4.3. 階層定理(続き)

定理4.4: 任意の制限時間 t_1, t_2 に対し,
$\forall c>0, \overset{\infty}{\forall} n\, [ct_1(n)^2 \le t_2(n)] \rightarrow \mathrm{TIME}(t_1) \subsetneq \mathrm{TIME}(t_2)$.

DIAG = {<a,w>: 次の3条件を満たす.
(a) *IsProgram*(a)
(b) $l < t$
(c) eval-in-time(a, <a,w>, $\overline{t}$) ≠ accept }
ただし, x=<a,w>, $l=|x|$, $t = [\sqrt{t_2(|x|)} / |a|]$
([ ]は切り捨て)

プログラムA= $\lfloor a \rfloor$に x=<a,w> を入力すると、
$|x| < \sqrt{t_2(|x|)} / |a|$ $t = \sqrt{t_2(|x|)} / |a|$ 以内にacceptしない

4.3. Hierarchy Theorem (Cont'd)

Theorem 4.4 For any time limits t_1 and t_2, we have
$\forall c>0, \overset{\infty}{\forall} n\, [ct_1(n)^2 \le t_2(n)] \rightarrow \mathrm{TIME}(t_1) \subsetneq \mathrm{TIME}(t_2)$.

DIAG = {<a,w>: the following three conditions are satisfied:
(a) *IsProgram*(a)
(b) $l < t$
(c) eval-in-time(a, <a,w>, $\overline{t}$) ≠ accept }
where, x=<a,w>, $l=|x|$, $t = [\sqrt{t_2(l)} / |a|]$ [ ] denotes round-off

If we input x=<a,w> to a program a as an input, $|x| < \sqrt{t_2(|x|)} / |a|$ and it does not accept before time $t=\sqrt{t_2(|x|)} / |a|$.

補題4.8: *DIAG* ∉ TIME(t_1) の証明:

DIAG∈TIME(t_1)として矛盾を導く.

- *DIAG* をO(t_1)時間で認識するプログラムをA_0, コードを a_0とする.
  → time_ $A_0(l) \leqq c_0\, t_1(l)$ .... (1) (c_0;定数)
- 定理の仮定 $\forall c > 0, \overset{\infty}{\forall} l [ct_1(l)^2 \le t_2(l)]$
  より、lを十分大きくとると、 $c_0 t_1(l) \le [\sqrt{t_2(l)} / |a_0|]$ ....(2)
- t_1は自然な制限時間であるから、l_0=|<a_0,w_0>|を十分長くすると
  $l_0 \le [\sqrt{t_2(l_0)} / |a_0|] (\equiv t_0)$ ....(3)
- (3) と*DIAG*の定義より
  <a_0,w_0>∈*DIAG* ↔ eval-in-time(a_0,<a_0,w_0>,$\overline{t_0}$) ≠ accept ....(4)

Proof of Lemma 4.8: *DIAG* ∉ TIME(t_1)

To derive contradictions, we assume *DIAG*∈TIME(t_1).

- Let A_0 be the program that recognizes *DIAG* in O(t_1) time, with code a_0. → time_ $A_0(l) \leqq c_0\, t_1(l)$ .... (1) (c_0;constant)
- By the assumption of theorem $\forall c > 0, \overset{\infty}{\forall} l [ct_1(l)^2 \le t_2(l)]$
  for sufficiently large l, $c_0 t_1(l) \le [\sqrt{t_2(l)} / |a_0|]$ ....(2)
- Since t_1 is a natural limit, for sufficiently long l_0=|<a_0,w_0>|,
  $l_0 \le [\sqrt{t_2(l_0)} / |a_0|] (\equiv t_0)$ ....(3)
- By (3) and definition of *DIAG*,
  <a_0,w_0>∈*DIAG* ↔ eval-in-time(a_0,<a_0,w_0>,$\overline{t_0}$) ≠ accept ....(4)

(1) $\mathrm{time\_A}_0(l) \le c_0 t_1(l)$ 一般の l について
(2) $c_0 t_1(l_0) \le [\sqrt{t_2(l_0)} / |a_0|] = t_0$ 十分長い l_0=<a_0,w_0>
(4) $<a_0, w_0> \in DIAG \leftrightarrow$ eval-in-time$(a_0, <a_0, w_0>, \overline{t_0}) \ne$ accept

