The Structure and Number of Global Roundings of a Graph

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Abstract

Given a connected weighted graph $G = (V, E)$, we consider a hypergraph $H_G = (V, \mathcal{P}_G)$ corresponding to the set of all shortest paths in G. For a given real assignment $\mathbf{a}$ on V satisfying $0 \leq \mathbf{a}(v) \leq 1$, a global rounding α with respect to H_G is a binary assignment satisfying that $|\sum_{v \in F} \mathbf{a}(v) - \alpha(v)| < 1$ for every $F \in \mathcal{P}_G$. We conjecture that there are at most $|V| + 1$ global roundings for H_G, and also the set of global roundings is an affine independent set. We give several positive evidences for the conjecture.

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1 Introduction

Given a real number a, an integer k is a *rounding* of a if the difference between a and k is strictly less than 1, or equivalently, if k is the floor $\lfloor a \rfloor$ or the ceiling $\lceil a \rceil$ of a. We extend this usual notion of rounding into that of *global rounding* on hypergraphs as follows.

Let $H = (V, \mathcal{F})$, where $\mathcal{F} \subset 2^V$, be a hypergraph on a set V of n nodes. Given a real valued function $\mathbf{a}$ on V, we say that an integer valued function α on V is a *global rounding* of $\mathbf{a}$ with respect to H, if $w_F(\alpha)$ is a rounding of $w_F(\mathbf{a})$ for each $F \in \mathcal{F}$, where $w_F(f)$ denotes $\sum_{v \in F} f(v)$. We assume in this paper that the hypergraph contains all the singleton sets as hyperedges; thus, $\alpha(v)$ is a rounding of $\mathbf{a}(v)$ for each v, and we can restrict our attention to the case where the ranges of $\mathbf{a}$ and α are $[0, 1]$ and $\{0, 1\}$ respectively.

This notion of global roundings on hypergraphs is closely related to that of *discrepancy* of hypergraphs[6,10,11,4]. Given $\mathbf{a}$ and $\mathbf{b} \in [0, 1]^V$, define the *discrepancy* $D_H(\mathbf{a}, \mathbf{b})$ between them on H by

$$D_H(\mathbf{a}, \mathbf{b}) = \max_{F \in \mathcal{F}} |w_F(\mathbf{a}) - w_F(\mathbf{b})|.$$

The supremum $\sup_{\mathbf{a} \in [0,1]^V} \min_{\alpha \in \{0,1\}^V} D_H(\mathbf{a}, \alpha)$ is called the linear (or inhomogeneous) discrepancy of H, and it is a quality measure of approximability of a real vector with an integral vector to satisfy constraints given by the linear system corresponding to H.

Thus, the set of global roundings of $\mathbf{a}$ is the set of integral points in the open unit ball around $\mathbf{a}$ where the distance is measured by the discrepancy D_H. It is known that the open ball always contains an integral point for any "input" $\mathbf{a}$ if and only if the hypergraph is unimodular (see [4,7]). This fact is utilized in digital halftoning applications [1,2]. It is NP-hard to decide whether the ball is empty (i.e. containing no integral point) or not even for some very simple hypergraphs [3].

In this paper, we are interested in the maximum number $\nu(H)$ of integral points in an open unit ball under the discrepancy distance.

This direction of research was initiated by Sadakane *et al.*[13] where the authors discovered a surprising fact that $\nu(I_n) \leq n+1$ where I_n is a hypergraph on $V = \{1, 2, .., n\}$ with edge set $\{[i, j]; 1 \leq i \leq j \leq n\}$ consisting of all subintervals of V. We can also see that $\nu(H) \geq n + 1$ for any hypergraph H: if we let $\mathbf{a}(v) = \epsilon$ for every v, where $\epsilon < 1/n$, then any binary assignment on V that assigns 1 to at most one vertex is a global rounding of H, and hence $\nu(H) \geq n + 1$.

Given this discovery, it is natural to ask for which class of hypergraphs this property $\nu(H) = n + 1$ holds. The understanding of such classes may well be related to algorithmic questions mentioned above. In fact, Sadakane *et al.* give an efficient algorithm to enumerate all the global roundings of a given input on I_n.

In this paper, we show that $\nu(H) = n+1$ holds for a considerably wider class of hypergraphs. Given a connected G in which edges are possibly weighted by a positive value, we define a *shortest-path hypergraph* H_G generated by G as follows: a set F of vertices of G is an edge of H_G if and only if F is the set of vertices of some shortest path[1] in G with respect to the given edge weights. In this notation, $I_n = H_{P_n}$ for the path P_n on n vertices. Note that we permit more than one shortest path between a pair of nodes if they have the same weight. We give several basic properties of the structure of a set of global roundings for H_G, and prove the following theorem:

Theorem 1.1 *$\nu(H_G) = n + 1$ holds for the shortest-path hypergraph H_G, if G is a tree, a cycle, a tree of cycles, an unweighted mesh, or an unweighted k-tree.*

Based on the positive evidence above and some failed attempts in creating counterexamples, we conjecture that the result holds for general connected graphs.

