# アルゴリズム論
# Theory of Algorithms

第2回講義
アルゴリズムの設計と解析の基礎(2)



# アルゴリズム論
# Theory of Algorithms

Lecture #2
Foundation of Design and Analysis of Algorithms(2)



**問題P3（最大区間和）:** n個のデータが配列a[]に蓄えられているとき,区間[p,q]に対する和（区間和）sum(p, q)を，その区間内の要素a[p]〜a[q]の和と定義する．このとき，区間和の最大値を求めよ．

すべての区間について対応する区間和を求めればよい．

**アルゴリズムP3-A0:**

```
maxsum=0;
for(p=0; p<n; p++)
  for(q=p; q<n; q++){
    // 区間[p,q]での和sumを求める
    sum=0;
    for(i=p; i<=q; i++)
      sum = sum + a[i];
    if(sum > maxsum) maxsum = sum;
  }
```

計算時間:
３重ループだから
$O(n^3)$時間



**Problem P3（Largest Sum Interval）:** Given n data in an array a[], a sum interval sum(p,q) for an interval [p,q] is defined as the sum of elements a[p]〜a[q]. Find a largest sum interval for a given array.

It can be computed by computing the interval sum for every interval.

**Algorithm P3-A0:**

```
maxsum=0;
for(p=0; p<n; p++)
  for(q=p; q<n; q++){
    // find the interval sum in an interval [p,q]
    sum=0;
    for(i=p; i<=q; i++)
      sum = sum + a[i];
    if(sum > maxsum) maxsum = sum;
  }
```

Computation time:
triple-loop=>
$O(n^3)$ time

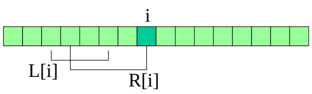


## 動的計画法に基づくアルゴリズム

配列を左から右に順に調べていく．
a[i]を調べているとき，[0,i-1]の範囲における最大の区間和をL[i]，a[i]を右端とする区間の中での最大区間和をR[i]とする．
このとき，

$$L[i] = \begin{cases} L[i-1] & L[i-1] \geqq R[i-1]\text{のとき}, \\ R[i-1] & \text{それ以外のとき}. \end{cases}$$

$$R[i] = \begin{cases} a[i] & R[i-1]+a[i]<a[i]\text{のとき}, \\ R[i-1] + a[i] & \text{それ以外のとき}. \end{cases}$$



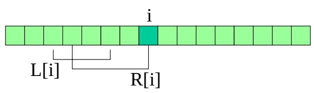



## Algorithm based on dynamic programming

The array is checked from left to right.
Let L[i] be the largest sum interval in the interval [0, i-1] and
R[i] be the largest sum interval for interval with a[i] in its right end.
Then, we have

$$L[i] = \begin{cases} L[i-1] & \text{if } L[i-1] \geqq R[i-1], \\ R[i-1] & \text{otherwise}. \end{cases}$$

$$R[i] = \begin{cases} a[i] & \text{if } R[i-1]+a[i]<a[i], \\ R[i-1] + a[i] & \text{otherwise}. \end{cases}$$



$$L[i] = \begin{cases} L[i-1] & L[i-1] \geqq R[i-1]\text{のとき}, \\ R[i-1] & \text{それ以外のとき}. \end{cases}$$

$$R[i] = \begin{cases} a[i] & R[i-1]+a[i]<a[i]\text{のとき}, \\ R[i-1] + a[i] & \text{それ以外のとき}. \end{cases}$$

最後にL[n-1]とR[n-1]の大きい方が最大値.

**アルゴリズムP3-A5:**

```
L[0] = R[0] = a[0];
for(i=1; i<n; i++){
  if( L[i-1] >= R[i-1] ) L[i] = L[i-1]; else L[i] = R[i-1];
  if( R[i-1] + a[i] < a[i] ) R[i] = a[i]; else R[i] = R[i-1] + a[i];
}
if( L[n-1] > R[n-1] ) return L[n-1]; else return R[n-1];
```

計算時間はO(n).

