計算量クラス間の定義を概観すると...

クラス$\mathcal{P}$の定義(5章)

集合Lがクラス$\mathcal{P}$に入る $\Leftrightarrow$

以下を満たす多項式時間計算可能述語Rが存在:

各 $x \in \Sigma^*$で$x \in L \Leftrightarrow R(x)$

クラス$\mathcal{NP}$の定義(定義5.2)

集合Lがクラス$\mathcal{NP}$に入る $\Leftrightarrow$

以下を満たす多項式qと多項式時間計算可能述語Rが存在:

各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \exists w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

クラスco-$\mathcal{NP}$の定義(定理5.5)

集合Lがクラスco-$\mathcal{NP}$に入る $\Leftrightarrow$

以下を満たす多項式qと多項式時間計算可能述語Rが存在:

各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \forall w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

Observation of the definitions of the classes…

Def: Class $\mathcal{P}$ (Chapter 5)

Set L is in the class $\mathcal{P}$ $\Leftrightarrow$

There exists a poly-time computable predicate R such that

for each $x \in \Sigma^*$, $x \in L \Leftrightarrow R(x)$

Def: Class $\mathcal{NP}$ (Def 5.2)

Set L is in the class $\mathcal{NP}$ $\Leftrightarrow$

There exists a poly q and a poly-time computable pred. R s.t.

for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \exists w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

Def: Class co-$\mathcal{NP}$ (Theorem 5.5)

Set L is in the class co-$\mathcal{NP}$ $\Leftrightarrow$

There exists a poly q and a poly-time computable pred. R s.t.

for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \forall w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

例5.9: 素数判定問題

$$\lceil n \rceil \notin \mathrm{PRIME} \leftrightarrow \exists m{:}1 < m < n[\ n \bmod m = 0]$$

したがって，$q_p(n) = n$とし，

$$R_p(x, w) \leftrightarrow [x \notin \mathsf{N}] \vee [[w \in \mathsf{N}] \wedge [1 < m < n] \wedge [n \ \mathsf{mod}\ m = 0]]$$

(ただし，n, mは各々x, wが表す自然数，
Nは自然数の2進表記全体）

と定義すると，

すべての$x \in \Sigma^*$に対し，　$x \notin \mathrm{PRIME} \leftrightarrow \exists q_p w[R_p(x, w)]$

これは，$x \notin \mathrm{PRIME}$に対する証拠

よって，$\overline{\mathrm{PRIME}} \in \mathcal{NP}$, i.e., $\mathrm{PRIME} \in$ co-$\mathcal{NP}$

実際，$Q(x, w) \leftrightarrow \neg R_p(x, w)$ とすると

$$\mathrm{PRIME} = \{x{:}\ \forall q_p w\ [Q_p(x, w)]\}$$

と表せる.

$\mathrm{PRIME} \in \mathcal{NP}$ も示せるが，その証明はもっと複雑.

Ex.5.9: Primality testing

$$\lceil n \rceil \notin \mathrm{PRIME} \leftrightarrow \exists m : 1 < m < n[\ n \bmod m = 0]$$

Therefore, for $q_p(n) = n$,

$$R_p(x, w) \leftrightarrow [x \notin \mathsf{N}] \vee [[w \in \mathsf{N}] \wedge [1 < m < n] \wedge [n \bmod m = 0]]$$

(where, n and m are natural numbers represented by x and w. N is a set of all natural numbers in the binary form)

This definition leads to

for every $x \in \Sigma^*$ we have $x \notin \mathrm{PRIME} \leftrightarrow \exists q_p w[R_p(x, w)]$

This is a witness to $x \notin \mathrm{PRIME}$

Thus, $\overline{\mathrm{PRIME}} \in \mathcal{NP}$, i.e., PRIME $\in$ co-$\mathcal{NP}$

In fact, using $Q(x, w) \leftrightarrow \neg R_p(x, w)$, PRIME can be expressed as

PRIME = $\{x: \forall q_p w\ [Q_p(x, w)]\}$

We can also show that PRIME $\in \mathcal{NP}$, but its proof is more complex.

$\mathcal{NP}$問題の例

・**合成数判定問題**(COMPOSITE)
入力：自然数n
質問：nは合成数か？（素数でないか？）

・**ナップサック問題**(KNAP)
入力：自然数の組$< a_1, a_2, \ldots , a_n, b >$
質問：$\sum_{i \in S} a_i = b$ となる添字の集合 $S \subseteq \{1, \ldots , n\}$があるか？

・**箱詰め問題**(BIN)
入力：自然数の組$< a_1, a_2, \ldots , a_n, b, k>$
質問：添字の集合$U=\{1, \ldots , n\}$を$U_1, \ldots , U_k$のk個に分割し，
各jで $\sum_{i \in U_j} a_i \le b$ とすることは可能か？

・**頂点被覆問題**(VC)
入力：無向グラフGと自然数kの組$<G, k>$
質問：Gにk頂点の頂点被覆が存在するか？

頂点被覆S:
どの辺(u,v)も
u,vの一方は
Sに含まれる

Examples of $\mathcal{NP}$ problems

- **Composite Number Testing Problem**(COMPOSITE)
  input : natural number n
  question : Is n composite? (Is it not prime?)