Conjecture 1.2 *$\nu(H_G) = n + 1$ for any connected graph G with n nodes.*

2 Preliminaries

We start with the following easy observations:

Lemma 2.1 *For hypergraphs $H = (V, \mathcal{F})$ and $H' = (V, \mathcal{F}')$ such that $\mathcal{F} \subset \mathcal{F}'$, $\nu(H) \geq \nu(H')$.*

Definition 1 *A set A of binary functions on V is called H-compatible if, for each pair α and β in A, $|w_F(\alpha) - w_F(\beta)| \leq 1$ holds for every hyperedge F of H.*

Lemma 2.2 *For a given input real vector $\mathbf{a}$, the set of global roundings with respect to H is H-compatible.*

Proof: Suppose that α and β are two different global roundings of an input $\mathbf{a}$ with respect to a hypergraph H. We have $|w_F(\alpha) - w_F(\beta)| \leq |w_F(\mathbf{a}) -$

[1] Precisely speaking, minimum weight path

$w_F(\alpha)| + |w_F(\mathbf{a}) - w_F(\beta)| < 2$. Since the value must be integral, we have the lemma. □

Thus, any set of global roundings of a given input is an H-compatible set. Conversely, we can prove that any H-compatible set is a set of global roundings for a suitable input vector.

Lemma 2.3 *Given an H-compatible set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, A is a set of global roundings of the center of gravity $\mathbf{g} = \frac{1}{m}\sum_{i=1}^{m} \alpha_i$ of A.*

Proof: It is clear that $\mathbf{g} \in [0,1]^V$. For each α_i, the set P_i of vectors $\mathbf{x}$ in $[0,1]^V$ satisfying that $D_H(\mathbf{x}, \alpha_i) < 1$ is a convex set (indeed, it is a convex polytope). Also, α_j is in the closure of P_i for all $j \neq i$. Thus, $\mathbf{g}_i = \frac{1}{m-1}\sum_{j\neq i} \alpha_j$ is in the closure of P_i, and $\mathbf{g} = \frac{1}{m}((m-1)\mathbf{g}_i + \alpha_i)$ is in the interior of P_i. Thus, $D_H(\mathbf{g}, \alpha_i) < 1$, and α_i is a global rounding of $\mathbf{g}$. □

Therefore, $\nu(H)$ equals the largest cardinality of an H-compatible set. The definition of an H-compatible set does not include the input vector $\mathbf{a}$, and facilitates the combinatorial analysis.

For a vector $\mathbf{q}$ in the n-dimensional real space $\mathcal{R}^n$, $\tilde{\mathbf{q}}$ is the vector in $\mathcal{R}^{n+1}$ obtained by appending 1 as the last coordinate value: i.e., $\tilde{\mathbf{q}} = (q_1, q_2, \ldots, q_n, 1)$ if $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. Vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_s$ are called *affine independent* in $\mathcal{R}^n$ if $\tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \ldots, \tilde{\mathbf{q}}_s$ are linearly independent in $\mathcal{R}^{n+1}$. If every H-compatible set is an affine independent set regarded as a set of vectors in the n-dimensional space, we call H satisfies the *affine independence property.*

Conjecture 2.4 *For any connected graph G, H_G satisfies the affine independence property.*

It is clear that Conjecture 2.4 implies Conjecture 1.2.

3 Properties of Compatible Sets of General Graphs

For a binary assignment α on V and a subset X of V, $\alpha|_X$ denotes the restriction of α on X. Let $V = X \cup Y$ be a partition of V into nonintersecting subsets X and Y of vertices. For binary assignments α on X and β on Y, $\alpha \oplus \beta$ is a binary assignment on V obtained by concatenating α and β: That is, $\alpha \oplus \beta(v) = \alpha(v)$ if $v \in X$, otherwise it is $\beta(v)$.

The following is a key lemma for our theory.

Lemma 3.1 *Let $G = (V, E)$ be a connected graph, and let $V = X \cup Y$ be a* *partition of V. Let α_1 and α_2 be different assignments on X and let β_1 and β_2 be different assignments on Y. Then, the set $\mathcal{F} = \{ \alpha_1 \oplus \beta_1, \alpha_1 \oplus \beta_2, \alpha_2 \oplus \beta_1, \alpha_2 \oplus \beta_2 \}$ cannot be H_G-compatible.*

Proof: Consider $x \in X$ satisfying $\alpha_1(x) \neq \alpha_2(x)$ and $y \in Y$ satisfying $\beta_1(y) \neq \beta_2(y)$. We choose such x and y with the minimum shortest path length. Thus, on each internal node of a shortest path $\mathbf{p}$ from x to y, all four assignments in $\mathcal{F}$ take the same value. Without loss of generality, we assume $\alpha_1(x) = \beta_1(y) = 0$ and $\alpha_2(x) = \beta_2(y) = 1$. Then, $w_{\mathbf{p}}(\alpha_2 \oplus \beta_2) = w_{\mathbf{p}}(\alpha_1 \oplus \beta_1) + 2$, and hence violate the compatibility. □