作業用の配列をなくす事はできるか？



$$L[i] = \begin{cases} L[i-1] & \text{if } L[i-1] \geqq R[i-1], \\ R[i-1] & \text{otherwise.} \end{cases}$$

$$R[i] = \begin{cases} a[i] & \text{if } R[i-1]+a[i]<a[i], \\ R[i-1] + a[i] & \text{otherwise.} \end{cases}$$

Finally, we take the larger of L[n-1] and R[n-1] as the maximum.

**Algorithm P3-A5:**

```
L[0] = R[0] = a[0];
for(i=1; i<n; i++){
  if( L[i-1] >= R[i-1] ) L[i] = L[i-1]; else L[i] = R[i-1];
  if( R[i-1] + a[i] < a[i] ) R[i] = a[i]; else R[i] = R[i-1] + a[i];
}
if( L[n-1] > R[n-1] ) return L[n-1]; else return R[n-1];
```

Computation time is O(n).

Is it possible to do without any auxiliary array?



L[i]の値はL[i-1]とR[i-1]だけで決まる.
R[i]の値はR[i-1], a[i]だけで決まる.
よって，配列を使う必要はない.

**アルゴリズムP3-A6:**

```
  L = R = a[0];
  for(i=1; i<n; i++){
    if( L >= R ) L = L; else L = R;
    if( R + a[i] < a[i] ) R = a[i]; else R = R + a[i];
  }
  if( L > R) return L; else return R;
```



L[i] is determined only by L[i-1] and R[i-1].
R[i] is determined only by R[i-1] and a[i].
Therefore, no auxiliary array is required.

**Algorithm P3-A6:**

```
  L = R = a[0];
  for(i=1; i<n; i++){
    if( L >= R ) L = L; else L = R;
    if( R + a[i] < a[i] ) R = a[i]; else R = R + a[i];
  }
  if( L > R) return L; else return R;
```



**問題P4**: n個のデータが配列a[]に蓄えられているとする．区間[p,q]における平均値をave[p,q]とするとき，この平均値を最大にする区間を求めよ.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| a | 10 | -9 | -5 | 12 | -3 | 10 | -8 | 11 | -8 | -2 |

ave[3,7] = 22/5=4.4　　ave[3,5]=19/3=6.33

実は，区間に制限がなければ，この問題は実に簡単.
配列内の最大値をa[i]とすると，平均値を最大にする区間は[i,i]ということになる.

質問：上記の事実を証明せよ.

では，区間の長さを2以上に限定した場合はどうだろう？

今度は単純な方法では解は求まらない.



**Problem P4**: Suppose n data are stored in an array a[]. By ave[p,q] we denote the average of elements in an interval [p,q]. Then, find an interval maximizing the average.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| a | 10 | -9 | -5 | 12 | -3 | 10 | -8 | 11 | -8 | -2 |

ave[3,7] = 22/5=4.4　　ave[3,5]=19/3=6.33

If there is no constraint in intervals, this problem is in fact quite easy. If a[i] is the maximum in the array, then the interval maximizing the average is obviously [i,i].

Exercise: Question: Prove the above fact.

Then, what about if the length of an interval must be at least two?

In this case there is no simple algorithm.

（腕力法）
すべての区間について平均値を求め，その最大値を求める

**アルゴリズムP4-A0:**
```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++)
  for(q=p+1; q<n; q++){
    sum = 0;
    for(i=p; i<=q; i++)
      sum = sum + a[i];
    ave = sum/(q-p+1);
    if(ave > maxave) maxave = ave;
  }
return maxave;
```

プログラムの構造が3重ループであるから，$O(n^3)$時間かかる．



**（Brute-force algorithm）**
computes average for every interval to find the largest value.

**Algorithm P4-A0:**
```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++)
  for(q=p+1; q<n; q++){
    sum = 0;
    for(i=p; i<=q; i++)
      sum = sum + a[i];
    ave = sum/(q-p+1);
    if(ave > maxave) maxave = ave;
  }
return maxave;
```

Triple-loop structure => $O(n^3)$ time.