(1)(2)より, $\mathrm{time\_A}_0(<a_0, w_0>) \le c_0 t_1(l_0) \le [\sqrt{t_2(l_0)} / |a_0|] = t_0$ .....(5)

(5) より、十分長い l_0=<a_0,w_0>をプログラムA_0に入力したとき,
計算は必ず t_0 時間以内に終わる。つまり
eval-in-timeでの制限時間 t_0 は本質的ではない.
→ eval-in-time(a_0, <a_0,w_0>, $\overline{t_0}$) = eval(a_0, <a_0,w_0>)

(4) に eval-in-time(a_0, <a_0,w_0>, $\overline{t_0}$) = eval(a_0, <a_0,w_0>) を代入:

<a_0,w_0> ∈ *DIAG* ↔ eval(a_0, <a_0,w_0>) ≠ accept
↔ A_0が<a_0,w_0>をacceptしない.

これは,「A_0が*DIAG*を認識する」という仮定に矛盾.(証明終)

(1) $\mathrm{time\_A}_0(l) \le c_0 t_1(l)$ For general l
(2) $c_0 t_1(l_0) \le [\sqrt{t_2(l_0)} / |a_0|] = t_0$ For sufficiently long l_0=<a_0,w_0>
(4) $<a_0, w_0> \in DIAG \leftrightarrow$ eval-in-time$(a_0, <a_0, w_0>, \overline{t_0}) \ne$ accept

By (1)(2), $\mathrm{time\_A}_0(<a_0, w_0>) \le c_0 t_1(l_0) \le [\sqrt{t_2(l_0)} / |a_0|] = t_0$ .....(5)

From (5), program A_0 halts in t_0 time for sufficiently long l_0=<a_0,w_0>, which means
the time limit t_0 in eval-in-time is not essential.
→ eval-in-time(a_0, <a_0,w_0>, t_0) = eval(a_0, <a_0,w_0>)

Substitute eval-in-time(a_0, <a_0,w_0>, t_0) = eval(a_0, <a_0,w_0>) to (4):

<a_0,w_0> ∈ *DIAG* ↔ eval(a_0, <a_0,w_0>) ≠ accept
↔ A_0 does not accept <a_0,w_0>.

Which contradicts the assumption that "A_0 recognizes *DIAG*." Q.E.D.

対角線論法に基づく解釈

$F_1 = O(t_1)$の時間計算量をもつ認識プログラムすべての集合
$= \{A_1, A_2, \dots\}$
それらのプログラムのコードを $a_1, a_2, \dots$ とする.
各a_iごとに適当な定数c_iを考えると,
$\mathrm{time\_A}_i(l) \leq c_i t_1(l)$
が成立. さらに, 各a_i, c_iに対し, 十分長いw_iを取ると,
$c_i t_1(l_i) \leq [\sqrt{t_2(l_i)} / |a_i|]$, $l_i \leq [\sqrt{t_2(l_i)} / |a_i|]$
とできる. 各プログラムA_iの入力x_iに対する出力の表を作ると,

$x=\langle a,w\rangle$, $l = |x|$

	x_1	x_2	x_3	....	x_k
A_1	A	R	A	....	A
A_2	R	R	R	....	A
A_3	A	A	A	....	R
......					
A_k	R	R	A	....	A

$A_i(x_i)$の値

x_1	x_2	x_3	....	x_k
R				
	A			
		R		
			....	
				R

$x_i \in DIAG$?の答

対角線で RとAが逆

*DIAG*を認識するプログラムはこの表に登場できない...矛盾

Interpretation based on Diagonalization

F_1 = a set of all recognizing programs of time complexity $O(t_1)$
$= \{A_1, A_2, \dots\}$
Let their program codes be $a_1, a_2, \dots$.
Considering an appropriate constant c_i for each a_i, we have
$\mathrm{time\_A}_i(l) \leq c_i t_1(l)$
Moreover, we can take sufficiently long w_i for each a_i and c_i s.t.
$c_i t_1(l_i) \leq [\sqrt{t_2(l_i)} / |a_i|]$, $l_i \leq [\sqrt{t_2(l_i)} / |a_i|]$
Putting the outputs of A_i for input x_i in the table:

$x=\langle a,w\rangle$, $l = |x|$

	x_1	x_2	x_3	....	x_k
A_1	A	R	A	....	A
A_2	R	R	R	....	A
A_3	A	A	A	....	R
......					
A_k	R	R	A	....	A

values of $A_i(x_i)$

x_1	x_2	x_3	....	x_k
R				
	A			
		R		
			....	
				R

answer to $x_i \in DIAG$?