- **Knapsack Problem**(KNAP)
  input : n+1 tuple of natural numbers $< a_1, a_2, \ldots, a_n, b >$
  question : Is there a set of indices $S \subseteq \{1, \ldots, n\}$ s.t. $\sum_{i \in S} a_i = b$?

- **Bin Packing Problem**(BIN)
  input : n+2 tuple of natural numbers $< a_1, a_2, \ldots, a_n, b, k >$
  question : Is there a partition of a set of indices $U=\{1, \ldots, n\}$
  into $U_1, \ldots, U_k$ such that $\sum_{i \in U_j} a_i \leq b$ for each j?

- **Vertex Cover Problem**(VC)
  input : pair of undirected graph G and natural number k $< G, k >$
  question : Is there a vertex cover of k vertices over G?

Vertex Cover S contains at least one of u and v for each edge (u,v).

5.3. 計算量クラス間の関係

定理5.6: $\mathcal{P} \subseteq \mathcal{E} \subseteq \mathcal{EXP}$.

定義より，明らか.

定理5.7: $\mathcal{P} \subsetneqq \mathcal{E} \subsetneqq \mathcal{EXP}$.

階層定理(定理4.4):
任意の制限時間 t_1, t_2 に対し、

$$\forall c > 0, \overset{\infty}{\forall} n[ct_1(n)^2 \le t_2(n)$$

$$\rightarrow \mathrm{TIME}(t_1) \subsetneqq \mathrm{TIME}(t_2)$$

証明:

(1) $\mathcal{P} \subsetneqq \mathcal{E}$.

$t_1(n)=2^n$, $t_2(n)=2^{3n}$とすると，<u>階層定理</u>より，

$$\mathrm{TIME}(2^n) \subsetneqq \mathrm{TIME}(2^{3n})$$

一方，$\mathcal{P} \subseteq \mathrm{TIME}(2^n) \subsetneqq \mathrm{TIME}(2^{3n}) \subseteq \mathcal{E}$だから，

$$\mathcal{P} \subsetneqq \mathcal{E}.$$

(2)も同様.

証明終

5.3. Relation in the Complexity Class

Theorem 5.6: $\mathcal{P} \subseteq \mathcal{E} \subseteq \mathcal{EXP}$.

Obvious from the definition.

Theorem 5.7: $\mathcal{P} \subsetneqq \mathcal{E} \subsetneqq \mathcal{EXP}$.

Hierarchy Thm. (Thm. 4.4):
For any times t_1, t_2 ,

$$\forall c > 0, \overset{\infty}{\forall} n[ct_1(n)^2 \leq t_2(n)) \rightarrow \mathrm{TIME}(t_1) \underset{\neq}{\subset} \mathrm{TIME}(t_2)$$

Proof:

(1) $\mathcal{P} \subsetneqq \mathcal{E}$.

For $t_1(n)=2^n$, $t_2(n)=2^{3n}$, from the <u>hierarchy theorem</u> we have

$$\mathrm{TIME}(2^n) \subsetneqq \mathrm{TIME}(2^{3n})$$

On the other hand, since $\mathcal{P} \subseteq \mathrm{TIME}(2^n) \subsetneqq \mathrm{TIME}(2^{3n}) \subseteq \mathcal{E}$

$$\mathcal{P} \subsetneqq \mathcal{E}.$$

(2) is similar.

Q.E.D.

定理5.8.
(1) $\mathcal{P} \subseteq \mathcal{NP}$, $\mathcal{P} \subseteq$ co-$\mathcal{NP}$ ($\therefore \mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$)
(2) $\mathcal{NP} \subseteq \mathcal{EXP}$, co-$\mathcal{NP} \subseteq \mathcal{EXP}$ ($\therefore \mathcal{NP} \cup$ co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

証明: (1) $\mathcal{P} \subseteq \mathcal{NP}$（$\mathcal{P} \subseteq$ co-$\mathcal{NP}$ も同様）

L: 任意の$\mathcal{P}$集合

➔ Lは多項式時間で認識可能

よって，多項式時間計算可能述語Pを用いて次のように書ける.

$$\forall x \in \Sigma^* : [x \in L \leftrightarrow P(x)] \quad \text{or} \quad P = \{x : P(x)\}$$

$R(x, w) = P(x)$と定義（第2引数は無視）

→ 任意の多項式qについて，

$$L = \{x : \exists_q w[\, R(x,w)]\}$$

よって，$\mathcal{NP}$の定義より，$L \in \mathcal{NP}$ i.e., $\mathcal{P} \subseteq \mathcal{NP}$.

