第6章 多項式時間計算可能性の分析

6.1. 多項式時間還元可能性

定義6.1:
AとBを任意の集合とする.
(1) 関数 h: A→B: 多項式時間還元(polynomial-time reduction)

$\Longleftrightarrow$ (a) h はΣ*からΣ*への全域的関数
(b) $x \in \Sigma^*[x \in A \leftrightarrow h(x) \in B]$
(c) h は多項式時間計算可能.

(2) AからBへの多項式時間還元が存在するとき,
AはBへ多項式時間還元可能という(polynomial time reducible).
このとき, 次のように書く:

$$A \le_m^P B$$

Chapter 6. Analysis on Polynomial-Time Computability

6.1. Polynomial-time Reducibility

Def.6.1:

Let A and B be arbitrary sets.

(1) function $h: A \rightarrow B$: polynomial-time reduction

$\Leftrightarrow$
(a) h is a total function from Σ^* onto Σ^*
(b) $x \in \Sigma^*[x \in A \leftrightarrow h(x) \in B]$
(c) h is polynomial-time computable.

(2) When there is a polynomial-time reduction from A to B,
we say *A* is polynomial-time reducible to *B*.
Then, we denote by

$$\boxed{A \leq_m^P B}$$

$A \le_m^P B$　多項式時間の範囲内では，Aの難しさ$\le$ Bの難しさ

定理6.1. $A \le_m^P B$ のとき，

(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

補注：クラス$\mathcal{E}$は例外．一般には，$B \in \mathcal{E} \rightarrow A \in \mathcal{E}$とはならない．

例6.2: ONE$\equiv$ {1}と定義するとき，クラス$\mathcal{P}$のすべての集合Lについて　$L \le_m^P$ ONE

が成り立つ．

$$h(x) \equiv \begin{cases} 1, & x \in L\text{のとき，} \\ 0, & \text{その他のとき} \end{cases}$$

と定義すると，(1) hはΣ^*からΣ^*への全域的関数．
(2) $\forall x \in \Sigma^*[x \in L \leftrightarrow h(x) \in \text{ONE}]$
(3) hは多項式時間計算可能($L \in \mathcal{P} \rightarrow x \in L$の判定も多項式時間内）

$A \leq_m^P B$ within polynomial time, hardness of A ≤ that of B

定理6.1 $A \leq_m^P B$ leads to,

(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

Note : class $\mathcal{E}$ is exceptional. Generally, $B \in \mathcal{E} \rightarrow A \in \mathcal{E}$ is not true.

Ex.6.2: If we define ONE≡ {1}, for each set L in $\mathcal{P}$ we have

$$L \leq_m^P \text{ONE}$$

If we define $h(x) \equiv \begin{cases} 1, & \text{if } x \in L, \\ 0, & \text{otherwise} \end{cases}$

(1) h is a total function from Σ^* onto Σ^*.
(2) $x \in \Sigma^*[x \in L \leftrightarrow h(x) \in \text{ONE}]$
(3) h is polynomial-time computable(so is computation $L \in \mathcal{P} \rightarrow x \in L$)

定理6.2: A, B, C: 任意の集合

(1) $A \leq_m^P A$

(2) $A \leq_m^P B \wedge B \leq_m^P C \rightarrow A \leq_m^P C$

定義 : $A \equiv_m^P B \leftrightarrow A \leq_m^P B \wedge B \leq_m^P A$

$\equiv_m^P$ は同値関係

Theorem 6.2: A, B, C: arbitrary sets

(1) $A \leq_m^P A$

(2) $A \leq_m^P B \wedge B \leq_m^P C \rightarrow A \leq_m^P C$

Def: $A \equiv_m^P B \leftrightarrow A \leq_m^P B \wedge B \leq_m^P A$

$\equiv_m^P$ is an equivalence relation.

命題論理式の充足可能性問題の間の関係

2SAT （命題論理式充足性問題：二和積形式）
3SAT （命題論理式充足性問題：三和積形式）
SAT （命題論理式充足性問題）
ExSAT （拡張命題論理式充足性問題）

$$2\mathrm{SAT} \le_m^P 3\mathrm{SAT}$$

- •高々k個…自明
- •ちょうどk個…
  - ➢同じリテラルを使ってよいなら簡単。
  - ➢だめなら…レポート(?)