（改良1）
区間和を求める計算と組み合わせると無駄が省ける

**アルゴリズムP3-A1:**
```
maxsum=a[0];
for(p=0; p<n-1; p++){
  sum=0;
  for(q=p; q<n; q++){
    sum = sum + a[q];
    if( sum > maxsum) maxsum = sum;
  }
return maxsum;
```

上記のアルゴリズムを区間平均用に変形すればよい．
ただし，線形時間のアルゴリズムを応用することはできない．
（何が違うか？）



(Improvement 1）
Application of algorithm for largest sum interval leads efficiency.

**Algorithm P3-A1:**
```
maxsum=a[0];
for(p=0; p<n-1; p++){
  sum=0;
  for(q=p; q<n; q++){
    sum = sum + a[q];
    if( sum > maxsum) maxsum = sum;
  }
return maxsum;
```

It suffices to convert the above algorithm for average in intervals.
Note that we cannot apply the linear-time algorithm.
(What is different?)



**アルゴリズムP4-A1:**
```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++){
  sum=a[p];
  for(q=p+1; q<n; q++){
    sum = sum + a[q];
    ave = sum/(q-p+1);
    if( ave > maxave) maxave = ave;
  }
}
return maxave;
```

区間平均の場合には，区間の長さが2以上でなければならないことに注意．

計算時間
今度は2重ループの構造なので，$O(n^2)$時間．

区間和問題のように線形時間アルゴリズムは存在するか？



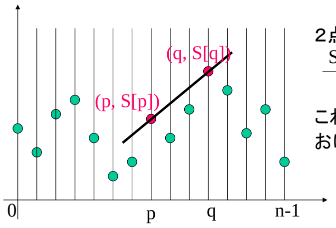
**Algorithm P4-A1:**
```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++){
  sum=a[p];
  for(q=p+1; q<n; q++){
    sum = sum + a[q];
    ave = sum/(q-p+1);
    if( ave > maxave) maxave = ave;
  }
}
return maxave;
```

For interval average, note that the length of interval must be at least 2.

Computation time
double loop structure => $O(n^2)$ time.

Is there a linear algorithm like in the largest sum interval problem?

区間和を最大にする問題P3と同様に，配列の0番目からの和
S[p] = a[0]+a[1]+...+a[p]をすべてのpについて求めて，(p, S[p])で
定まる点を平面上にプロットしてみよう．




２点を通る直線の傾き

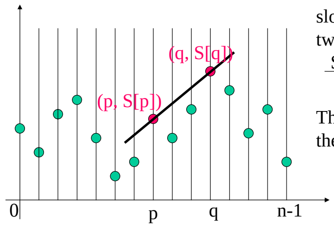
$$\frac{S[q]-S[p]}{q-p}$$

これは区間[p+1,q]に
おける平均値に等しい

したがって，上記のn点の内，どの２点を通る直線を引けば傾きが
最大になるかを求める問題となる．ただし，区間の長さを2以上に
制限しているので，横座標は2以上離れていなければならない．



As in the largest sum interval problem P3, compute the sum
S[p] = a[0]+a[1]+...+a[p] for each p and plot it at (p, S[p]) in
the plane.




slope of line through the
two points

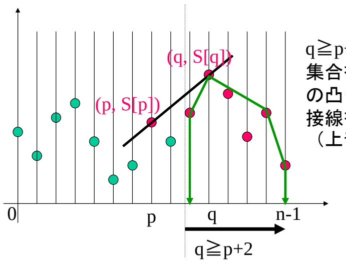
$$\frac{S[q]-S[p]}{q-p}$$

This is the average in
the interval [p+1,q].

Thus, the problem is to find two points among n points so that their
associated line has a largest slope.  Here, since the length of each
interval must be at least 2, their x-coordinates must differ at least by 2.