Theorem 5.8.
(1) $\mathcal{P} \subseteq \mathcal{NP}$, $\mathcal{P} \subseteq$ co-$\mathcal{NP}$ ($\therefore \mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$)
(2) $\mathcal{NP} \subseteq \mathcal{EXP}$, co-$\mathcal{NP} \subseteq \mathcal{EXP}$ ($\therefore \mathcal{NP} \cup$ co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

Proof:

(1) $\mathcal{P} \subseteq \mathcal{NP}$ ($\mathcal{P} \subseteq$ co-$\mathcal{NP}$ is similar)

L: arbitrary $\mathcal{P}$ set

➔ L is recognizable in polynomial time

Thus, we have the following description using a polynomial-time computable predicate P.

$$\forall x \in \Sigma^* : [x \in L \leftrightarrow P(x)] \text{ or } P = \{x : P(x)\}$$

We define $R(x, w) = P(x)$ (neglecting the second argument)

→ for any polynomial q,

$$L = \{x : \exists_q w[\, R(x,w)]\}$$

Thus, from the definition of $\mathcal{NP}$, $L \in \mathcal{NP}$ i.e., $\mathcal{P} \subseteq \mathcal{NP}$.

(2) $\mathcal{NP} \subseteq \mathcal{EXP}$ (co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

L: 任意の$\mathcal{NP}$集合

➔多項式qと多項式時間計算可能述語Rが存在して，

$$L = \{x : \exists_q w[R(x,w)]\} = \{x : \exists_q w[|w| \leq q(|x|) \wedge R(x,w)]\}$$

qとRを用いて，Lを認識するプログラムを作る．

```
prog L(input x);
begin
    for each w ∈ Σ^{≤q(|x|)} do
      if R(x, w) then accept end-if
    end-for;
    reject
end.
```

長さlの入力に対するプログラムの時間計算量:

Rは多項式時間計算可能だったから，ある多項式pに対し，

Rの計算時間$=p(|x| + |w|) \leq p(l + q(l)) \longleftarrow l$ の多項式

全体では，$\{p(l+q(l)) + cq(l)\}2^{q(l)} + d = O(2^{l+q(l)})$

よって，$L \in \mathcal{EXP}$ → $\mathcal{NP} \subseteq \mathcal{EXP}$ 証明終

(2) $\mathcal{NP} \subseteq \mathcal{EXP}$ (co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

L: any $\mathcal{NP}$ set

➔There is some polynomial q and polynomial-time computable predicate R such that

$$L = \{x : \exists_q w[R(x, w)]\} = \{x : \exists_q w[|w| \le q(|x|) \wedge R(x, w)]\}$$

program recognizing L using q and R

prog L(input x);
begin
for each $w \in \Sigma^{\le q(|x|)}$ do
if R(x, w) then accept end-if
end-for;
reject
end.

time complexity of the program for an input of length *l*:

Since R is polynomial-time computable, for some polynomial q

time of R=$p(|x| + |w|) \le p(l + q(l))$ ⟵ polynomial of l

In total, $\{p(l+q(l)) + cq(l)\}2^{q(l)} + d = O(2^{l+q(l)})$

Hence, $L \in \mathcal{EXP}$ → $\mathcal{NP} \subseteq \mathcal{EXP}$ Q.E.D.

定理5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$
(2) co-$\mathcal{NP} \subseteq \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$.

補注: (3)より, $\mathcal{NP} \neq$co-$\mathcal{NP}$の証明は, $\mathcal{P} \neq \mathcal{NP}$の証明より難しい.

証明: (1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$ ((2)の証明も同様)
任意の$L \in$ co-$\mathcal{NP}$に対して$L \in \mathcal{NP}$が示せれば,
co-$\mathcal{NP} \subseteq \mathcal{NP}$が証明できるので,
仮定の$\mathcal{NP} \subseteq$ co-$\mathcal{NP}$と合わせて$\mathcal{NP} =$ co-$\mathcal{NP}$が言える.

$L \in$ co-$\mathcal{NP} \rightarrow \overline{L} \in \mathcal{NP}$ (定義5.3より)
$\rightarrow \overline{L} \in$ co-$\mathcal{NP}$ ($\mathcal{NP} \subseteq$ co-$\mathcal{NP}$より)
$\rightarrow L \in \mathcal{NP}$ (定義5.3と$\overline{\overline{L}}=L$より)

Theorem 5.9

(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

(2) co-$\mathcal{NP} \subseteq \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$.

Note: from (3) the proof for $\mathcal{NP} \neq$ co-$\mathcal{NP}$ is harder than that for $\mathcal{P} \neq \mathcal{NP}$.

Proof: (1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$ (proof of (2) is similar)

Since co-$\mathcal{NP} \subseteq \mathcal{NP}$ is shown if we prove $L \in \mathcal{NP}$ for any $L \in$ co-$\mathcal{NP}$

Combining it with the assumption $\mathcal{NP} \subseteq$ co-$\mathcal{NP}$, we have $\mathcal{NP} =$ co-$\mathcal{NP}$ and so

$L \in$ co-$\mathcal{NP} \rightarrow \overline{L} \in \mathcal{NP}$ (by Definition 5.3)

$\rightarrow \overline{L} \in$ co-$\mathcal{NP}$ ($\mathcal{NP} \subseteq$ co-$\mathcal{NP}$)

$\rightarrow L \in \mathcal{NP}$ (Definition 5.3 and $\overline{\overline{L}}=L$)

(3) $\mathcal{NP} \neq$ co-$\mathcal{NP}$ $\rightarrow \mathcal{P} \neq \mathcal{NP}$.

