^m

At the start we have a copy of polygon $\mathcal{P}_m$ centred at A_0 touching point A (at vertex v_0) and B_0 (at vertex v_{2m}). (Another copy centred at B_0 touches B and A_0). We trace the locus of the centre of this first copy. As our polygon moves away from A_0 it is supported by A along edge $\overline{v_0v_1}$, while vertex v_{2m} follows edge $\overline{B_0B_1}$. It follows that the lines $\overline{A_0A_1}$ and $\overline{B_0B_1}$ trisect the sector formed by lines ℓ_1 through A and ℓ_{2m} through B. Points A_1 and B_1 are defined by the intersection of these trisecting lines and the rays r_1 and r_{2m-1} through A and B respectively. At point A_1

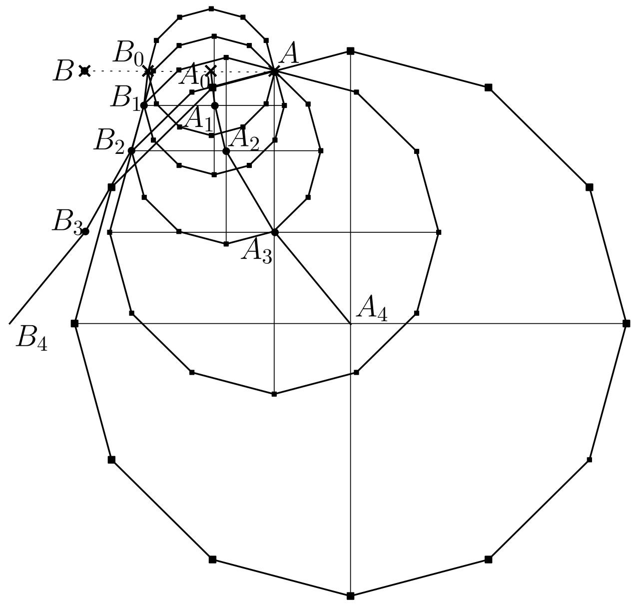


Figure 9. The distance trisector curves for the case $m = 3$.

the point A enters sector 2 of $\mathcal{P}_m$. As C_A^m continues the associated copy of $\mathcal{P}_m$ contacts A along edge $\overline{v_1 v_2}$ and point B_1 along edge $\overline{v_{2m-1} v_{2m}}$. Thus the line from A_1 towards A_2 bisects these contact lines and forms an angle θ_1 with the vertical. Point A_2 is defined by the (simultaneous) intersection of this line with the ray r_{2m+2} from A and the ray r_{4m-1} from B_1.

5.2. Continuing the curves

In general the curve C_A^m (and, by symmetry, C_B^m) continues in this fashion always tracing the bisector of two lines one determined by the sector contact at point A and the other by the partner point on curve C_B^m. If A changes its sector contact (to a higher indexed sector), the angle of the trisector curve C_A^m, with respect to the y-axis, increases.

Otherwise, there are four cases depending on the nature of the polygon-curve contact at the partner point.

vertex-edge contact In this case, polygon $\mathcal{P}_m$ slides along this contact edge retaining contact with the same vertex of $\mathcal{P}_m$.

edge-vertex contact In this case, polygon $\mathcal{P}_m$ slides along its polygonal contact edge.

vertex-vertex contact In this case, polygon $\mathcal{P}_m$ instantaneously changes contact to one of the preceding two types. Consequently, the angle of C_A^m with respect to the y-axis decreases at this point (i.e. we encounter a new vertex A_i).

double vertex-edge contact Because C_B^m is not convex, it is possible for there to be two distinct vertex partner points. As above, this signals a change in the angle of C_A^m with respect to the y-axis (in this case, an increase) and a new C_A^m vertex.

The piecewise linearity of C_A^m is clear from this case analysis. Other properties follow from a more careful (quantitative) analysis of the way angles change along C_A^m. For any non-vertex point p on C_A^m (resp. C_B^m) let $\alpha(p)$ denote the angle formed by the edge of C_A^m(resp. C_B^m) containing p with the y-axis. Similarly, let $\sigma(p)$ denote the angle formed by the ray from p to A (resp. B) with the positive (resp. negative) x-axis, and let $\lambda(p)$ denote the angle formed by the ray from p to $t(p)$ with the x-axis.