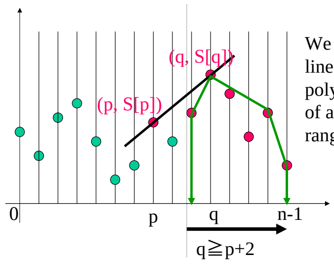
区間の左端pを固定したとき，
どの点(q, S[q])を選べば傾きが最大になるか？




q≧p+2の範囲の点
集合を包含する最小
の凸多角形への
接線を求めればよい．
（上部凸包）



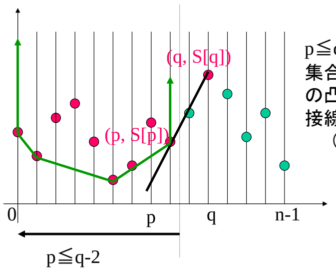
Fixing the left endpoint p,
which point (q, S[q]) should be selected to maximize the slope?




We should find a tangent
line to the smallest convex
polygon (upper convex hull)
of a set of point in the
range q≧p+2.



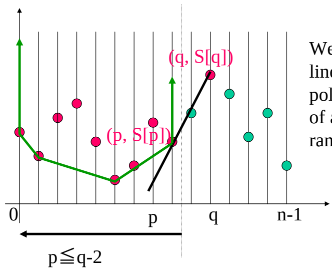
区間の右端qを固定したとき，
どの点(p, S[p])を選べば傾きが最大になるか？




p≦q-2の範囲の点
集合を包含する最小
の凸多角形への
接線を求めればよい．
（下部凸包）

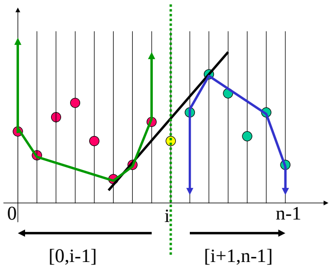


Fixing the right endpoint p,
which point (p, S[p]) should be selected to maximize the slope?



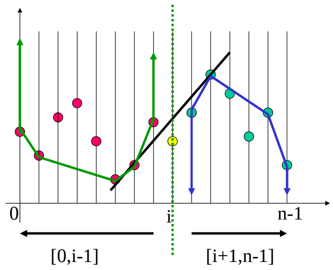

We should find a tangent
line to the smallest convex
polygon (lower convex hull)
of a set of point in the
range p≦q-2.

結局，各iについて，区間[0,i-1]の点の下部凸包LCH(0,i-1)と
区間[i+1,n-1]の点の上部凸包UCH(i+1,n-1)を求めておき，
両者の共通接線を求めればよい．



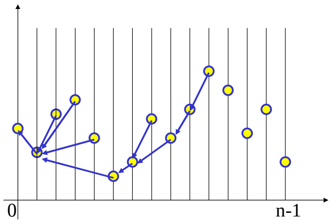



After all we should compute for each i the lower convex hull LCH(0,i-1) for an interval [0,i-1] and the upper convex hull UCH(i+1, n-1) for an interval [i+1, n-1] and then find their common tangent.



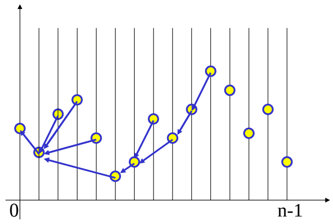



区間[0,i-1]の点の下部凸包LCH(0,i-1)の計算
　iを順に増やしながらLCH(0,i)を順に更新



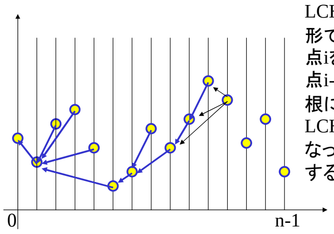



Compute the lower convex hull LCH(0,i-1) for an interval [0,i-1].
　Update LCH(0,i) while increasing i.