同様に,

$$3\mathrm{SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT}$$

$$2\mathrm{SAT} \le_m^P 3\mathrm{SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT} \quad (6.1)$$

ここで

$$\mathrm{ExSAT} \le_m^P 3\mathrm{SAT}$$

であることを示せると,

$$3\mathrm{SAT} \equiv_m^P \mathrm{SAT} \equiv_m^P \mathrm{ExSAT}$$

となる.

Relation among satisfiability problems of propositional expressions

2SAT （propositional satisfiability problem）
3SAT
SAT
ExSAT （extended propositional satsifiability problem）

$$2\mathrm{SAT} \le_m^P 3\mathrm{SAT}$$

- at most *k*…trivial
- exactly *k*…
  - easy if you can repeat the same literal.
  - report for the other case (?)

Similarly,

$$3\mathrm{SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT}$$

$$2\mathrm{SAT} \le_m^P 3\mathrm{SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT} \quad (6.1)$$

Here, if we can show

$$\mathrm{ExSAT} \le_m^P 3\mathrm{SAT}$$

then we have

$$3\mathrm{SAT} \equiv_m^P \mathrm{SAT} \equiv_m^P \mathrm{ExSAT}$$

例6.3: ExSATから3SATへの還元

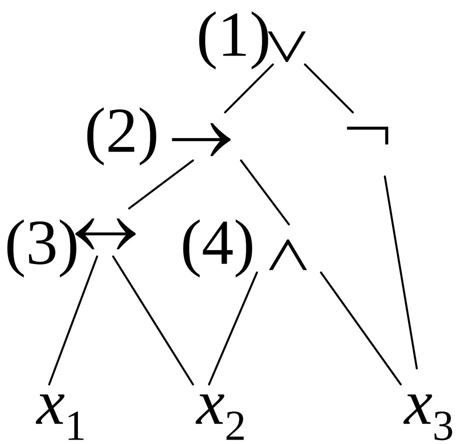
$$E_1(x_1,x_2,x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1,x_2,x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]]$$
$$\wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

このとき, $[E_1$が充足可能$] \leftrightarrow [F_1$が充足可能$]$ (6.2)

F_1は三和積形式に直しやすい形になっている.

F_1の構成方法



(1)$V_1 \equiv V_2 \vee \neg x_3$

(2)$V_2 \equiv [V_3 \rightarrow V_4]$

(3)$V_3 \equiv [x_1 \leftrightarrow x_2]$

(4)$V_4 \equiv x_2 \wedge x_3$

F_1を構成するために, $V_i \rightarrow U_i$とし, V_iの定義式を$\wedge$で結ぶ

Ex. 6.3: Reduction from ExSAT to 3SAT

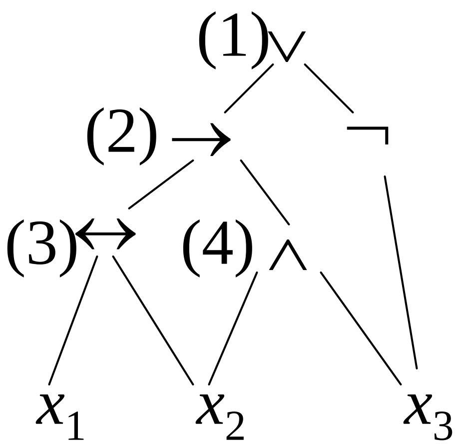
$$E_1(x_1, x_2, x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1, x_2, x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]] \wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

Then, [E_1 is satisfiable] $\leftrightarrow$ [F_1 is satisfiable] (6.2)

F_1 is easier to be converted to 3SAT form.

How to construct F_1



(1) $V_1 \equiv V_2 \vee \neg x_3$

(2) $V_2 \equiv [V_3 \rightarrow V_4]$

(3) $V_3 \equiv [x_1 \leftrightarrow x_2]$

(4) $V_4 \equiv x_2 \wedge x_3$

To construct $\boldsymbol{F_1}$ we let $V_i \rightarrow U_i$, and connect expressions of V_i by $\wedge$

F_1 の構成方法より,
(1)各U_i の値を$V_i(x_1, x_2, x_3)$としない限り, F_1は真にはならない.
(2)各U_iの値を$V_i(x_1, x_2, x_3)$としたとき, F_1 ＝E_1

上の性質が成り立つことは, 帰納法を用いるなどして証明可能.
証明は省略.