対偶: $\mathcal{P} = \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

$\mathcal{P}$=$\mathcal{NP}$と仮定すると，すべてのLに対し

$L \in \mathcal{NP} \leftrightarrow L \in \mathcal{P}$　（$\mathcal{P} = \mathcal{NP}$ より）

$\leftrightarrow \overline{L} \in \mathcal{P}$　(演習問題5.5)

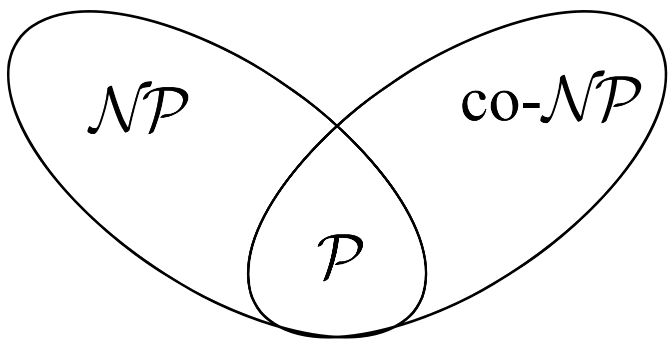
$\leftrightarrow \overline{L} \in \mathcal{NP}$　($\mathcal{P} = \mathcal{NP}$ より)

$\leftrightarrow L\ (= \overline{\overline{L}}) \in$ co-$\mathcal{NP}$　(定義5.3より)

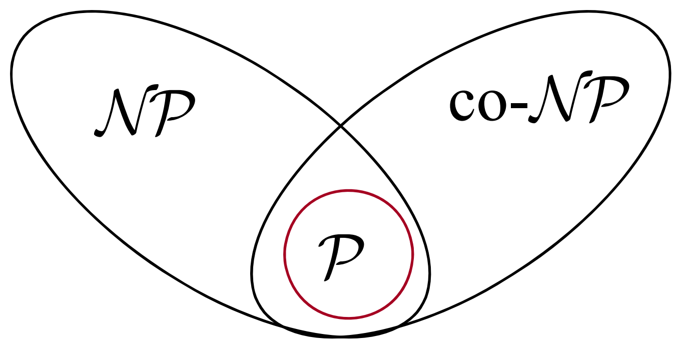
$\therefore \mathcal{NP} =$ co-$\mathcal{NP}$

証明終

$\mathcal{NP} \neq$ co-$\mathcal{NP}$が正しいと



or



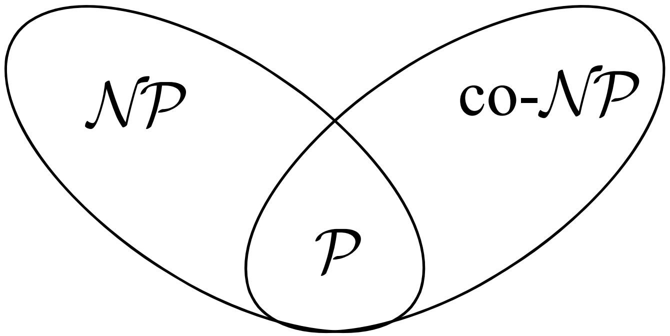
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP}$ $\rightarrow$ $\mathcal{P} \neq \mathcal{NP}$.

Contraposition: $\mathcal{P} = \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

If we assume $\mathcal{P}=\mathcal{NP}$, for any L we have

$$
\begin{array}{lll}
L \in \mathcal{NP} & \leftrightarrow L \in \mathcal{P} & (\mathcal{P} = \mathcal{NP}) \\
& \leftrightarrow \overline{L} \in \mathcal{P} & \text{(Exercise 5.5)} \\
& \leftrightarrow \overline{L} \in \mathcal{NP} & (\mathcal{P} = \mathcal{NP}) \\
& \leftrightarrow L\ (= \overline{L}\ ) \in \text{co-}\mathcal{NP} & \text{(Definition 5.3)} \\
& \therefore \mathcal{NP} = \text{co-}\mathcal{NP} & \text{Q.E.D.}
\end{array}
$$

If $\mathcal{NP} \neq$ co-$\mathcal{NP}$ is true,



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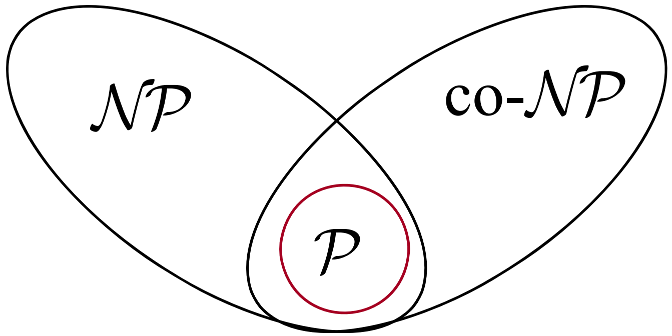


第6章 多項式時間計算可能性の分析

6.1. 多項式時間還元可能性

定義6.1:

AとBを任意の集合とする.

