I618 Advanced Computer Science II (Part II)

12/21 11:00-12:30
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I will give you some report problems on January.

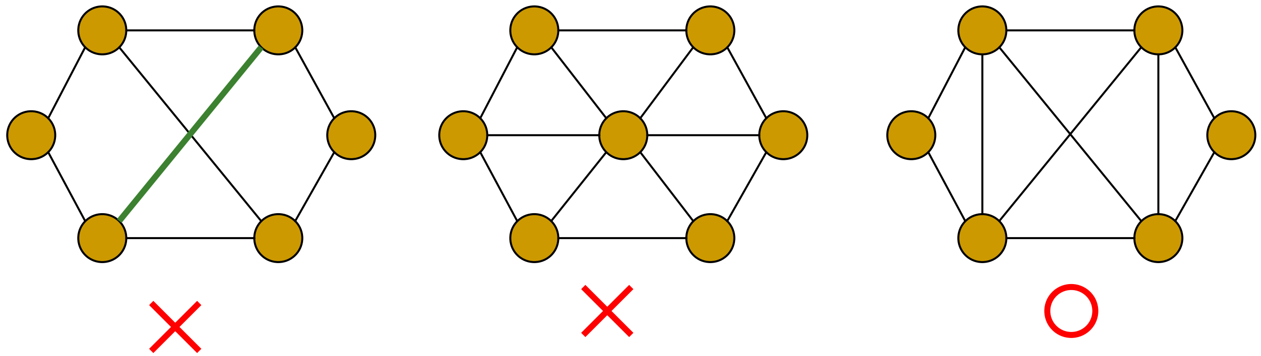
Chordal Graph

- Well known as "chordal graph," "triangulated graph," and "rigid circuit graph."
  - since it has many applications
    - matrix manipulation, graphical modeling, architecture, …
  - (typical intersection graphs)
  - many useful graph theoretic properties
  - many "hard" problems can be solved efficiently

Chordal Graph

[Notation]
A *chord* of a cycle is an edge that joins two non-consecutive vertices on the cycle.

[Definition 3] A graph $G=(V,E)$ is *chordal* if and only if any cycle of length at least 4 has a *chord*.



- Well known as "chordal graph," "triangulated graph," and "rigid circuit graph."
  - (typical intersection graphs?)
  - many useful graph theoretic properties?

An interval graph is a special chordal graph.

[Today's Goal] Properties of a chordal graph, especially, following two characterizations.

1. Perfect Elimination Ordering
2. Intersection graph of subtrees of a tree.

Chordal Graph

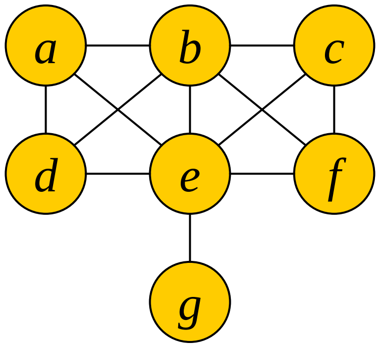
[Notation]
A *chord* of a cycle is an edge that joins two non-consecutive vertices on the cycle.

[Definition 3] A graph $G=(V,E)$ is *chordal* if and only if any cycle of length at least 4 has a *chord*.

[Definition 4] For a graph $G=(V,E)$:

For two vertices u, v and a subset $S \subset V$, S is a *uv-separator* if S is a separator of G and u and v are separated by S.

Set S is a *minimal separator* if no proper subset of S separates the graph.



$S=\{b,e\}$ is a minimal dc-separator, but,
S is not a minimal separator of G
since $\{e\} \subset S$ is also a separator of G.

Chordal Graph

[Theorem 6] (Dirac 1961) A graph $G=(V,E)$ is chordal if and only if every minimal separator is a clique.

(Proof) "if" part: every min. separator is a clique → G is chordal

Let $C=(v_0,v_1,\ldots,v_k,v_0)$ be <u>any</u> cycle with $k \geqq 3$.

If $\{v_0,v_2\}$ is in E, we have a chord. Hence assume $\{v_0,v_2\} \notin E$.

Then there exist v_0v_2-separators.

Among them, we take a minimal v_0v_2-separator S.