Corollary 3.2 *Let A be an H_G-compatible set, and let $A|_X$ and $A|_Y$ be the set obtained by restricting assignments of A to X and Y, respectively, for a partition (X, Y) of V. If $A|_X$ and $A|_Y$ have ν_X and ν_Y elements, respectively, and $\nu_X \geq \nu_Y$, $|A| \leq \min\{\nu_Y(\nu_Y - 1)/2 + \nu_X, \nu_Y\sqrt{\nu_X} + \nu_X\}$.*

Proof: If we construct a bipartite graph with node sets corresponding to $A|_X$ and $A|_Y$, in which two nodes $\alpha \in A|_X$ and $\beta \in A|_Y$ are connected by an edge if $\alpha \oplus \beta \in A$, Lemma3.1 implies the K_{22}-free property of the graph. Thus, the corollary follows from a famous result in extremal graph theory ([5], Lemma 9). □

If we consider the case where $|X| = n - 1$ and $|Y| = 1$, we have $|A| \leq 1 + \nu_X$, since $\nu_Y \leq 2$. However, although the recursive formula $f(n) \leq 1 + f(n-1)$ gives a linear upper bound of $f(n)$, this does not imply that $\nu(G) = O(n)$, since the restriction $A|_X$ is not always an $H_{G'}$-compatible set where G' is the induced subgraph by X of G.

The affine independence of an H-compatible set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ means that any linear relation $\sum_{1 \leq i \leq m} c_i \alpha_i = 0$ satisfying that $\sum_{1 \leq i \leq m} c_i = 0$ implies that $c_i = 0$ for $1 \leq i \leq m$. We can prove its special case as follows: [2]

Proposition 3.3 *If α, β, α', and β' are mutually distinct elements of an H_G-compatible set for some graph G, then it cannot happen that $\alpha - \beta = \alpha' - \beta'$.*

Proof: Assume on the contrary that $\alpha - \beta = \alpha' - \beta'$. Let X be the subset of V consisting of u satisfying $\alpha(u) = \beta(u)$, and let Y be its complement in V. Let $\alpha = \xi \oplus \eta$, where ξ and η are the parts of α on X and Y, respectively. Thus, $\beta = \xi \oplus \bar{\eta}$, where $\bar{\eta}$ is obtained by flipping all the entries of η.

Let $\alpha' = \xi' \oplus \eta'$. Then, since $\alpha - \beta = \alpha' - \beta'$, $\beta' = \xi' \oplus \bar{\eta'}$. Moreover, $\eta - \bar{\eta}$ and $\eta' - \bar{\eta'}$ are vectors whose entries are 1 and -1 because of the definition of Y, and hence $\eta - \bar{\eta} = \eta' - \bar{\eta'}$ implies that $\eta = \eta'$.

[2] This fact was suggested by Günter Rote.

Thus, all of $\xi \oplus \eta$, $\xi \oplus \bar{\eta}$, $\xi' \oplus \eta$, and $\xi' \oplus \bar{\eta}$ are in A; this contradicts with Lemma 3.1. □

4 Graphs for Which The Conjectures Hold

4.1 Graphs with Path-Preserving Ordering

Given a connected graph $G = (V, E)$, consider an ordering $v_1, v_2, \ldots, v_n$ of nodes of V. Let $V_i = \{v_1, v_2, \ldots, v_i\}$, and let G_i be the induced subgraph of G by V_i. The ordering is path-preserving if G_i is connected for each i, and every shortest path in G_i is a shortest path in G. It is clear that a tree with arbitrary edge lengths and a complete graph with a uniform edge length have path-preserving orderings. More generally, a k-tree with a uniform edge length has a path-preserving ordering by its definition. A d-dimensional mesh, where each edge has unit length, is also a typical example.

Theorem 4.1 *If G has a path-preserving ordering, H_G satisfies the affine independence property.*

Proof: We prove the statement by induction on $n = |V|$. If $n = 1$, the statement is trivial, since $(0, 1)$ and $(1, 1)$ are linearly independent. If G has a path-preserving ordering, it gives a path-preserving ordering for G_{n-1} that has $n-1$ nodes. Thus, from the induction hypothesis, we assume that any $H_{G_{n-1}}$-compatible set of binary assignments is an affine independent set. Let π be the restriction map from $\{0, 1\}^V$ to $\{0, 1\}^{V_{n-1}}$ defined by $\pi(\alpha)(v) = \alpha(v)$ for every $v \in V_{n-1}$. Let A be an H_G-compatible set, and let $\pi(A) = \{\pi(\alpha) : \alpha \in A\}$ be the set obtained by restricting A to V_{n-1} and removing the multiplicities. The set $\pi(A)$ must be an $H_{G_{n-1}}$-compatible set: otherwise, there must be a shortest path in G_{n-1} violating the compatibility condition for A, which cannot happen since the path is also a shortest path in G.