(1) 関数 $h: A \to B$: 多項式時間還元(polynomial-time reduction)

$$\Leftrightarrow \begin{cases} \text{(a) } h \text{ は}\Sigma^* \text{から}\Sigma^* \text{への全域的関数} \\ \text{(b) } x \in \Sigma^* [x \in A \leftrightarrow h(x) \in B] \\ \text{(c) } h \text{ は多項式時間計算可能.} \end{cases}$$

(2) AからBへの多項式時間還元が存在するとき,
AはBへ多項式時間還元可能という(polynomial time reducible).
このとき, 次のように書く:

$$\boxed{A \leq_m^P B}$$

Chapter 6. Analysis on Polynomial-Time Computability

6.1. Polynomial-time Reducibility

Def.6.1:

Let A and B be arbitrary sets.

(1) function $h: A \rightarrow B$: polynomial-time reduction

$\Longleftrightarrow$
- (a) h is a total function from Σ^* onto Σ^*
- (b) $x \in \Sigma^* [x \in A \leftrightarrow h(x) \in B]$
- (c) h is polynomial-time computable.

(2) When there is a polynomial-time reduction from A to B, we say A is polynomial-time reducible to B.
Then, we denote by

$$\boxed{A \leq_m^P B}$$

$A \le_m^P B$　多項式時間の範囲内では，Aの難しさ$\le$ Bの難しさ

定理6.1. $A \le_m^P B$ のとき，

(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in \text{co-}\mathcal{NP} \rightarrow A \in \text{co-}\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

補注：クラス$\mathcal{E}$は例外．一般には，$B \in \mathcal{E} \rightarrow A \in \mathcal{E}$とはならない．

例6.2: ONE$\equiv \{1\}$と定義するとき，クラス$\mathcal{P}$のすべての集合Lについて　$L \le_m^P \text{ONE}$

が成り立つ．
$$h(x) \equiv \begin{cases} 1, & x \in L\text{のとき}, \\ 0, & \text{その他のとき} \end{cases}$$
と定義すると，(1) hはΣ^*からΣ^*への全域的関数．
(2) $\forall x \in \Sigma^* [x \in L \leftrightarrow h(x) \in \text{ONE}]$
(3) hは多項式時間計算可能($L \in \mathcal{P} \rightarrow x \in L$の判定も多項式時間内）

$A \leq_m^P B$ within polynomial time, hardness of $A \leq$ that of B

定理6.1 $A \leq_m^P B$ leads to,

(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

Note : class $\mathcal{E}$ is exceptional. Generally, $B \in \mathcal{E} \rightarrow A \in \mathcal{E}$ is not true.

Ex.6.2: If we define ONE$\equiv \{1\}$, for each set L in $\mathcal{P}$ we have

$$L \leq_m^P \text{ONE}$$

If we define $h(x) \equiv \begin{cases} 1, & \text{if } x \in L, \\ 0, & \text{otherwise} \end{cases}$

(1) h is a total function from Σ^* onto Σ^*.
(2) $x \in \Sigma^*[x \in L \leftrightarrow h(x) \in \text{ONE}]$
(3) h is polynomial-time computable(so is computation $L \in \mathcal{P} \rightarrow x \in L$)

定理6.2: A, B, C: 任意の集合

(1) $A \le_m^P A$

(2) $A \le_m^P B \wedge B \le_m^P C \rightarrow A \le_m^P C$

定義 : $A \equiv_m^P B \leftrightarrow A \le_m^P B \wedge B \le_m^P A$

$\equiv_m^P$ は同値関係

Theorem 6.2: A, B, C: arbitrary sets

(1) $A \le_m^P A$

(2) $A \le_m^P B \land B \le_m^P C \to A \le_m^P C$

Def: $A \equiv_m^P B \leftrightarrow A \le_m^P B \land B \le_m^P A$

$\equiv_m^P$ is an equivalence relation.

命題論理式の充足可能性問題の間の関係

2SAT　（命題論理式充足性問題：二和積形式）
3SAT　（命題論理式充足性問題：三和積形式）
SAT　（命題論理式充足性問題）
ExSAT　（拡張命題論理式充足性問題）

$$\mathrm{2SAT} \le_m^P \mathrm{3SAT}$$

- 高々k個…自明
- ちょうどk個…
  - 同じリテラルを使ってよいなら簡単。
  - だめなら…考えてみよう！

同様に,

$$\mathrm{3SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT}$$

$$\mathrm{2SAT} \le_m^P \mathrm{3SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT} \quad (6.1)$$

ここで

$$\mathrm{ExSAT} \le_m^P \mathrm{3SAT}$$

であることを示せると,

$$\mathrm{3SAT} \equiv_m^P \mathrm{SAT} \equiv_m^P \mathrm{ExSAT}$$

となる.

Relation among satisfiability problems of propositional expressions

2SAT (propositional satisfiability problem)
3SAT
SAT
ExSAT (extended propositional satisfiability problem)

$$2\mathrm{SAT} \le_m^P 3\mathrm{SAT}$$

- at most k…trivial
- exactly k…
  - easy if you can repeat the same literal.
  - the other case … good exercise!