Then S has to contain v_1 and v_i with $3 \leqq i \leqq k$.

By assumption, $\{v_1,v_i\}$ is in E, which is a chord of C.

Chordal Graph

[Theorem 6] (Dirac 1961) A graph $G=(V,E)$ is chordal if and only if every minimal separator is a clique.

(Proof) "only if" part: G is chordal→every min. separator is a clique.

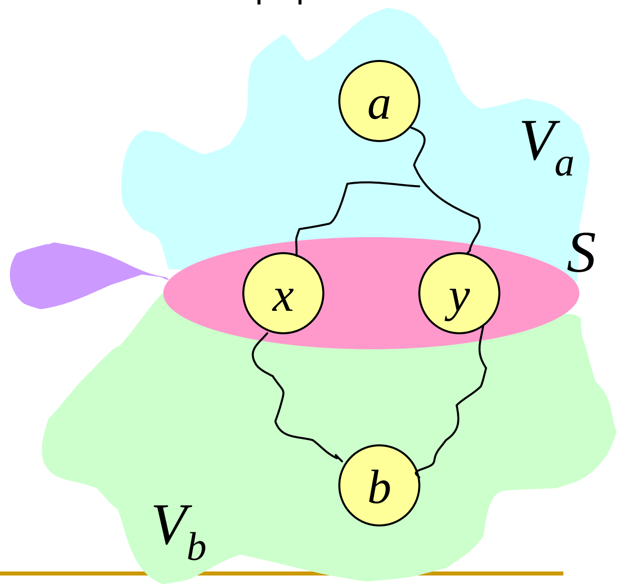
Let a and b be <u>any</u> non-adjacent vertices, and

let S be a minimal ab-separator. W.l.o.g., assume $|S|>1$.

Let x and y be <u>any</u> distinct vertices in S.

It is sufficient to show that $\{x,y\}$ is in E.

Let V_a and V_b be the vertex sets of two connected components in $G[V-S]$ that contain a and b, resp.



Chordal Graph

[Theorem 6] (Dirac 1961) A graph $G=(V,E)$ is chordal if and only if every minimal separator is a clique.

(Proof) "only if" part: G is chordal→every min. separator is a clique.

Let V_a and V_b be the vertex sets of two connected components in $G[V\text{-}S]$ that contain a and b, resp.

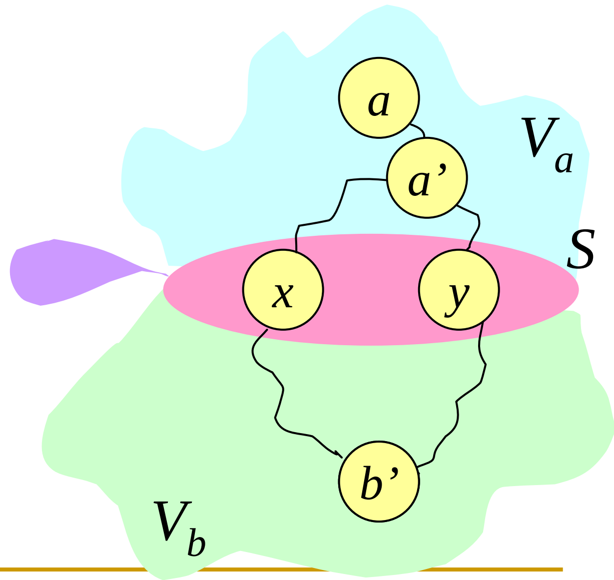
Let P_{ij} denote any path between i and j.

We take four paths P_{ax}, P_{ay}, P_{bx}, P_{by}.

We further take a' that is

1. common on P_{ax} and P_{ay}
2. no other vertices closer to x, y.

Take b' similarly.