三和積形式への変換

$a \to b = \neg a \vee b$

$a \leftrightarrow b = (a \to b) \wedge (b \to a) = [\neg a \vee b] \wedge [\neg b \vee a]$であることを用いる.

$$
\begin{aligned}
U_1 \leftrightarrow [U_2 \vee \neg x_3] &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg[U_2 \vee \neg x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2]
\end{aligned}
$$

他も同様.
よって, すべて三和積形式に変形できることがわかる.

From the construction of F_1
(1) F_1 is never true unless each U_i is $V_i(x_1, x_2, x_3)$.
(2) If each U_i is $V_i(x_1, x_2, x_3)$, we have $F_1 = E_1$

The above properties are proved by using induction.

proof is omitted.

Conversion to 3SAT form

$a \rightarrow b = \neg a \vee b$

$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a) = [\neg a \vee b] \wedge [\neg b \vee a]$: useful relations

$$U_1 \leftrightarrow [U_2 \vee \neg x_3] = [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg [U_2 \vee \neg x_2]]$$
$$= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]]$$
$$= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2]$$
$$= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2]$$

Others are similar.
Thus, every 3SAT form is converted.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

定義6.2: 計算量クラス$\mathcal{C}$に対し，集合Aが次の条件を満たすとき，それを（$\leq_m^P$の下で）$\mathcal{C}$-完全という.

(a) $\forall L \in \mathcal{C}\ [L \leq_m^P A]$

(b) $A \in \mathcal{C}$

補注：条件(a)を満たす集合は$\mathcal{C}$-困難.

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Def.6.2: For a class $\mathcal{C}$, if a set A satisfies the following conditions, then it is called $\mathcal{C}$-complete (under $\leq_m^P$)

(a) $\forall L \in \mathcal{C}\ [L \leq_m^P A]$

(b) $A \in \mathcal{C}$

Note : Sets satisfying the condition (a) are called $\mathcal{C}$-hard.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

例6.5. クラス$\mathcal{NP}$の完全集合の例

3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VCなど

クラス$\mathcal{EXP}$の完全集合

EVAL-IN-E, HALT-IN-Eなど

EVAL-IN-E :

入力: $< a, x, \bar{t} >$

a :1入力プログラムのコード, $x \in \Sigma^*, \bar{t} \geq 0$

出力: $\textit{eval-in-time}(a, x, \overline{2}^{\bar{t}}) = \textit{accept}$?

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Ex.6.5. Examples of $\mathcal{NP}$-complete sets

3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VC, etc

$\mathcal{EXP}$-complete sets

EVAL-IN-E, HALT-IN-E, etc.

EVAL-IN-E:

Input: $< a, x, \bar{t} >$

a : the code of a program with 1 input, $x \in \Sigma^*, \bar{t} \geq 0$

Output: $\mathit{eval\text{-}in\text{-}time}(a, x, \overline{2}^{\bar{t}}) = \mathit{accept}$?

定理6.3. 任意の$\mathcal{C}$-困難集合(含:$\mathcal{C}$-完全集合)Aに対し,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ 対偶は $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ 対偶は $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in \text{co-}\mathcal{NP} \rightarrow \mathcal{C} \subseteq \text{co-}\mathcal{NP}$ 対偶は $\mathcal{C} \not\subset \text{co-}\mathcal{NP} \rightarrow A \notin \text{co-}\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ 対偶は $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

証明:

(1) Bを任意の$\mathcal{C}$集合とすると, Aは$\mathcal{C}$-困難だから,

$B \le_m^P A$ 一方, $A \in \mathcal{P}$の仮定より, $B \in \mathcal{P}$ (定理6.1)

(2), (3), (4)も同様

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subset$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

Proof: CP: contraposition

(1) Let B be any $\mathcal{C}$-set. Then, since A is $\mathcal{C}$-hard,

$B \leq_m^P A$ and by the assumption $A \in \mathcal{P}$ we have $B \in \mathcal{P}$ (Th. 6.1)

(2), (3), (4) are similar.