Compare Diagonals

This table can't include a program recognizing *DIAG*...contradiction.

例4.12: $\mathrm{TIME}(n^2) \subsetneqq \mathrm{TIME}(n^5)$
$\Leftarrow \forall c>0, \forall n \geq c\ [\ c(n^2)^2 \leq n^5\ ]$

要するに,
$$\liminf_{n\to\infty} \frac{t_2(n)}{t_1(n)^2} = \infty$$
となれば, 階層定理より$\mathrm{TIME}(t_1) \subsetneqq \mathrm{TIME}(t_2)$

Ex.4.12: $\mathrm{TIME}(n^2) \subsetneqq \mathrm{TIME}(n^5)$
$\Leftarrow \forall c>0, \forall n \geq c\ [\ c(n^2)^2 \leq n^5\ ]$

If we have
$$\liminf_{n\to\infty} \frac{t_2(n)}{t_1(n)^2} = \infty$$
then the hierarchy theorem tells us that $\mathrm{TIME}(t_1) \subsetneqq \mathrm{TIME}(t_2)$

第5章 代表的な計算量クラス

5.1. 代表的な時間計算量クラス

$\mathrm{P} \equiv \bigcup_{p:\text{多項式}} \mathrm{TIME}(p(l))$

$\mathrm{E} \equiv \bigcup_{c>1} \mathrm{TIME}(2^{cl})$

$\mathrm{EXP} \equiv \bigcup_{p:\text{多項式}} \mathrm{TIME}(2^{p(l)})$

C集合: 計算量クラスCに入る集合.
C問題: C集合の認識問題

ある問題がPに入っていないなら、
現実的には手に負えない…

Chapter 5
Representative Complexity Classes

5.1. Representative time complexity classes

$\mathrm{P} \equiv \bigcup_{p:\text{polynomial}} \mathrm{TIME}(p(l))$

$\mathrm{E} \equiv \bigcup_{c>1} \mathrm{TIME}(2^{cl})$

$\mathrm{EXP} \equiv \bigcup_{p:\text{polynomial}} \mathrm{TIME}(2^{p(l)})$

C set: set in the complexity class C.
C problem: problem of recognizing a C set.

Problems not in P are intractable
from the practical viewpoint…

例5.1: クラスP, E, EXPでは，多項式時間程度の違いは問題ではない．

P: 多項式 × 多項式→多項式
E: 2の線形乗 × 多項式 → 2の線形乗
EXP: 2の多項式乗 × 多項式 → 2の多項式乗

例5.2: PRIMEの計算量クラス
例4.7 → PRIME ∈ TIME(2^l)
故に，PRIME ∈ E

余談: 2002年に $O(l^6)$ のアルゴリズムが考案されたので、今ではP

定義5.1. T: 制限時間の集合

$\bigcup_{t\in T}$TIME(t): T時間計算量クラス
→これをTIME(T)と表す．

定理5.1: (1) P = $\bigcup_{c>0}$TIME(l^c), (2) EXP = $\bigcup_{c>0}$ TIME(2^{l^c})

Ex.5.1: Polynomial makes no serious difference in the classes P, E, EXP.

P: polynomial × polynomial→polynomial
E: linear power of 2 × polynomial → linear power of 2
EXP: poly. power of 2 × poly. → poly. power of 2

Ex.5.2: Complexity class of PRIME
Ex.4.7 → PRIME ∈ TIME(2^l)
Thus, PRIME ∈ E

$O(l^6)$ time algorithm put it in P!!

Def.5.1: T: set of time limits

$\bigcup_{t\in T}$TIME(t): T time complexity class
→It is denoted by TIME(T).

Theorem5.1 (1) P = $\bigcup_{c>0}$TIME(l^c), (2) EXP = $\bigcup_{c>0}$TIME(2^{l^c})

定理5.1: (1) P = $\bigcup_{c>0}$TIME(l^c), (2) EXP = $\bigcup_{c>0}$TIME(2^{l^c})

証明: (2)の証明は省略．
T_1: l^cという形の多項式の集合．
T_2: 多項式の全体
→ $T_1 \subseteq T_2$ なので，TIME(T_1) ⊆ TIME(T_2)
p: 任意の多項式 （pはT_2の任意の要素）
多項式pの最大次数をkとすると，$p(l) = O(l^k)$
定理4.3より，
TIME($p(l)$) ⊆ TIME(l^k) ⊆ TIME(T_1)
したがって，TIME(T_1) ＝ TIME(T_2)