Chordal Graph

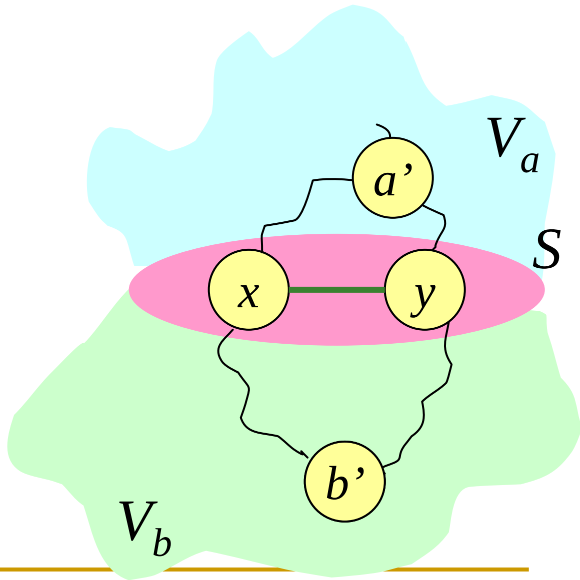
[Theorem 6] (Dirac 1961) A graph G=(V,E) is chordal if and only if every minimal separator is a clique.

(Proof) “only if” part: G is chordal→every min. separator is a clique.

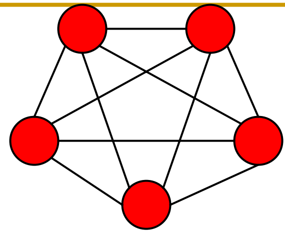
There exists a cycle C of length at least *four* that contains a', x, b', y by joining $P_{a'x}$, $P_{b'x}$, $P_{b'y}$, and $P_{a'y}$.

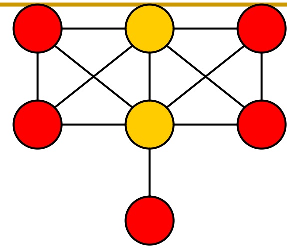
Among them, no pair of vertices in $G[V_a]$ and $G[V_b]$ is joined by an edge since S is a separator.

Hence, since G is chordal, $\{x,y\} \in E$. □



Chordal Graph





[Definition 5] A vertex v is *simplicial* if $N(v)$ induces a clique.

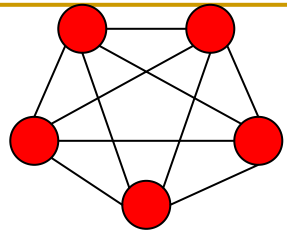
[Theorem 7] (Dirac 1961) Every chordal graph G has a simplicial vertex. If G is not complete, it has at least two non-adjacent simplicial vertices.

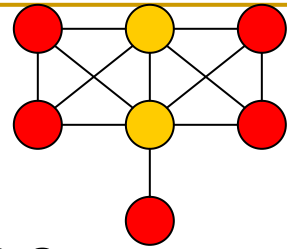
(Proof) When G is complete, it is clear. Hence we assume that G is not complete. We proceed by induction on the number n of vertices. Since the cases $n<3$ is easy, suppose $n \geqq 3$.

Since G is not complete, there are two non-adjacent vertices a and b.

Then, by Theorem 6, there exists an ab-separator S which induces a clique.

Chordal Graph





[Theorem 7] (Dirac 1961) Every chordal graph G has a simplicial vertex. If G is not complete, it has at least two non-adjacent simplicial vertices.

(Proof)

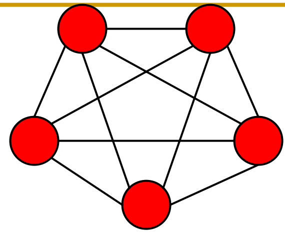
Since G is not complete, there are two non-adjacent vertices a and b.

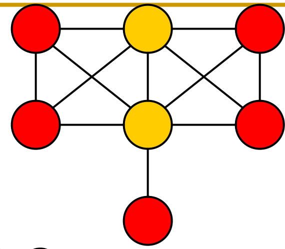
Then, by Theorem 6, there exists an ab-separator S which induces a clique.

Let V_a and V_b be vertex sets such that $G[V_a]$ and $G[V_b]$ contain a and b in $G[V\text{-}S]$, respectively.

Then, $G[V_a \cup S]$ and $G[V_b \cup S]$ are chordal graphs with fewer vertices than G (since $V_a \neq \phi$ and $V_b \neq \phi$).

Chordal Graph





[Theorem 7] (Dirac 1961) Every chordal graph G has a simplicial vertex. If G is not complete, it has at least two non-adjacent simplicial vertices.