For each $\beta \in \pi(A)$, let $\beta \oplus 0$ and $\beta \oplus 1$ be its extension in $\{0, 1\}^V$ obtained by assigning 0 and 1 to v_n, respectively. Naturally, $\pi^{-1}(\beta)$ is a subset of $\{\beta \oplus 0, \beta \oplus 1\}$. For any two different assignments β and γ in $\pi(A)$, it cannot happen that all of $\beta \oplus 0$, $\beta \oplus 1$, $\gamma \oplus 0$, and $\gamma \oplus 1$ are in A. Indeed, this is a special case of Lemma 3.1 for $X = V_{n-1}$ and $Y = \{v_n\}$. Thus, there is at most one rounding in $\pi(A)$ satisfying that its inverse image by π contains two elements.

List the elements of A as $\alpha_1, \ldots, \alpha_k$ where $\alpha_1 = \beta \oplus 0$ and $\alpha_2 = \beta \oplus 1$ for some $\beta \in \pi(A)$. Suppose a linear relation $\sum_{1 \le i \le k} c_i \alpha_i = 0$ holds with $\sum_{1 \le i \le k} c_i = 0$. By the induction hypothesis that $\pi(A)$ is affine independent, we have $c_1 + c_2 = 0$ and $c_i = 0$ for $3 \le i \le k$. Because of the last components of the vectors, it follows that $c_1 = c_2 = 0$ as well. □

Corollary 4.2 *For a connected graph G, if we consider the hypergraph H associated with the set of all paths in G (irrespective of their lengths), H satisfies the affine independence property.*

Proof: Consider a spanning tree T of G. Then the hypergraph associated with the set of all paths in G has the same node set as H, and its hyperedge set is a subset of that of H. Hence, it suffices to prove the statement for T, which has a path-preserving ordering. Every path in T is a shortest path in T; hence, the set is H_T-compatible, and consequently, affine independent. □

4.2 Connecting Two Graphs

An edge e in a connected graph G is called a *bridge* if the graph is separated into two connected components by removing e from G.

Theorem 4.3 *Suppose that a graph G has a bridge e separating $G - \{e\}$ into two connected components G_1 and G_2. Then, $\nu(G) \leq \nu(G_1) + \nu(G_2) - 1$. Moreover, if both of H_{G_1} and H_{G_2} satisfy the affine independence property, H_G does.*

Proof: Consider an H_G-compatible set A. Let $A_i = \{\alpha|_{V_i} : \alpha \in A\}$, where V_i are vertex sets of G_i for $i = 1, 2$. It is clear that A_i is an H_{G_i}-compatible set for each $i = 1, 2$. We construct a bipartite graph M whose vertex set corresponds to A_1 and A_2, where an edge is given between two members $\beta \in A_1$ and $\gamma \in A_2$ if and only if $\beta \oplus \gamma \in A$. We claim that the M is a forest. From this claim, it is straightforward to see that $\nu(G) \leq \nu(G_1) + \nu(G_2) - 1$, since $|A|$ is the number of edges in M.

In order to prove the claim, consider the endpoint v_1 of the bridge e in G_1. We construct a shortest-path tree T from v_1 in G_1, and give the breadth-first ordering $v_1, v_2, \ldots, v_t$ of vertices of G_1 along this tree. This ordering is a path-preserving ordering of T, although it may not be a path-preserving ordering of G_1. Let $U^j = \{v_1, v_2, \ldots, v_j\}$, and let A_1^j be $A_1|_{U^j}$. We consider a bipartite graph M_j whose vertex set corresponds to A_1^j and A_2, where an edge is given between two members $\beta^j \in A_1^j$ and $\gamma \in A_2$ if and only if there exists $\beta \in A_1$ such that $\beta^j = \beta|_{U^j}$ and $\beta \oplus \gamma \in A$.

It suffices to show that M_j is a forest for every j, since $M_t = M$. The graph M_0 is defined to be a star graph connecting all the nodes corresponding to assignments in A_2 to a node (representing the empty assignment). We can construct M_j from M_{j-1} by splitting each node $x(\alpha)$ corresponding to an assignment in $\alpha \in A_1^{j-1}$ into two nodes $x(\alpha \oplus 0)$ and $x(\alpha \oplus 1)$, one assigns 0 and the other assigns 1 to v_j. The neighbors of $x(\alpha)$ is connected to $x(\alpha \oplus 0)$ and/or $x(\alpha \oplus 1)$ by definition.

Analogously to the proof of Lemma 3.1, we show that at most one neighbor of x can be connected to both of $x(\alpha \oplus 0)$ and $x(\alpha \oplus 1)$. Assume on the contrary that there exist $\gamma \neq \gamma' \in A_2$ such that both of the corresponding nodes $y(\gamma)$ and $y(\gamma')$ are adjacent to both of $x(\alpha \oplus 0)$ and $x(\alpha \oplus 1)$. From the adjacency relation, there exists β_0, β_1, β_2, β_3 such that all of $\beta_0 \oplus \gamma$, $\beta_1 \oplus \gamma$, $\beta_2 \oplus \gamma'$, and $\beta_3 \oplus \gamma'$ are in A, and the restriction of β_i to U^j is $\alpha \oplus 0$ if i is even, otherwise $\alpha \oplus 1$.