Similarly,

$$3\mathrm{SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT}$$

$$2\mathrm{SAT} \le_m^P 3\mathrm{SAT} \le_m^P \mathrm{SAT} \le_m^P \mathrm{ExSAT} \quad (6.1)$$

Here, if we can show

$$\mathrm{ExSAT} \le_m^P 3\mathrm{SAT}$$

then we have

$$3\mathrm{SAT} \equiv_m^P \mathrm{SAT} \equiv_m^P \mathrm{ExSAT}$$

例6.3: ExSATから3SATへの還元

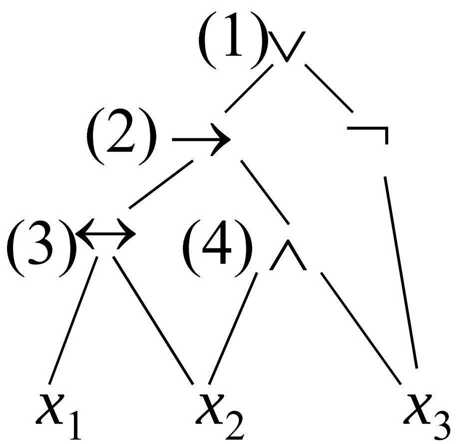
$$E_1(x_1, x_2, x_3) \equiv [[x_1 \leftrightarrow x_2] \to [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1, x_2, x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \to U_4]] \\ \wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

このとき, [E_1が充足可能] $\leftrightarrow$ [F_1が充足可能] (6.2)

F_1は三和積形式に直しやすい形になっている.

F_1の構成方法



$(1) V_1 \equiv V_2 \vee \neg x_3$

$(2) V_2 \equiv [V_3 \to V_4]$

$(3) V_3 \equiv [x_1 \leftrightarrow x_2]$

$(4) V_4 \equiv x_2 \wedge x_3$

F_1を構成するために, $V_i \to U_i$とし, V_iの定義式を $\wedge$で結ぶ

Ex. 6.3: Reduction from ExSAT to 3SAT

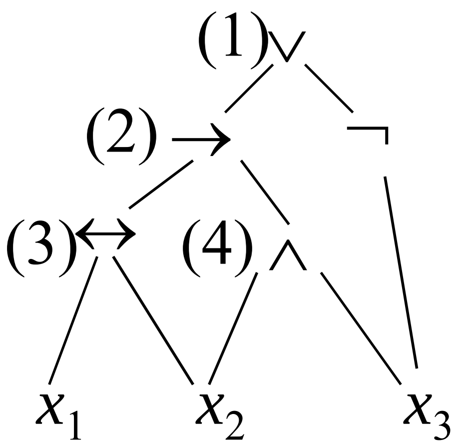
$$E_1(x_1, x_2, x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1, x_2, x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]]$$
$$\wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

Then, $[E_1$ is satisfiable$] \leftrightarrow [F_1$ is satisfiable$]$ (6.2)

F_1 is easier to be converted to 3SAT form.

How to construct $\boldsymbol{F_1}$



$(1) V_1 \equiv V_2 \vee \neg x_3$

$(2) V_2 \equiv [V_3 \rightarrow V_4]$

$(3) V_3 \equiv [x_1 \leftrightarrow x_2]$

$(4) V_4 \equiv x_2 \wedge x_3$

To construct $\boldsymbol{F_1}$ we let $V_i \rightarrow U_i$, and connect expressions of V_i by $\wedge$

F_1 の構成方法より，
(1)各 U_i の値を $V_i(x_1, x_2, x_3)$ としない限り，F_1 は真にはならない．
(2)各 U_i の値を $V_i(x_1, x_2, x_3)$ としたとき，$F_1 = E_1$

上の性質が成り立つことは，帰納法を用いるなどして証明可能．
証明は省略．

三和積形式への変換

$a \rightarrow b = \neg a \vee b$
$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a) = [\neg a \vee b] \wedge [\neg b \vee a]$ であることを用いる．

$$
\begin{aligned}
U_1 \leftrightarrow [U_2 \vee \neg x_3] &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg [U_2 \vee \neg x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2]
\end{aligned}
$$

他も同様．
よって，すべて三和積形式に変形できることがわかる．

From the construction of F_1
(1) F_1 is never true unless each U_i is $V_i(x_1, x_2, x_3)$.
(2) If each U_i is $V_i(x_1, x_2, x_3)$, we have $F_1 = E_1$

The above properties are proved by using induction.

proof is omitted.

Conversion to 3SAT form

$a \to b = \neg a \vee b$

$a \leftrightarrow b = (a \to b) \wedge (b \to a) = [\neg a \vee b] \wedge [\neg b \vee a]$: useful relations

$$
\begin{aligned}
U_1 \leftrightarrow [U_2 \vee \neg x_3] &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg[U_2 \vee \neg x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2]
\end{aligned}
$$

Others are similar.
Thus, every 3SAT form is converted.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

定義6.2: 計算量クラス$\mathcal{C}$に対し，集合Aが次の条件を満たすとき，それを（$\leq_m^P$の下で）$\mathcal{C}$-完全という.