定理6.3. 任意の$\mathcal{C}$-困難集合（含：$\mathcal{C}$-完全集合）Aに対し,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ 対偶は $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ 対偶は $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in \text{co-}\mathcal{NP} \rightarrow \mathcal{C} \subseteq \text{co-}\mathcal{NP}$ 対偶は $\mathcal{C} \not\subset \text{co-}\mathcal{NP} \rightarrow A \notin \text{co-}\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ 対偶は $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

例6.6. 定理6.3の意味（クラス$\mathcal{NP}$）

Aを$\mathcal{NP}$-完全集合とする.

定理5.9.
(1) $\mathcal{NP} \subseteq \text{co-}\mathcal{NP} \rightarrow \mathcal{NP} = \text{co-}\mathcal{NP}$

定理6.3(1)の対偶より,

$\mathcal{NP} \neq \mathcal{P} \rightarrow A \notin \mathcal{P}$

定理6.3(3)の対偶と定理5.9(1)の対偶より,

$A \notin \text{co-}\mathcal{NP}$

つまり, $\mathcal{NP}$-完全集合は$\mathcal{P} \neq \mathcal{NP}$である限り,

多項式時間では認識できない.

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subset$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

Theorem 5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

Ex.6.6: Meaning of Theorem 6.3 (class $\mathcal{NP}$)

Let A be $\mathcal{NP}$-complete set.

By the contraposition of Theorem 6.3(1) we have

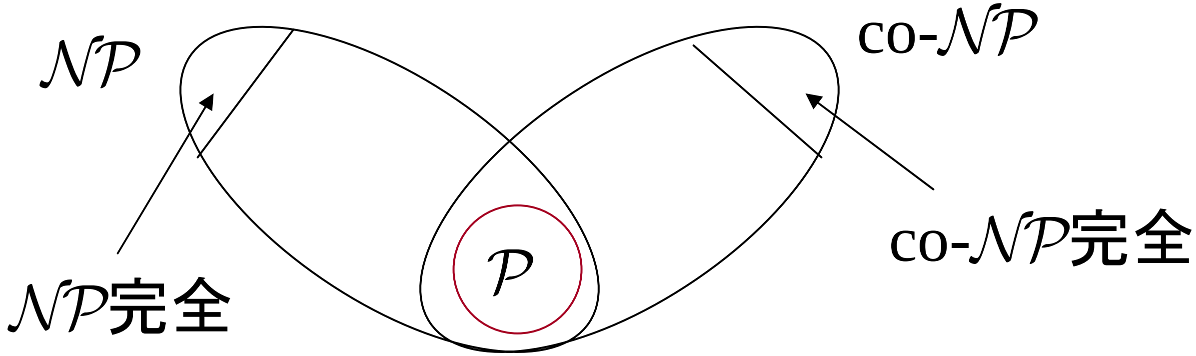
$\mathcal{NP} \neq \mathcal{P} \rightarrow A \notin \mathcal{P}$

By the contraposition of Theorem 6.3(3) and that of Theorem 5.9(1),

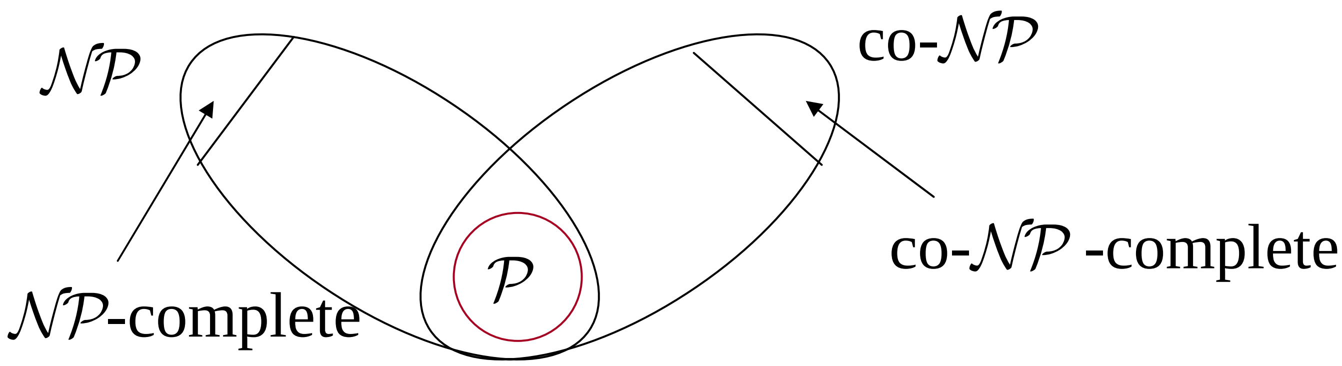
$A \notin$ co-$\mathcal{NP}$

That is, $\mathcal{NP}$-complete sets are $\mathcal{NP}$-sets that cannot be recognized in polynomial time unless $\mathcal{P} = \mathcal{NP}$.