区間[0,i-1]の点の下部凸包LCH(0,i-1)の計算
　iを順に増やしながらLCH(0,i)を順に更新

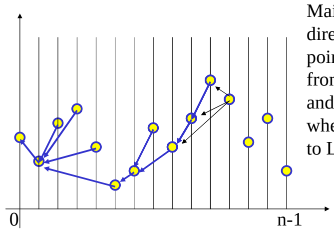



LCH(0,i-1)を有向木の形で管理しておく．
点iを加えるときは，点i-1から始めて，木を根にむけて辿り，LCH(0,i-1)への接線になったところで辺を確定する．

演習問題：上記の計算は線形時間O(n)でできることを示せ．



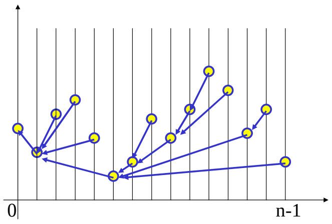
Compute the lower convex hull LCH(0,i-1) for an interval [0,i-1].
　Update LCH(0,i) while increasing i.




Maintain LCH(0,i-1) as a directed tree. To insert a point i, traverse the tree from point i-1 to the root and determine the edge when it becomes a tangent to LCH(0, i-1).

Exercise：Show that the above computation is done in O(n) time.

区間[0,i-1]の点の下部凸包LCH(0,i-1)の計算

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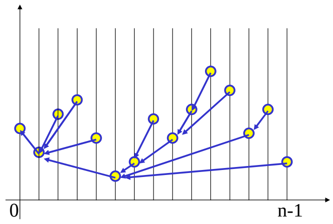
Compute the lower convex hull LCH(0,i-1) for an interval [0,i-1].

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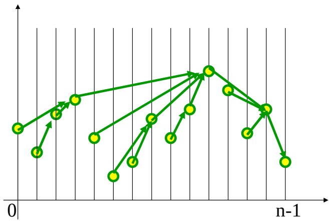
区間[i+1,n-1]の点の上部凸包UCH(i+1, n-1)の計算
　iを順に減らしながらUCH(i+1,n-1)を順に更新

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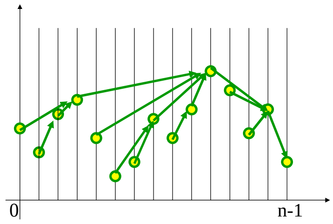
Compute the upper convex hull UCH(i+1,n-1) for an interval [i+1,n-1].
　Update UCH(i+1,n-1) while decreasing i.

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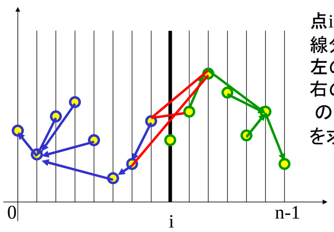
iに関して区間平均の最大値を求める.
　[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.




点i-1と点i+1を結ぶ
線分から始めて,
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り, 共通接線
を求める.

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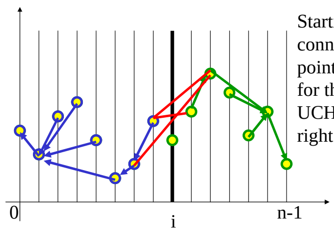
Find an interval maximizing the average for each i.
　Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].




Starting from the line
connecting point i-1 and
point i+1, trace LCH(0,i-1)
for the left points and
UCH(i+1,n-1) for the
right points.