証明終

Theorem 5.1: (1) P = $\bigcup_{c>0}$ TIME(l^c), (2) EXP = $\bigcup_{c>0}$TIME(2^{l^c})

Proof: The proof of (2) is omitted.
T_1: set of polynomials of the form of l^c.
T_2: set of all polynomials
→ since $T_1 \subseteq T_2$, TIME(T_1) ⊆ TIME(T_2)
p: arbitrary polynomial （p is any element of T_2）
if the maximum degree of a polynomial p is k, $p(l) = O(l^k)$
From Theorem 4.3,
TIME($p(l)$) ⊆ TIME(l^k) ⊆ TIME(T_1)
Therefore, TIME(T_1) ＝ TIME(T_2)

Q.E.D.

例5.3. 命題論理式評価問題(PROP-EVAL)
入力 : <F, < $a_1, a_2, \dots, a_n$ >>
Fは拡張命題論理式 ∧ ∨ ￢ → ↔
($a_1, a_2, \dots, a_n$)は F に対する真理値割り当て
質問 : $F(a_1, a_2, \dots, a_n) = 1$?

(x,y)	x→y (￢x∨y)	x ↔ y ((x→y)∧(y→x))
(0,0)	1	1
(0,1)	1	0
(1,0)	0	0
(1,1)	1	1

Ex.5.3. Problem of evaluating propositional expression(PROP-EVAL)
Input : <F, < $a_1, a_2, \dots, a_n$ >>
F is an extended prop. expression
($a_1, a_2, \dots, a_n$) is a truth assignment to F
Question : $F(a_1, a_2, \dots, a_n) = 1$?

(x,y)	x→y (￢x∨y)	x ↔ y ((x→y)∧(y→x))
(0,0)	1	1
(0,1)	1	0
(1,0)	0	0
(1,1)	1	1

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例5.3. 命題論理式評価問題(PROP-EVAL)

入力: $<F, < a_1, a_2, \dots, a_n >>$

Fは拡張命題論理式 $\wedge \vee \neg \rightarrow \leftrightarrow$

$(a_1, a_2, \dots, a_n)$はFに対する真理値割り当て

質問: $F(a_1, a_2, \dots, a_n) = 1$?

拡張命題論理式 F がコード化されたもの $\ulcorner F \urcorner$ から計算木を作る.

計算木は$O(|\ulcorner F \urcorner|^3)$時間で構成できる.

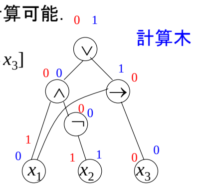
計算木が得られていれば, ボトムアップ式で $F(a_1, a_2, \dots, a_n)$ の値は容易に計算可能.

例: $F(x_1, x_2, x_3) = [x_1 \wedge \neg x_2] \vee [x_1 \rightarrow x_3]$

$F(0,1,0)=1$

$F(1,1,0)=0$

よって PROP-EVAL ∈ P



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Ex.5.3. Problem of evaluating propositional expression(PROP-EVAL)

Input: $<F, < a_1, a_2, \dots, a_n >>$

F is an extended prop. expression

$(a_1, a_2, \dots, a_n)$ is a truth assignment to F

Question: $F(a_1, a_2, \dots, a_n) = 1$?

Construct a computation tree from a code $\ulcorner F \urcorner$ of ext. prop. expression

It is built in time $O(|\ulcorner F \urcorner|^3)$.

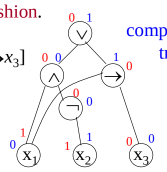
If computation tree is available, we can easily obtain the value $F(a_1, a_2, \dots, a_n)$ in a bottom-up fashion.

Ex.: $F(x_1, x_2, x_3) = [x_1 \wedge \neg x_2] \vee [x_1 \rightarrow x_3]$

$F(0,1,0)=1$

$F(1,1,0)=0$

Hence PROP-EVAL ∈ P



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例5.3. 命題論理式充足性問題:2和積形(2SAT)

入力: $<F>$ Fは2和積形命題論理式

質問: $F(a_1, a_2, \dots, a_n) = 1$を満たす割り当てがあるか?