$\mathcal{NP}$-完全集合は$\mathcal{P}\neq\mathcal{NP}$である限り, $\mathcal{NP}\cap$ co-$\mathcal{NP}$には入らない$\mathcal{NP}$集合である.



$\mathcal{NP}$-compete sets are $\mathcal{NP}$-sets that do not belong to $\mathcal{NP} \cap$ co-$\mathcal{NP}$ unless $\mathcal{P} = \mathcal{NP}$.



例6.7. 定理6.3の意味(クラス$\mathcal{EXP}$)

Dを$\mathcal{EXP}$-完全集合とする.

定理6.3(1)の対偶($\mathcal{C} \not\subset \mathcal{P}$ → $A \notin \mathcal{P}$, ここでは$\mathcal{EXP} \not\subset \mathcal{P}$ →$D \notin \mathcal{P}$)

$\mathcal{P} \neq \mathcal{EXP}$ → $\mathcal{EXP} \not\subset \mathcal{P}$ $(\because \mathcal{P} \subseteq \mathcal{EXP})$ → $D \notin \mathcal{P}$

定理6.3(2)の対偶($\mathcal{C} \not\subset \mathcal{NP}$ → $A \notin \mathcal{NP}$,

ここでは$\mathcal{EXP} \not\subset \mathcal{NP}$ →$D \notin \mathcal{NP}$)

$\mathcal{NP} \neq \mathcal{EXP}$ → $\mathcal{EXP} \not\subset \mathcal{NP}$ $(\because \mathcal{NP} \subseteq \mathcal{EXP})$ → $D \notin \mathcal{NP}$

定理6.3(3)の対偶($\mathcal{C} \not\subset$ co-$\mathcal{NP}$ → $A \notin$ co-$\mathcal{NP}$,

ここでは$\mathcal{EXP} \not\subset$ co-$\mathcal{NP}$ →$D \notin$ co-$\mathcal{NP}$)

co-$\mathcal{NP} \neq \mathcal{EXP}$ →$\mathcal{EXP} \not\subset$ co-$\mathcal{NP}$ →$D \notin$ co-$\mathcal{NP}$

ところが定理5.7から $\mathcal{P} \subsetneq \mathcal{EXP}$ であるから, $D \notin \mathcal{P}$.

$\mathcal{EXP}$-完全集合は多項式時間では計算不可能.

Ex. 6.7. Meaning of Theorem 6.3（class $\mathcal{EXP}$)

Let D be an $\mathcal{EXP}$-complete set.

Contraposition of Theorem 6.3(1)
（$\mathcal{C} \not\subset \mathcal{P}$ → $A \notin \mathcal{P}$， where $\mathcal{EXP} \not\subset \mathcal{P}$ → $D \notin \mathcal{P}$)
$\mathcal{P} \neq \mathcal{EXP}$ → $\mathcal{EXP} \not\subset \mathcal{P}$ (∵$\mathcal{P} \subseteq \mathcal{EXP}$) → $D \notin \mathcal{P}$
Contraposition of Theorem 6.3(2)（$\mathcal{C} \not\subset \mathcal{NP}$ → $A \notin \mathcal{NP}$,
Here, $\mathcal{EXP} \not\subset \mathcal{NP}$ → $D \notin \mathcal{NP}$)
$\mathcal{NP} \neq \mathcal{EXP}$ → $\mathcal{EXP} \not\subset \mathcal{NP}$ (∵$\mathcal{NP} \subseteq \mathcal{EXP}$) → $D \notin \mathcal{NP}$
Contraposition of Theorem 6.3(3)（$\mathcal{C} \notin$ co-$\mathcal{NP}$ → $A \not\subset$ co-$\mathcal{NP}$,
here, $\mathcal{EXP} \not\subset$ co-$\mathcal{NP}$ → $D \notin$ co-$\mathcal{NP}$)
co-$\mathcal{NP} \neq \mathcal{EXP}$ → $\mathcal{EXP} \not\subset$ co-$\mathcal{NP}$ → $D \notin$ co-$\mathcal{NP}$
But，by Theorem 5.7， since we know $\mathcal{P} \subsetneq \mathcal{EXP}$ ， we have $D \notin \mathcal{P}$.