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具体的な手続きは次の通り

```
p=i-1; q=i+1;
do{
  change=false;
  while((p,q,UCH(i+1,n-1)におけるqの親)が反時計回り){
    q=UCH(i+1,n-1)におけるqの親; change=true;
  }
  while((LCH(0,i-1)におけるpの親,p,q,)が時計回り){
    p=LCH(0,i-1)におけるpの親; change=true;
  }
}
```

上記の手続きはO(n)時間でできる.
これをn回繰り返すと全体では$O(n^2)$になってしまう.
線形時間に改善できるか？
あるいは，本当に$O(n^2)$時間かかるのか？

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The concrete procedure is as follows:

```
p=i-1; q=i+1;
do{
  change=false;                    ccw: counter-clockwise
  while((p,q,parent of q in UCH(i+1,n-1)) is ccw){
    q=parent of q in UCH(i+1,n-1); change=true;
  }
  while((parent of p in LCH(0,i-1),p,q,) is cw){
    p=parent of p in LCH(0,i-1); change=true;
  }
}
```

The above procedure is done in O(n) time.
If we iterate it n times, the overall time becomes $O(n^2)$.
Can we improve it into linear time?
Or, does it really take $O(n^2)$ time?

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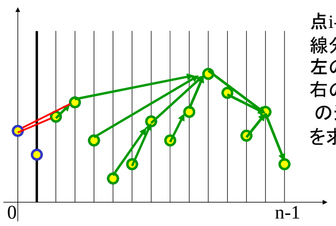
iに関して区間平均の最大値を求める.
[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.




*i=1*

点i-1と点i+1を結ぶ線分から始めて，
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り，共通接線を求める.

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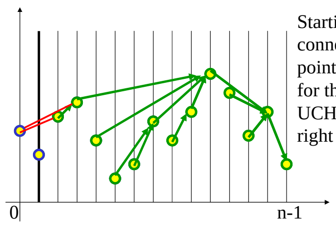
Find an interval maximizing the average for each i.
Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].




*i=1*

Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

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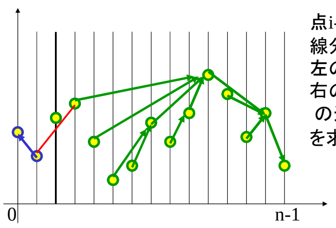
iに関して区間平均の最大値を求める.
[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.




*i=2*

点i-1と点i+1を結ぶ線分から始めて，
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り，共通接線を求める.

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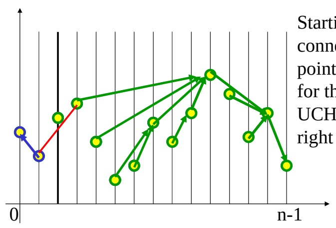
Find an interval maximizing the average for each i.
Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].




*i=2*

Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

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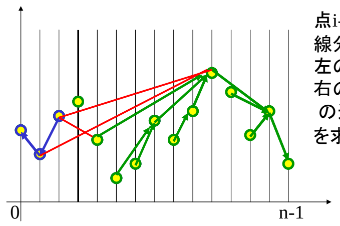
iに関して区間平均の最大値を求める.
[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.

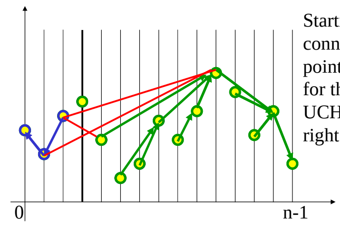



点i-1と点i+1を結ぶ線分から始めて，左の点はLCH(0,i-1) 右の点はUCH(i+1,n-1) の辺を辿り，共通接線を求める.

*i=3*



Find an interval maximizing the average for each i.
Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].

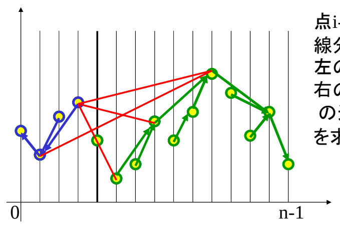



Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

*i=3*



iに関して区間平均の最大値を求める.
[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.

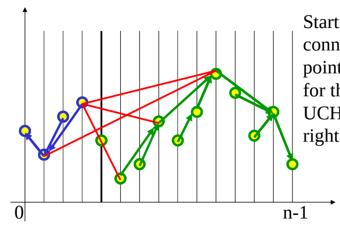



点i-1と点i+1を結ぶ線分から始めて，左の点はLCH(0,i-1) 右の点はUCH(i+1,n-1) の辺を辿り，共通接線を求める.