和積形:

F = (●∨●∨…∨●)∧(●∨…∨●)∧…∧(…)

- リテラルの論理和の論理積で表現されたもの

k和積形(k SAT)

- 和積形の各論理和が k 個のリテラルを含む

ちょうど/たかだか

- 3SAT, 4SAT も同様に定義できる。
- SAT: 各論理和のリテラルの個数に制限がないもの
- ExSAT: 入力が拡張命題論理式(→や↔も許す)

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Ex. 5.3. 2-Satisfiability (2SAT)

Input: $<F>$ F is 2-conjunctive normal form

Question: Is there any assignment such that $F(a_1, a_2, \dots, a_n) = 1$?

Conjunctive Normal Form (CNF)

F = (●∨●∨…∨●)∧(●∨…∨●)∧…∧(…)

- described by ∧ of ∨ of literals.

exactly/at most

k SAT

- Each closure contains k literals

- We can define 3SAT, 4SAT similarly.
- SAT consists of any CNF.
- ExSAT consists of any extended propositional expression.

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例5.4: 到達可能性問題(ST-CON)

入力: $<G,s,t>$: 無向グラフG, $1 \leqq s,t \leqq n(=|G|)$

質問: G上で s から t への道があるか?

- 閉路とは、始点と終点が同じである路
- オイラー閉路とは、すべての辺を一度づつ通る閉路
- ハミルトン閉路とは、すべての頂点を一度づつ通る閉路

例5.4: 一筆書き閉路問題(DEULER)

入力: $<G>$: 有向グラフG

質問: Gはオイラー閉路をもつか?

例5.5: ハミルトン閉路問題(DHAM)

入力: $<G>$: 有向グラフG

質問: Gはハミルトン閉路をもつか?

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Ex. 5.4: Graph reachability problem (ST-CON)

Input: $<G,s,t>$: an undirectd graph G, $1 \leqq s,t \leqq n(=|G|)$

Question: Does G have a path from s to t?

- Cycle is a path that shares two endpoints.
- Euler cycle is a cycle that visits all edges once.
- Hamiltonian cycle is a cycle that visits all vertices once.

Ex. 5.4: Euler cycle problem (DEULER)

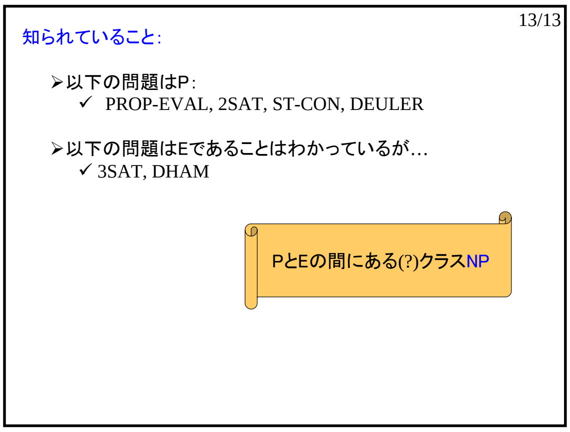
Input: $<G>$: a directed graph G

Question: Does G have an Euler cycle?

Ex. 5.5 Hamiltonian cycle problem (DHAM)

Input: $<G>$: a directed graph G

Question: Does G have a Hamiltonian cycle?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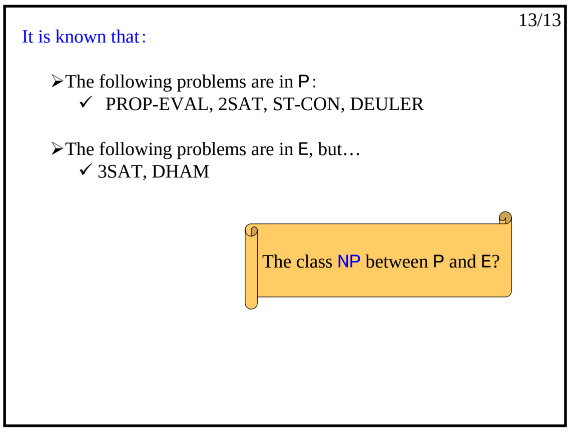
知られていること:
➢以下の問題はP:
✓ PROP-EVAL, 2SAT, ST-CON, DEULER
➢以下の問題はEであることはわかっているが…
✓ 3SAT, DHAM
PとEの間にある(?)クラスNP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It is known that:
➢The following problems are in P:
✓ PROP-EVAL, 2SAT, ST-CON, DEULER
➢The following problems are in E, but…
✓ 3SAT, DHAM
The class NP between P and E?