Let u be the nearest vertex in V_2 from the bridge e satisfying that $\gamma(u) \neq \gamma'(u)$. We can assume that $\gamma(u) = 0$ and $\gamma'(u) = 1$. Since T is the shortest path tree, at least one shortest path (in G) between u and v_j is a path in $T \cup G_2$. On the path, the entry sums of $\beta_0 \oplus \gamma$ and $\beta_3 \oplus \gamma'$ differs by two from each other. This contradicts the property of a compatible set.

Thus, we have shown that at most one neighbor of x can be connected to both of $x(\alpha \oplus 0)$ and $x(\alpha \oplus 1)$. If M_{j-1} is a forest. we can see that such a splitting operation keeps the graph to be a forest, and accordingly, M_j is a forest. Thus, we can prove the claim by induction.

The affine independence also follows from this claim in a routine way: Suppose that A does not satisfy the affine independence. Then, there exists real numbers c_α for $\alpha \in A$ such that $\sum_{\alpha \in A} c_\alpha = 0$, $\sum_{\alpha \in A} c_\alpha \alpha = \mathbf{0}$, and at least one c_α is nonzero. Because of affine independence of A_1, $\sum_{\alpha, \alpha|_{V_1} = \beta} c_\alpha = 0$ for each fixed $\beta \in A_1$. Similarly, $\sum_{\alpha, \alpha|_{V_2} = \gamma} c_\alpha = 0$ for each fixed $\gamma \in A_2$.

Let us take a leaf node z in M. Without loss of generality, we assume z corresponds to a member β of A_1. Then, we can conclude that $c_\alpha = 0$ for the unique member α of A such that $\alpha\, |_{V_1} = \beta$. We remove z from M and continue this process iteratively to see that $c_\alpha = 0$ for all $\alpha \in A$. □

A graph G is *series connection* of two graphs G_1 and G_2 if $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$ (implying that they share no edge), where v is called the *separator*. We have the following:

Theorem 4.4 *Suppose that a graph G is a series connection of two connected graphs G_1 and G_2, and let v be the separator. Then, we have the following:*
(1) If there is an H_G compatible set A such that $|A| = \nu(G)$ and there exist α and β in A satisfying $\alpha(v) \neq \beta(v)$, $\nu(G) \leq \nu(G_1) + \nu(G_2) - 2$.
(2) Otherwise, let $\nu^0(G_i)$ be the maximum size of a compatible set A_i for H_{G_i} such that $\alpha(v) = 0$ for every $\alpha \in A_i$. Then, $\nu(G) \leq \nu^0(G_1) + \nu^0(G_2) - 1$.
(3)If both of H_{G_1} and H_{G_2} satisfy the affine independence property, so does

H_G.

Proof: Consider an H_G-compatible set A. Let $A_i = \{\alpha|_{V_i} : \alpha \in A\}$, where V_i are vertex sets of G_i for $i = 1, 2$. It is clear that A_i is an H_{G_i}-compatible set for each $i = 1, 2$. Let $A^0 = \{\alpha \in A : \alpha(v) = 0\}$ and $A^1 = \{\alpha \in A : \alpha(v) = 1\}$. If $A^0 = \emptyset$, we have another H_G-compatible set by changing the value at v to be 0 for all elements in A. Thus, we can assume that $A^0 \neq \emptyset$.

Let $A_i^0 = \{\alpha \in A_i : \alpha(v) = 0\}$ and $A_i^1\{\alpha \in A_i : \alpha(v) = 1\}$ for $i = 1, 2$. Then, analogous to the proof of Theorem 4.3, we can construct a forest to prove that $|A^0| \leq |A_1^0| + |A_2^0| - 1$. Similarly, $|A^1| \leq |A_1^1| + |A_2^1| - 1$ if $A^1 \neq \emptyset$. Thus, we have (1) and (2).

(3) can be proved analogously to Theorem 4.3. □

4.3 The Case of a Cycle

Let C_n be a cycle on n vertices $V = \{1, 2, \ldots, n\}$ with edge set $\{e_1, \ldots, e_n\}$ where $e_i = (i, i+1)$, $1 \leq i \leq n$. The arithmetics on vertices are cyclic, i.e., $n + 1 = 1$. We sometimes refer to the edge e_n as e_0 as well. For $i, j \in V$, let $P(i, j)$ denote the path from i to j containing the nodes $v_i, v_{i+1}, \ldots v_j$ in this cyclic order. Note that $P(j, i)$ is different from $P(i, j)$ if $i \neq j$. $P(i, i)$ is naturally interpreted as an empty path consisting of a single vertex and no edge. Let $\mathcal{P} = \mathcal{P}_{C_n}$ be a set of shortest paths on C_n. Note that for any given edge lengths, $\mathcal{P}$ satisfies the following conditions: (1) $P(i, i) \in \mathcal{P}$ for every $i \in V$, (2) if $P \in \mathcal{P}$ then every subpath of P is in $\mathcal{P}$, and (3) for every pair i, j of distinct vertices of C_n, at least one of $P(i, j)$ and $P(j, i)$ is in $\mathcal{P}$.