(a) $\forall L \in \mathcal{C}\ [L \leq_m^P A]$

(b) $A \in \mathcal{C}$

補注：条件(a)を満たす集合は$\mathcal{C}$-困難.

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Def.6.2: For a class $\mathcal{C}$, if a set A satisfies the following conditions, then it is called $\mathcal{C}$-complete (under $\leq_m^P$)

(a) $\forall L \in \mathcal{C}\ [L \leq_m^P A]$

(b) $A \in \mathcal{C}$

Note: Sets satisfying the condition (a) are called $\mathcal{C}$-hard.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

例6.5. クラス$\mathcal{NP}$の完全集合の例

3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VCなど

クラス$\mathcal{EXP}$の完全集合

EVAL-IN-E, HALT-IN-Eなど

EVAL-IN-E:

入力: $< a, x, \bar{t} >$

a: 1入力プログラムのコード, $x \in \Sigma^*, \bar{t} \geq 0$

出力: $\textit{eval-in-time}(a, x, \overline{2}^{\bar{t}}) = \textit{accept}$?

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Ex.6.5. Examples of $\mathcal{NP}$-complete sets

3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VC, etc

$\mathcal{EXP}$-complete sets

EVAL-IN-E, HALT-IN-E, etc.

$\mathrm{EVAL\text{-}IN\text{-}E}:$

$\text{Input}: < a, x, \bar{t} >$

a : the code of a program with 1 input, $x \in \Sigma^*, \bar{t} \geq 0$

$\text{Output}: \textit{eval-in-time}(a, x, \overline{2}^{\bar{t}}) = \textit{accept}\,?$

定理6.3. 任意の$\mathcal{C}$-困難集合(含:$\mathcal{C}$-完全集合)Aに対し,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$　対偶は $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$　対偶は $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in \text{co-}\mathcal{NP} \rightarrow \mathcal{C} \subseteq \text{co-}\mathcal{NP}$　対偶は $\mathcal{C} \not\subset \text{co-}\mathcal{NP} \rightarrow A \notin \text{co-}\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$　対偶は $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

証明:

(1) Bを任意の$\mathcal{C}$集合とすると, Aは$\mathcal{C}$-困難だから,

$B \leq_m^P A$　一方, $A \in \mathcal{P}$の仮定より, $B \in \mathcal{P}$ (定理6.1)

(2), (3), (4)も同様

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subset$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

Proof: CP: contraposition

(1) Let B be any $\mathcal{C}$-set. Then, since A is $\mathcal{C}$-hard,

$B \leq_m^P A$ and by the assumption $A \in \mathcal{P}$ we have $B \in \mathcal{P}$ (Th. 6.1)

(2), (3), (4) are similar.

定理6.3. 任意の$\mathcal{C}$-困難集合(含:$\mathcal{C}$-完全集合)Aに対し,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in \text{co-}\mathcal{NP} \rightarrow \mathcal{C} \subseteq \text{co-}\mathcal{NP}$ 対偶は $\mathcal{C} \not\subseteq \text{co-}\mathcal{NP} \rightarrow A \notin \text{co-}\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

例6.6. 定理6.3の意味(クラス$\mathcal{NP}$)

Aを$\mathcal{NP}$-完全集合とする.

定理6.3(1)の対偶より,

$\mathcal{NP} \neq \mathcal{P} \rightarrow A \notin \mathcal{P}$

定理6.3(3)の対偶と定理5.9(1)の対偶より,

$A \notin \text{co-}\mathcal{NP}$

つまり, $\mathcal{NP}$-完全集合は$\mathcal{P} \neq \mathcal{NP}$である限り,
多項式時間では認識できない.

定理5.9.

(1) $\mathcal{NP} \subseteq \text{co-}\mathcal{NP} \rightarrow \mathcal{NP} = \text{co-}\mathcal{NP}$

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subset$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

Theorem 5.9.

(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

Ex.6.6: Meaning of Theorem 6.3 (class $\mathcal{NP}$)

Let A be $\mathcal{NP}$-complete set.

By the contraposition of Theorem 6.3(1) we have

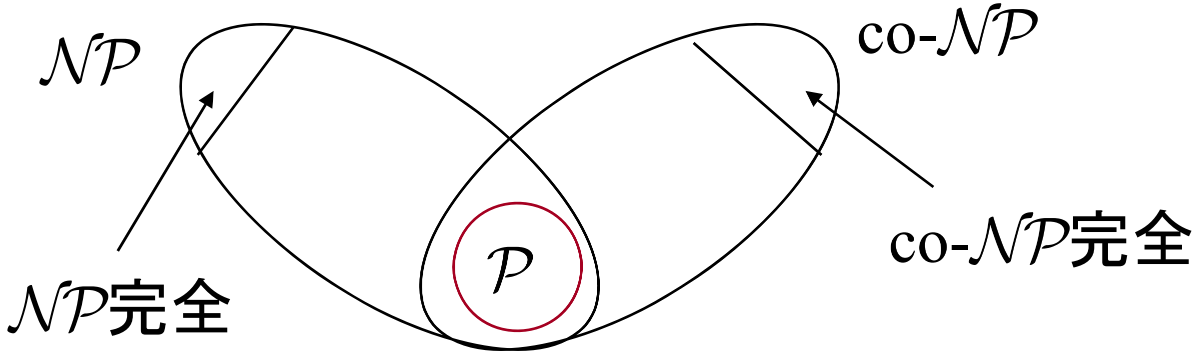
$$\mathcal{NP} \neq \mathcal{P} \rightarrow A \notin \mathcal{P}$$