(Proof)

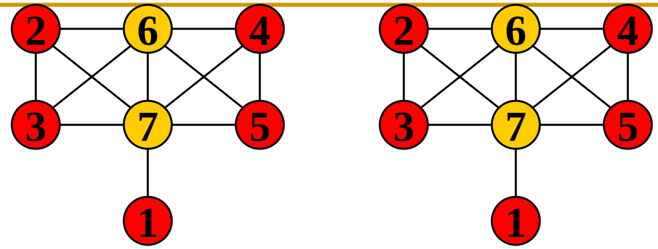
Then, $G[V_a \cup S]$ and $G[V_b \cup S]$ are chordal graphs with fewer vertices than G (since $V_a \neq \phi$ and $V_b \neq \phi$).

By inductive hypothesis, $G[V_a \cup S]$ ($G[V_b \cup S]$) have two simplicial vertices a_1, a_2 (b_1, b_2) as follows;

1. if it is complete, we take at least one from V_a (V_b),
2. if it is not complete, we take two non-adj. vertices.

Thus we can take two non-adjacent simplicial vertices a_i and b_j for some i and j. □

Chordal Graph



[Definition 6] Let $G=(V,E)$ with $|V|=n$. Then a vertex ordering $v_1,v_2,\ldots,v_n$ is called a *perfect elimination ordering* (*PEO*) if v_i is simplicial in $G_i:=G[\{v_i,v_{i+1},\ldots,v_n\}]$.

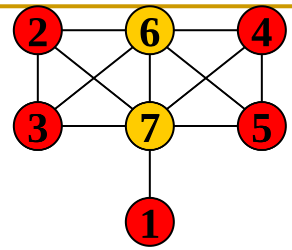
[Theorem 8] (Fulkerson and Gross 1965) A graph G is chordal if and only if G has a PEO.

(Proof) "only if" part: G is chordal → G has a PEO.

By Theorem 6, G has at least one simplicial vertex if $n>0$. Hence let v_1 be the simplicial vertex, and remove it from G.

Since "chordality" is *hereditary* for vertex deletion, the resultant graph is still chordal, and we can repeat this process until V becomes empty.

Chordal Graph



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[Theorem 8] (Fulkerson and Gross 1965) A graph G is chordal if and only if G has a PEO.

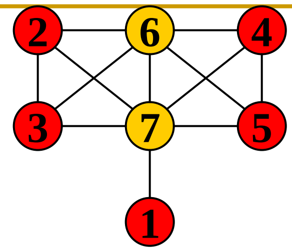
(Proof) "if" part: G has a PEO → G is chordal.

To derive a contradiction, we assume that G has a PEO and G is *not* chordal. We then have a chordless cycle C of length at least 4. Let $C=(v_1,v_2,\ldots,v_k,v_1)$.

Without loss of generality, we suppose v_i is *j*-th element in the PEO, and it is the smallest index in C.

Then, v_i is not simplicial in G_j since $\{v_{i-1},v_{i+1}\}$ is not in E, which contradicts the definition of PEO. □

Chordal Graph



[Definition 7] Let S be any family of sets $s_1, s_2, \ldots, s_n$.

We say S has ***Helly property*** if any subfamily $S' \subseteq S$ satisfies the following condition:

for any $s_i, s_j \in S'$, $s_i \cap s_j \neq \phi$ implies $\bigcap_{i=1}^{n} s_i \neq \phi$.

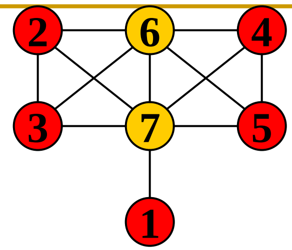
[Fact 1] In the following cases, we have Helly property;

1. S is a set of intervals
2. S is a set of subtrees of a tree

[Note 1]

1. We can find a similar idea in the proof of Theorem 3.

Chordal Graph



[Theorem 9] A graph G is chordal if and only if G is an intersection graph of subtrees of a tree.