Lemma 5.1 *(a) If point p on C_A^m sees point A in the interior of sector i and its partner point $t(p)$ in the interior of sector $2m - j$, then (i) $\sigma(p) \in [(i-1)\omega_m, i\omega_m]$, (ii) $\lambda(p) \in [j\omega_m, (j+1)\omega_m]$, and (iii) $\alpha(p) = (i-j-1)\omega_m/2$. (b) If point p on C_A^m sees point A in the interior of sector i and its partner point $t(p)$ along ray r_{2m-j} then $\alpha(p) = (\alpha(t(p)) + (2i-1)\omega_m/2)/2$.*

Proof: Follows directly from the case analysis above. □

As a direct consequence of the above lemma, it follows by induction on i that if p sees A in the interior of sector i then $(2i-1)\omega_m/8 \le \alpha(p) \le (2i-1)\omega_m/4$. Hence, $(\sigma(p) - \omega_m/2)/4 \le \alpha(p) \le (\sigma(p) + \omega_m/2)/2$.

Lemma 5.2 *If point p lies on C_A^m and $\sigma(t(p)) \ge 9\omega_m$, then the y-coordinate of $t(p)$ is at most a fraction $8/9$ of the y-coordinate of p.*

Proof: Since $\lambda(p) \in [\alpha(t(p)) - \omega_m, \alpha(t(p)) + \omega_m]$, it follows that $\lambda(p) \geq (\sigma(t(p)) - 9\omega_m/2)/4$. Hence $\lambda(p) \geq \sigma(t(p))/8$, provided $\sigma(t(p)) \geq 9\omega_m$. It follows that the difference in the y-coordinates of p and $t(p)$ is at least $1/8$-th of the y-coordinate of $t(p)$). □

As a direct consequence of the above lemma, it follows that the polygonal chain C_A^m from point A_0 to point p consists of $O(m)$ segments.

6. Existence, uniqueness and approximation of the Euclidean trisector curves

Since D_m approaches Euclidean distance in the limit as m goes to infinity, it seems intuitively clear that the D_m-trisector should approach the Euclidean trisector. Indeed, if we define the distance $\tilde{D}_m$ to be D_m scaled by a factor $\cos(\pi/4m)$ (so that $d(p,q) \geq \tilde{D}_m(p,q) \geq d(p,q)cos(\pi/4m)$), we can observe empirically that the D_m and $\tilde{D}_m$ trisectors bound and converge to the Euclidean trisectors).

However, it is not immediately obvious how to demonstrate this formally. In this section, we sketch a proof based on D_m-trisectors.

First, we modify the D_m-trisector curve C_A^m (and similarly C_B^m) by placing a copy of $\mathcal{P}_m$ with radius $D_m(p,A)/m$ at every point p of C_A^m. This defines a variable width neighbourhood of C_A^m that we denote by $\hat{C}_A^m$.

Next, we argue that points on the A side of $\hat{C}_A^m$ have the property that their D_k-distance to A is less than the average of their D_k distances to the boundaries of $\hat{C}_B^m$, for all $k \geq m$ (including $k = \infty$, i.e Euclidean distance). Similarly, points on the opposite (B) side of $\hat{C}_A^m$ have the property that their D_k-distance to A is greater than the average of their D_k distances to the boundaries of $\hat{C}_B^m$. Thus, by continuity, for every $\delta > D_k(A,B)/3$ there exists a point in $\hat{C}_A^m$ of D_k distance δ to A and average D_k distance δ to the boundaries of $\hat{C}_B^m$.

It follows that the intersections of neighbourhoods $\hat{C}_A^m$, for $m \leq k$, converge to the Euclidean trisector C_A as k approaches infinity. The details, together with a discussion of an efficient Euclidean trisector approximation procedure based on this scheme, will appear in an expended version of this paper.

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