We can also assume that $P(i, i+1)$ is always a shortest path between v_i and v_{i+1}; otherwise, e_i is contained in no shortest path, and we can simply remove the edge e_i from the consideration to reduce the problem to the case where the graph is a path.

Theorem 4.5 $\nu(H_{C_n}) = n + 1$.

A graph G is a *tree of cycles* if either (1) G is a tree or (2) G has a proper subgraph G' that is a tree of cycles and a subgraph C that is either a cycle or an edge such that G is obtained by series connection of G' and C.

As a corollary of Theorem 4.5 and Theorem 4.4, we have the following:

Corollary 4.6 $\nu(H_G) = n + 1$ *if* G *is a tree of cycles with* n *vertices.*

Proof: We prove by induction on the number of vertices. If G is a tree, we have already proven that the statement holds. Suppose that G is obtained as a series connection of G' and C. We focus on the case where C is a cycle, since it is easy to handle the case where C itself is an edge. Let n_1 and n_2 be number of vertices in G' and C, respectively. By definition, $n = n_1 + n_2 - 1$ is the number of vertices in G and $n_1 < n$.

If $\nu(G) \le \nu(G') + \nu(C) - 2$, we have $\nu(G) \le n_1 + 1 + n_2 + 1 - 2 = n + 1$.

Otherwise, $\nu(G) \le \nu^0(G') + \nu^0(C) - 1$. We consider a cycle C' with $n_2 - 1$ vertices by replacing the joint node v in C and its adjacent edges $e = (prev(v), v)$ and $e'(v, succ(v))$ by an edge $e'' = (prev(v), succ(v))$ whose edge weight is the sum of those of e and e'. Then, it is routine to verify that $\nu^0(C) = \nu(C')$. Thus, we have $\nu(G) \le n_1 + 1 + n_2 - 1 = n + 1$. □

We often write $\mathcal{P}$ for H_{C_n} identifying the hypergraph and the set of hyperedges in this section for abbreviation. We devote the rest of this section for proving Theorem 4.5.

For $n \le 2$ the theorem is trivial to verify, so we will assume $n \ge 3$ in the sequel. For an assignment α, we define $w(\alpha) = w_V(\alpha) = \sum_{v \in C_n} \alpha(v)$ to be the weight of α over all vertices in C_n.

Lemma 4.7 *Let α and β be $\mathcal{P}$-compatible assignments on C_n. Then, $w(\alpha)$ and $w(\beta)$ differ by at most 1.*

Proof: Suppose that assignments α and β are $\mathcal{P}$-compatible and $w(\beta) \ge w(\alpha) + 2$. Let's say that each vertex v has type $(\alpha(v), \beta(v))$. Cyclically list the vertices of types $(0, 1)$ and $(1, 0)$ in the direction of the cycle, ignoring those of types $(0, 0)$ or $(1, 1)$. Then, since the number of vertices of type $(0, 1)$ is greater than that of the vertices of type $(1, 0)$ by at least 2, there are at least two places where type $(0, 1)$ vertices appear consecutively. At least one of such consecutive pairs forms (together with $(0, 0)$ and $(1, 1)$ entries between the vertices of the pair) a path in $\mathcal{P}$, on which α and β are not compatible. □

Lemma 4.8 *Suppose $w(\alpha) = w(\beta)$ for assignments α and β. Then, if α and β are $\mathcal{P}$-compatible they are compatible on* every *path of C_n.*

Proof: We show that α and β are compatible on an arbitrary path $P(i, j)$. If $i = j + 1$ then the path consists of all the vertices of C_n and, because α and β are of the same weight, they are compatible on $P(i, j)$. Suppose $i \ne j + 1$. Then, path $P(j + 1, i - 1)$ is the complement of path $P(i, j)$ in terms of their vertex sets. At least one of $P(i, j)$ and $P(j + 1, i - 1)$ is in $\mathcal{P}$ and hence a and b are compatible on at least one of these paths. But the compatibility on one of these paths implies the compatibility on the other, since the vertex sets of these paths are the complement of each other. Therefore α and β are compatible on $P(i,j)$. □

From the above observations and Corollary 4.2, it is clear that $\nu(H_{C_n}) \leq 2(n+1)$. We need some more tools in order to sharpen this bound.

The following notion of edge opposition is one of our main tools. Let e_i and e_j be two edges of C_n. We say e_i *opposes* e_j (and vice versa) if paths $P(i+1,j)$ and $P(j+1,i)$ are both in $\mathcal{P}$. Note that when $P(i+1,i) \in \mathcal{P}$, e_i opposes itself in this definition. However, in this case, the length of e_i is so large that it does not appear in any shortest path, and we can cut the cycle into a path at e_i to reduce the problem into the sequence rounding problem. Thus, we assume this does not happen.