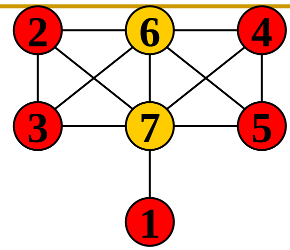
[Corollary 1] Any interval graph is a chordal graph.

(Proof of Theorem 9)

"if" part; any intersection graph G of subtrees $T_1, T_2, \ldots, T_n$ of $\mathcal{T}$ is chordal.

To derive a contradiction, we assume that G is *not* chordal. Then we have a chordless cycle C of length at least 4. Without loss of generality, let $C=(T_1, T_2, \ldots, T_k, T_1)$.

Chordal Graph



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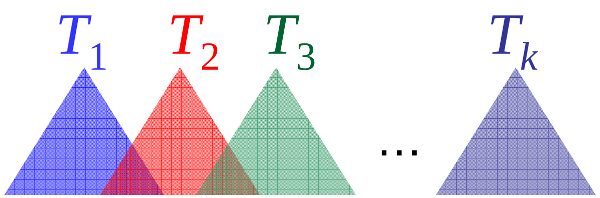
(Proof of Theorem 9) "if" part; any intersection graph G of subtrees $T_1, T_2, \ldots, T_n$ of $\mathcal{T}$ is chordal.

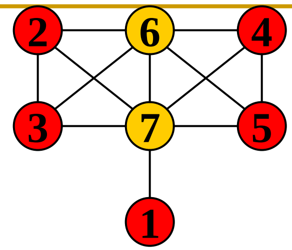
To derive a contradiction, we assume that G is *not* chordal. Then we have a chordless cycle C of length at least 4. Without loss of generality, let $C=(T_1, T_2, \ldots, T_k, T_1)$.

To make $T_1 \cap T_2 \neq \phi$ and $T_2 \cap T_3 \neq \phi$ and $T_1 \cap T_3 = \phi$, and so on, we have to arrange…



Then we cannot make $T_k \cap T_1 \neq \phi$ without intersecting one of them and $T_2, T_3, \ldots, T_{k-1}$ which contradicts that C is chordless.

Chordal Graph



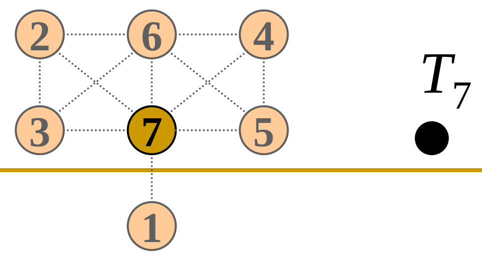
[Theorem 9] A graph G is chordal if and only if G is an intersection graph of subtrees of a tree.

(Proof of Theorem 9) "only if" part; chordal graph G can be an intersection graph of subtrees $T_1,T_2,\ldots,T_n$ of $\mathcal{T}$.

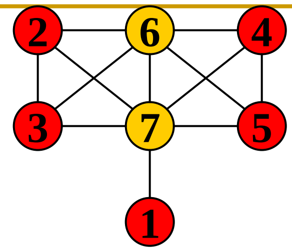
For any chordal graph G, we construct a tree representation. By Theorem 8, G has a PEO $v_1,v_2,\ldots,v_n$.

For $i=n,n$-$1,\ldots,2,1$, we construct T_n, $T_{n-1},\ldots$, T_2, T_1 (with $\mathcal{T}$), where T_i corresponds to v_i, as follows.

1. When $i=n$, we initialize T_n by single vertex of $\mathcal{T}$.



Chordal Graph



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For $i=n, n\text{-}1, \ldots, 2, 1$, we construct $T_n, T_{n\text{-}1}, \ldots, T_2, T_1$ (with $\mathcal{T}$), where T_i corresponds to v_i, as follows.

2. When $i<n$, since v_i is *simplicial* in $G[v_i, v_{i+1}, \ldots, v_n]$, all subtrees corresponding to vertices in $N(v_i)$ have a common vertex u in $\mathcal{T}$ by Fact 1.
3. Add a new neighbor w of u and set $T_i := \{w\}$, and extend subtrees corresponding to vertices in $N(v_i)$.
4. Repeat steps 2-3 and obtain T_is and $\mathcal{T}$. □