$\mathcal{EXP}$-complete sets are not computable in polynomial time.

定理6.4. A: 任意の$\mathcal{C}$-完全集合

すべての集合Bに対し,

(1) $A \leq_m^P B$ → Bは$\mathcal{C}$-困難.

(2) $A \leq_m^P B \wedge B \in \mathcal{C}$ → Bは$\mathcal{C}$-完全.

証明:

定義6.2より, $\forall L \in \mathcal{C}[L \leq_m^P A]$

定理6.2より, $L \leq_m^P A \wedge A \leq_m^P B \to L \leq_m^P B$

したがって, $\forall L \in \mathcal{C}[L \leq_m^P B]$

すなわち, Bは$\mathcal{C}$-困難.

Theorem 6.4. A: any $\mathcal{C}$-complete set

For any set B we have

(1) $A \le_m^P B \rightarrow B$ is $\mathcal{C}$-hard.

(2) $A \le_m^P B \wedge B \in \mathcal{C} \rightarrow B$ is $\mathcal{C}$-complete.

Proof:

By Def. 6.2 $\quad \forall L \in \mathcal{C}[L \le_m^P A]$

By Theorem 6.2, $\quad L \le_m^P A \wedge A \le_m^P B \to L \le_m^P B$

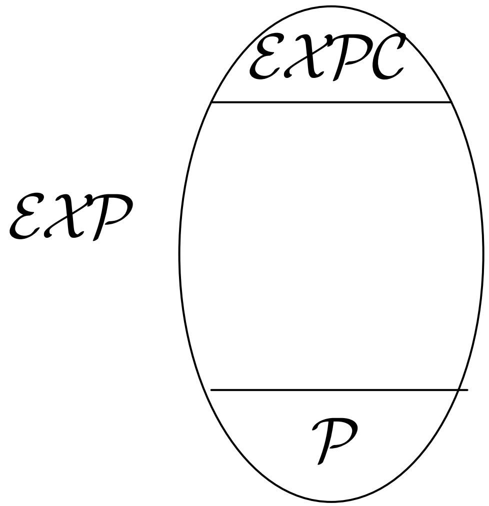
Therefore, $\quad \forall L \in \mathcal{C}[L \le_m^P B]$

That is, B is $\mathcal{C}$-hard.

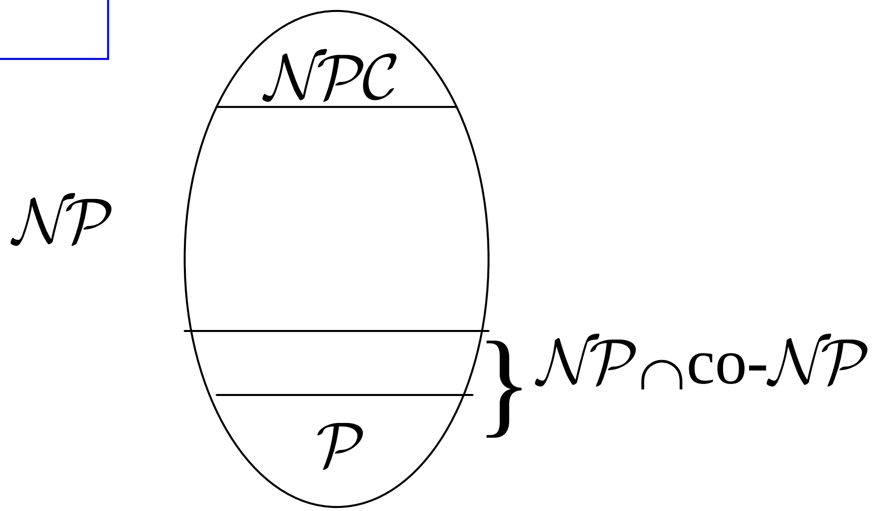
$\mathcal{EXPC}$ ≡{L: Lは$\mathcal{EXP}$-完全}
$\mathcal{NPC}$ ≡ {L: Lは$\mathcal{NP}$-完全}

とすると，次の定理が成り立つ.