*i=4*



Find an interval maximizing the average for each i.
Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].




Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

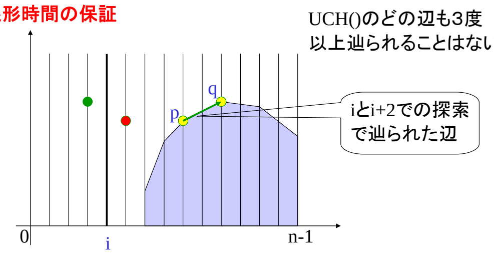
*i=4*



この方法では同じ辺を何度も辿ってしまうことがあり，
計算時間が$O(n)$であることを保証できない.
$O(n^2)$時間かかってしまう例はあるか？

**線形時間の保証**

UCH()のどの辺も3度以上辿られることはない.



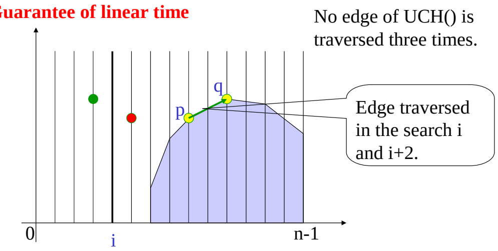

2点i-1, i+1とも直線pqより上になければならない．しかし，そうすると，iでの探索で辺pqを辿ることはないから矛盾（何故か？）



The same edge can be traversed more than once in this algorithm and thus it cannot guarantee linear computation time.
Is there any example requiring $O(n^2)$ time?

**Guarantee of linear time**

No edge of UCH() is traversed three times.




The two points i-1 and i+1 must lie above the line pq. But, then the edge is never traversed in the search at i, a contradiction (Why?).

同様に，k≧2に対して，点i-1からの探索と点i+kからの探索でUCHの同じ辺が辿られることはない．

注意：点i-1からの探索と点iからの探索でUCHの同じ辺が辿られることはある．

結局，UCHのどの辺も3度以上辿られることはないことが証明された．LCHについても同様である．



Similarly, for k≧2, the same edge in UCH is never traversed in the search from point i-1 and from i+k.

Remark：It happens that the same edge in UCH is traversed in the search from point i-1 and point i.

At last, it has been proven that no edge in UCH is traversed three times. Same for LCH.



最終的にアルゴリズムは次のようになる．

**アルゴリズムP4-A2:**
最初に，S[i]=a[0]+a[1]+...+a[i]を求める(i=0, 1, ... , n-1)
次に点集合(i, S[i]), i=0, ..., n-1を構成．
LCH(0, i-1)とUCH(i+1, n-1)を順に構成．
maxslope=(a[0]+a[1])/2;
for(i=1; i<n-1; i++){
　点i-1と点i+1を結ぶ線分から始めて，
　左の点はLCH(0,i-1), 右の点はUCH(i+1,n-1) の辺を辿り，
　共通接線(p, S[p])-(q, S[q])を求める．
　 if( 共通接線の傾き>maxslope) maxslope=共通接線の傾き;
}
maxslopeを出力;

演習問題：このアルゴリズムを実装せよ．



Finally we have the following algorithm.

**Algorithm P4-A2:**
First, we compute S[i]=a[0]+a[1]+...+a[i] (i=0, 1, ... , n-1).
Then, construct a set of points (i, S[i]), i=0, ..., n-1.
Construct LCH(0, i-1) and UCH(i+1, n-1) in order.
maxslope=(a[0]+a[1])/2;
for(i=1; i<n-1; i++){
　Starting from the line connecting point i-1 and point i+1,
　traverse edges in LCH(0,i-1) for the left point and in UCH(i+1,n-1)
　for the right point to find the common tangent (p, S[p])-(q, S[q]).
　 if( slope of tangent line>maxslope) maxslope=slope of tangent line;
}
output maxslope;

Exercise: Implement this algorithm.