On an empty triangle with the maximum area in planar point sets

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1 Introduction

We deal with only finite point sets P in the plane in general position. A point set is *convex* or *in convex position* if it determines a convex polygon. A convex subset Q of P is said to be *empty* if no point of P lies inside the convex hull of Q. An empty convex subset of P with k elements is also called a *k-hole* of P.

Let P be an n planar point set in general position. For a subset Q of P, denote the area of the convex hull of Q by $A(Q)$. In [3], we considered the ratio between the maximum area of 3-holes (empty triangles) T of P and the whole area $A(P)$. Namely, let

$$F(P) = \max_{T \subset P} \frac{A(T)}{A(P)}$$

and define $f(n)$ as the minimum value of $F(P)$ over all sets P with n points.

Then we obtained the following result where c is some constant:

Theorem A.

$$\frac{23}{(37+3\sqrt{5})n+c} \leq f(n) \leq \frac{1}{n-1} \quad \text{for any } n \geq 25.$$

In this talk, we improve on the lower bound of $f(n)$. To achieve the aim we consider the existence of 5-holes of point sets in the next section.

2 The existence of 5-holes

Let $V(P)$ be a set of vertices; a subset of P on the boundary of the convex hull of P. Then we obtain the next proposition.

Proposition 1. If $|V(P)| \geq 5$ for any *7* or *8* point set P, we have a 5-hole of P.

It is well-known that Harborth [2] proves that any 10 point set contains a 5-hole and the bound is tight. The next result by using Proposition 1 shows a sufficient condition for the existence of a 5-hole of a 9 point set. An *ear* of P is three consecutive vertices on the convex hull boundary of P.

Proposition 2. Any *9* point set P contains a 5-hole if it has an empty ear.

Figure 1 gives an 8 point set in general position with an empty ear, containing no 5-hole. And we obtain the following lemma by using Proposition 2.

Lemma 1. *Any 17 point set P contains two 5-holes with disjoint interiors or one 6-hole.*

We give a 12 point set in general position in Fig. 2 which contains neither two 5-holes with disjoint interiors nor one 6-hole.

3 Result

For n point sets P in convex position, the value $f^{\mathrm{conv}}(n)$ is defined in a similar way to $f(n)$. The following lemma is proved in [1].

Lemma A. For point sets in convex position with 5 elements and with 6 elements,

$$f^{\mathrm{conv}}(5) = \frac{1}{\sqrt{5}} \quad \text{and} \quad f^{\mathrm{conv}}(6) = \frac{4}{9}.$$

From the *Polar Partition* of P with the apex a vertex of P we have $\lfloor \frac{n-2}{15} \rfloor$ convex cones with disjoint interiors, each of which contains exactly 17 points of P. By Lemma 1 each such convex cone contains two 5-holes with disjoint interiors or one 6-hole. We improve on the lower bound by virtue of Lemma A:

Theorem 1.

$$f(n) \geq \frac{1}{2n - 5 - (6 - 2\sqrt{5}) \left\lfloor \frac{n-2}{15} \right\rfloor} \geq \frac{15}{(24 + 2\sqrt{5})n + c} \quad \text{for any } n \geq 17.$$

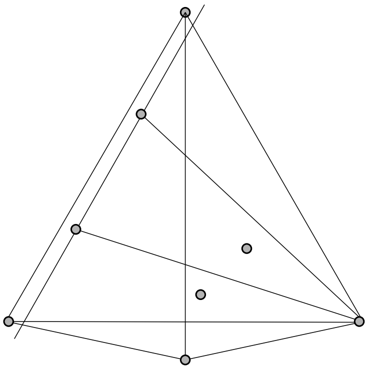

Fig. 1.

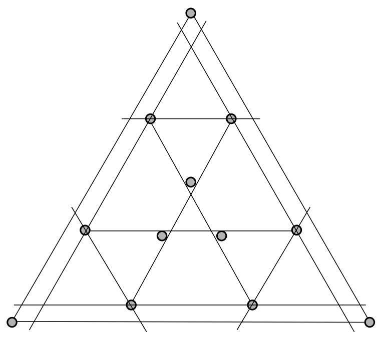

Fig. 2.

References

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[3] K. Hosono, F. Hurtado, M. Urabe and J. Urrutia, On a triangle with the maximum area in a planar point set, *LNCS* 3330 (2004) 102–107.