Lemma 4.9 *For every edge e_i of C_n, there is at least one edge e_j that opposes e_i.*

Proof: Fix an edge e_i. Let P be the maximal path in $\mathcal{P}$ ending at i and let P' be the maximal path in $\mathcal{P}$ starting at $i+1$. We claim that $V(P) \cup V(P') = V(C_n)$: if there is some k in neither P nor P', then neither $P(k,i)$ nor $P(i,k)$ is in $\mathcal{P}$, contradicting the definition of the shortest path system. Therefore, there is an edge e_j such that $j \in V(P')$ and $j+1 \in V(P)$. These conditions imply $P(i+1,j) \in \mathcal{P}$ and $P(j+1,i) \in \mathcal{P}$, that is, e_j opposes e_i. □

Lemma 4.10 *If edges e_i and e_j oppose each other, then, either e_{i+1} opposes e_j or e_{j+1} opposes e_i.*

Proof: We start with the special case where $i+1=j$. Since $n \geq 3$, $j+1 \neq i$ in this case. Since $P(j+1,i) \in \mathcal{P}$, $\mathcal{P}$ also contains $P(j+2,i)$, which is well-defined because $j+1 \neq i$. From our assumption on the graph C_n, $P(j,j+1) = P(i+1,j+1) \in \mathcal{P}$; thus e_{j+1} opposes e_i. The case where $j+1=i$ is similar, so assume $i+1 \neq j$ and $j+1 \neq i$. Then, we have both $P(j+2,i)$ and $P(i+2,j)$ in $\mathcal{P}$ similarly to the above. Therefore, if $P(i+1,j+1) \in \mathcal{P}$ then e_{i+1} opposes e_j. Otherwise, $P(j+1,i+1) \in \mathcal{P}$ and therefore e_{j+1} opposes e_i. □

Define the *opposition graph*, denoted by $opp(\mathcal{P})$, to be the graph on $E(C_n)$ in which $\{e_i, e_j\}$ is an edge if and only if e_i and e_j oppose each other. By Lemma 4.9 and Lemma 4.10, we obtain the following:

Lemma 4.11 *The opposition graph $opp(\mathcal{P})$ is connected.*

We next prove a lemma regarding two equivalence relations on the vertex set of a graph. Let G be a graph. We say that a pair (R_1, R_2) of equivalence relations on $V(G)$ *honors* G, if for every edge $\{u,v\}$ of G, u and v are equivalent either in R_1 or in R_2. For an equivalence relation R, denote by $ec(R)$ the number of equivalence classes of R.

Lemma 4.12 *Let G be a connected graph on n vertices and suppose a pair (R_1, R_2) of equivalence relations on $V(G)$ honors G. Then $ec(R_1) + ec(R_2) \leq n + 1$.*

Proof: Fix an arbitrary spanning tree T of G. We assume (R_1, R_2) honors G and hence it honors T. We grow tree S from a singleton tree towards T, and consider $f(S)$ that is the sum of the number of equivalence classes for R_1 and R_2 among the nodes of S. Initially, we have one node, and hence $f(S) = 2$. If we add an edge and a vertex, $f(S)$ increases by at most one, since the vertex is equivalent to the vertex it attached to for at least one of the equivalence relations. Hence, $f(T) \leq n + 1$. □

A $\mathcal{P}$-compatible set A is called *uniform* if there is a fixed integer w such that $w(\alpha) = w$ for every $\alpha \in A$. We call w the *weight* of A.

The following equivalence relation on the edge set of C_n plays a central role in our proof.

Let A be a uniform $\mathcal{P}$-compatible set. We say that two edges e_i, e_j of C_n are *A-equivalent* and write $e_i \sim_A e_j$ if and only if either $i = j$ or $w_{P(i+1,j)}(\alpha)$ is the same for every $\alpha \in A$. This relation is symmetric since the assignments in A have the same weight on the entire cycle and $P(i + 1, j)$ and $P(j + 1, i)$ are complement to each other in terms of their vertex sets. It is indeed straightforward to check the transitivity to confirm that A-equivalence is an equivalence relation.

Lemma 4.13 *Let A and B be uniform $\mathcal{P}$-compatible sets on C_n such that $A \cup B$ is a $\mathcal{P}$-compatible set. We assume that A and B has weights w and $w+1$, respectively. Then, for any pair of edges e_i and e_j opposing each other with respect to $\mathcal{P}$, either $e_i \sim_A e_j$ or $e_i \sim_B e_j$; in other words, the pair $(\sim_A, \sim_B)$ honors the opposition graph $opp(\mathcal{P})$.*