定理6.5.
(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$



定理6.6: $\mathcal{P} \neq \mathcal{NP}$**を仮定すると**
(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$

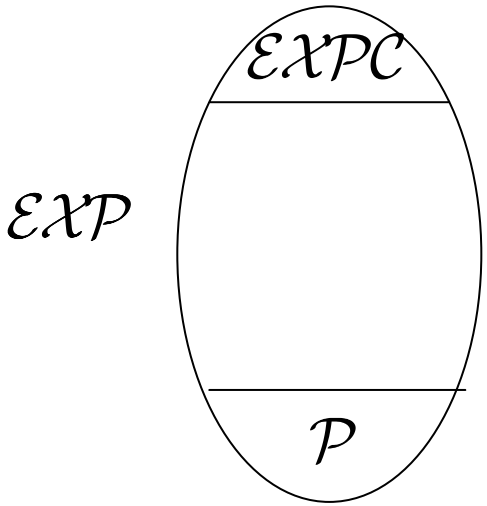


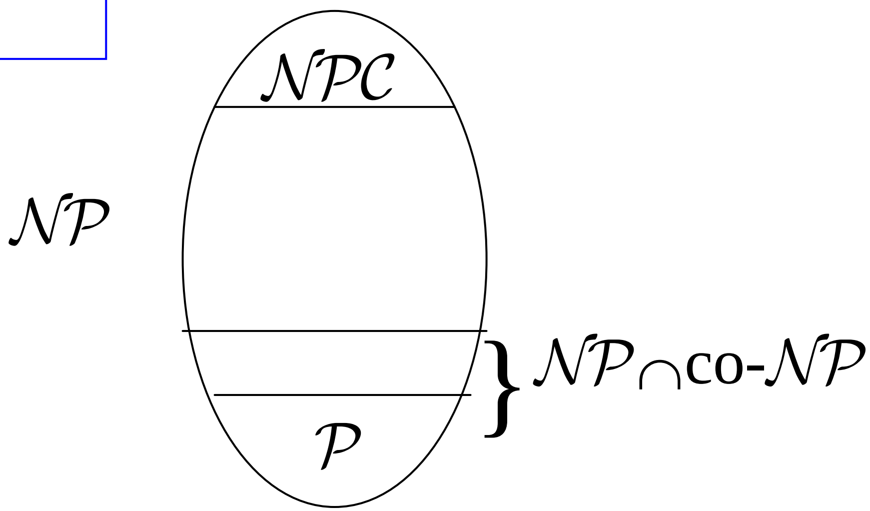
$\mathcal{EXPC} \equiv \{L: L \text{ is } \mathcal{EXP}\text{-complete}\}$
$\mathcal{NPC} \equiv \{L: L \text{ is } \mathcal{NP}\text{-complete}\}$

Then, we have the following theorems.

Theorem 6.5.
(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$

Theorem 6.6: Assuming $\mathcal{P} \neq \mathcal{NP}$
(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$





6.2.2 完全性の証明

定理6.7: EVAL-IN-Eは$\mathcal{EXP}$-完全

証明：例5.6より，EVAL-IN-E $\in \mathcal{EXP}$, よって，

$$\forall L \in \mathcal{EXP}[\, L \leq_m^P \text{EVAL-IN-E}]$$

を示せばよい．

L：任意の$\mathcal{EXP}$集合とする．

Lを$2^{p(l)}$時間で認識するプログラムが存在($p(l)$は多項式)

そのプログラムをA_Lとする．このとき，

$x \in L \leftrightarrow A_L(x)$=accept

time_A_L $(x) \leq 2^{p(|x|)}$

LからEVAL-IN-Eへの還元として次の関数hを考える．

$$h(x) \equiv <\ulcorner A_L \urcorner, x, \overline{p(|x|)}> \quad \text{for } \forall x \in \Sigma^*$$

すると，hは全域的で，多項式時間計算可能．

6.2.2 Proof of Completeness

Theorem 6.7: EVAL-IN-E is $\mathcal{EXP}$-completeness.

Proof : By Example 5.6, we have EVAL-IN-E $\in \mathcal{EXP}$. Thus, it suffices to prove

$$\forall L \in \mathcal{EXP}[\, L \leq_m^P \text{EVAL-IN-E}]$$

L : any $\mathcal{EXP}$ set.

There is a program recognizing L in time $2^{p(l)}$ ($p(l)$ is polynomial)

Let the program be A_L. Then, we have

$$x \in L \leftrightarrow A_L(x)\text{=accept}$$

$$\text{time\_}A_L\,(x) \leq 2^{p(|x|)}$$

Consider the following function h to reduce from L to EVAL-IN-E.

$$h(x) \equiv <\ulcorner A_L \urcorner, x, \overline{p(|x|)}> \quad \text{for } \forall x \in \Sigma^*$$

Then, h is total and computable in polynomial time.

また，すべての$x \in \Sigma^*$に対し

$$
\begin{aligned}
x \in L \leftrightarrow\ & A_L(x) = \text{accept} \\
\leftrightarrow\ & \text{eval}(\lceil A_L \rceil,\ x) = \text{accept} \\
\leftrightarrow\ & \text{eval\_in\_time}(\lceil A_L \rceil,\ x,\ \overline{2^{p(|x|)}}) = \text{accept} \\
\leftrightarrow\ & < \lceil A_L \rceil,\ x,\ \overline{2^{p(|x|)}} > \in \text{EVAL-IN-E} \\
\leftrightarrow\ & h(x) \in \text{EVAL-IN-E}
\end{aligned}
$$

ゆえに，hはLからEVAL-IN-Eへの多項式時間還元.

$$\therefore L \le_m^P \text{EVAL-IN-E} \quad \text{for } \forall L \in \mathcal{EXP}$$

すなわち，EVAL-IN-Eは$\mathcal{EXP}$-完全.

証明終

Moreover, for each $x \in \Sigma^*$ we have

$$
\begin{aligned}
x \in L \leftrightarrow\ & A_L(x) = \text{accept} \\
\leftrightarrow\ & \text{eval}(\lceil A_L \rceil,\ x) = \text{accept} \\
\leftrightarrow\ & \text{eval\_in\_time}(\lceil A_L \rceil,\ x,\ \overline{2^{p(|x|)}}) = \text{accept} \\
\leftrightarrow\ & < \lceil A_L \rceil,\ x,\ \overline{2^{p(|x|)}} > \in \text{EVAL-IN-E} \\
\leftrightarrow\ & h(x) \in \text{EVAL-IN-E}
\end{aligned}
$$

Thus, h is a polynomial-time reduction from L to EVAL-IN-E.

$$\therefore L \le_m^P \text{EVAL-IN-E} \ \text{ for } \forall L \in \mathcal{EXP}$$

That is, EVAL-IN-E is $\mathcal{EXP}$-complete.

Q.E.D.

定理6.8.
(1) EVAL-IN-E $\notin \mathcal{P}$
(2) EVAL-IN-Eは$\mathcal{NP}$-困難
(3) HALT-IN-Eは$\mathcal{EXP}$-完全.

証明:
(1) EVAL-IN-Eは$\mathcal{EXP}$-完全集合で, $\mathcal{EXP}$-完全集合$\notin \mathcal{P}$.
(2) $\forall L \in \mathcal{EXP} \quad [A \leq_m^P \text{EVAL-IN-E}]$ と

$\mathcal{NP} \subseteq \mathcal{EXP}$ より.

Theorem 6.8.
(1) EVAL-IN-E $\notin \mathcal{P}$
(2) EVAL-IN-E is $\mathcal{NP}$-hard.
(3) HALT-IN-E is $\mathcal{EXP}$-complete.

Proof:
(1) EVAL-IN-E is $\mathcal{EXP}$-complete and any $\mathcal{EXP}$-complete set $\notin \mathcal{P}$.
(2) It follows from

$$\forall L \in \mathcal{EXP} \quad [A \leq_m^P \text{EVAL-IN-E}] \quad \text{and}$$

$$\mathcal{NP} \subseteq \mathcal{EXP}$$