Proof: Let A, B, e_i, e_j be as in the statement of the lemma and suppose that neither $e_i \sim_A e_j$ nor $e_i \sim_B e_j$ holds. Since e_i is not A-equivalent to e_j, there are $\alpha_1, \alpha_2 \in A$ such that $w_{P(i+1,j)}(\alpha_1) < w_{P(i+1,j)}(\alpha_2)$. Similarly, there are $\beta_1, \beta_2 \in B$ such that $w_{P(i+1,j)}(\beta_1) < w_{P(i+1,j)}(\beta_2)$. First suppose that $w_{P(i+1,j)}(\alpha_2) \leq w_{P(i+1,j)}(\beta_1)$. Then we have $w_{P(i+1,j)}(\beta_2) \geq w_{P(i+1,j)}(\alpha_1) + 2$, and hence α_1 and β_2 are incompatible on path $P(i + 1, j)$. This is a contradiction because, since e_i and e_j are opposing each other, path $P(i + 1, j)$ must be in $\mathcal{P}$. Next suppose $w_{P(i+1,j)}(\alpha_2) > w_{P(i+1,j)}(\beta_1)$. Since $w(\beta_1) = w(\alpha_2) + 1$, we then have $w_{P(j+1,i)}(\beta_1) \geq w_{P(j+1,i)}(\alpha_2) + 2$, the incompatibility of β_1 and α_2 on path $P(j + 1, i) \in \mathcal{P}$. □

Consider a uniform $\mathcal{P}$-compatible set A. From Lemma 4.8, A is an I_n-compatible set, where I_n is the hypergraph on V associated with all the intervals on the graph obtained by cutting C_n at the edge $e_0 = (v_n, v_1)$. Let $V_i = \{1, 2, \ldots, i\} \subset V$, and let $A(V_i)$ be the set of assignments on V_i obtained by restricting A to V_i.

Lemma 4.14 $|A(V_i)| \leq |A(V_{i-1})| + 1$.

Proof: $V_i = V_{i-1} \cup \{v_i\}$. Applying Lemma 3.1, there is at most one assignment α in $A(V_{i-1})$ such that both of $\alpha \oplus 0$ and $\alpha \oplus 1$ are in $A(V_i)$. Thus, we obtain the lemma. □

We call the index i a *branching index* of A if $|A(V_i)| = |A(V_{i-1})| + 1$ holds. Note that for a branching index, there must be an assignment α in $A(V_{i-1})$ such that both of $\alpha \oplus 0$ and $\alpha \oplus 1$ are in $A(V_i)$.

Lemma 4.15 *Let A be a uniform $\mathcal{P}$-compatible set. Then, i is a branching index in A only if the edge $e_i = (i, i+1)$ is A-equivalent to none of $e_0, e_1, e_2, \ldots, e_{i-1}$.*

Proof: Suppose level i is a branching index. Then, we have $\alpha \in A(V_{i-1})$ such that $\alpha \oplus 0$ and $\alpha \oplus 1$ are in $A(V_i)$. If e_i is A-equivalent to e_j for $j < i$, the assignments $\alpha \oplus 0$ and $\alpha \oplus 1$ must have the same total weight on $V_i \setminus V_j = \{j+1, \ldots, i\}$. This is impossible, since two assignments are the same on $V_i \setminus V_j$ except on i. □

Consider the number $ec(\sim_A)|_{V_i}$ of equivalence classes in V_i. Lemma 4.15 implies that $|A(V_i)| - |A(V_{i-1})| \leq ec(\sim_A)|_{V_i} - ec(\sim_A)|_{V_{i-1}}$. Thus, we have the following corollary:

Corollary 4.16 *Let A be a uniform $\mathcal{P}$-compatible set. Then $|A| \leq ec(\sim_A)$.*

We are now ready to prove Theorem 4.5. Let $\mathcal{P}$ be a shortest path system on C_n and let A be a $\mathcal{P}$-compatible set on C_n. By Lemma 4.7, the assignments of A have at most two weights. If there is only one weight, then $|A| \leq ec(\sim_A)$ by Corollary 4.16 and hence $|A| \leq n+1$. Suppose A consists of two subsets A_1 and A_2, with the assignments in A_1 having weight w and those in A_2 having weight $w+1$. By Lemma 4.13, the pair of equivalence relations $(\sim_{A_1}, \sim_{A_2})$ honors the opposition graph $opp(\mathcal{P})$. Since $opp(\mathcal{P})$ is connected (Lemma 4.11), we have $ec(\sim_{A_1}) + ec(\sim_{A_2}) \leq n+1$ by Lemma 4.12. We are done, since $|A_i| \leq ec(\sim_{A_i})$ for $i = 1, 2$ by Corollary 4.16.

5 Concluding remarks

We have proven the conjectures only for special graphs. It will be nice if the conjectures are proven for wider classes of graphs such as series parallel graphs [3]. Also, the affine independence property for the cycle graph has not been proven in this paper.

For a general graph, we do not even know whether $\nu(H_G)$ is polynomially bounded by the number of vertices. It is plausible that the number of roundings can become large if the entries have some middle values (around 0.5). For a special input $\mathbf{a}$ consisting of entries with a same value $0.5 + \epsilon$, we can show that the number of global roundings of $\mathbf{a}$ is bounded by $n+1$ if G is bipartite; otherwise by $m+1$, where m is the number of edges in G [9].

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