By the contraposition of Theorem 6.3(3) and that of Theorem 5.9(1),

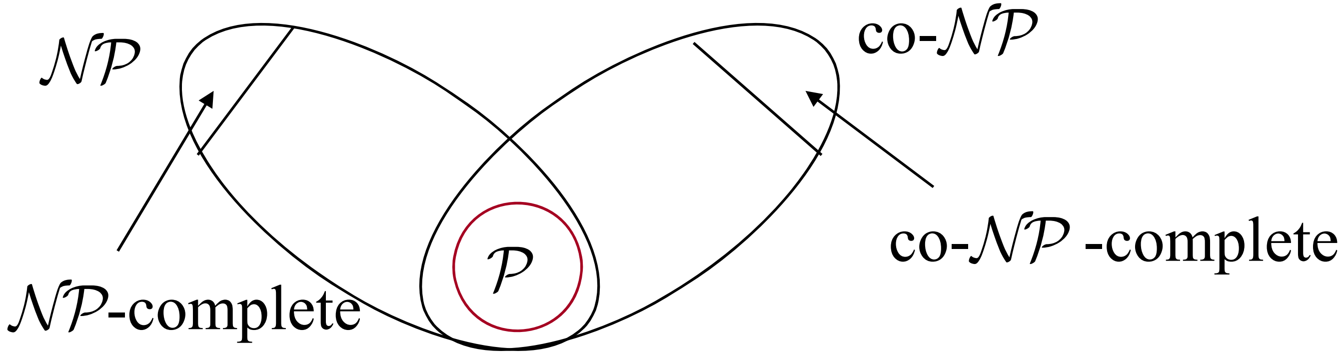
$$A \notin \text{co-}\mathcal{NP}$$

That is, $\mathcal{NP}$-complete sets are $\mathcal{NP}$-sets that cannot be recognized in polynomial time unless $\mathcal{P} = \mathcal{NP}$.

$\mathcal{NP}$-完全集合は$\mathcal{P}\neq \mathcal{NP}$である限り，$\mathcal{NP}\cap$ co-$\mathcal{NP}$には入らない$\mathcal{NP}$集合である.



$\mathcal{NP}$-compete sets are $\mathcal{NP}$-sets that do not belong to $\mathcal{NP} \cap$ co-$\mathcal{NP}$ unless $\mathcal{P} = \mathcal{NP}$.



例6.7. 定理6.3の意味（クラス$\mathcal{EXP}$)

Dを$\mathcal{EXP}$-完全集合とする.

定理6.3(1)の対偶（$\mathcal{C} \not\subset \mathcal{P} \to A \notin \mathcal{P}$，ここでは$\mathcal{EXP} \not\subset \mathcal{P} \to D \notin \mathcal{P}$)

$\mathcal{P} \neq \mathcal{EXP} \to \mathcal{EXP} \not\subset \mathcal{P}$ $(\because \mathcal{P} \subseteq \mathcal{EXP}) \to D \notin \mathcal{P}$

定理6.3(2)の対偶（$\mathcal{C} \not\subset \mathcal{NP} \to A \notin \mathcal{NP}$,

ここでは$\mathcal{EXP} \not\subset \mathcal{NP} \to D \notin \mathcal{NP}$)

$\mathcal{NP} \neq \mathcal{EXP} \to \mathcal{EXP} \not\subset \mathcal{NP}$ $(\because \mathcal{NP} \subseteq \mathcal{EXP}) \to D \notin \mathcal{NP}$

定理6.3(3)の対偶（$\mathcal{C} \not\subset \text{co-}\mathcal{NP} \to A \notin \text{co-}\mathcal{NP}$,

ここでは$\mathcal{EXP} \not\subset \text{co-}\mathcal{NP} \to D \notin \text{co-}\mathcal{NP}$)

$\text{co-}\mathcal{NP} \neq \mathcal{EXP} \to \mathcal{EXP} \not\subset \text{co-}\mathcal{NP} \to D \notin \text{co-}\mathcal{NP}$

ところが定理5.7から $\mathcal{P} \subsetneq \mathcal{EXP}$ であるから, $D \notin \mathcal{P}$.

$\mathcal{EXP}$-完全集合は多項式時間では計算不可能.

Ex. 6.7. Meaning of Theorem 6.3 (class $\mathcal{EXP}$)

Let D be an $\mathcal{EXP}$-complete set.

Contraposition of Theorem 6.3(1)
($\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$, where $\mathcal{EXP} \not\subset \mathcal{P} \rightarrow D \notin \mathcal{P}$)
$\mathcal{P} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \not\subset \mathcal{P}$ $(\because \mathcal{P} \subseteq \mathcal{EXP}) \rightarrow D \notin \mathcal{P}$
Contraposition of Theorem 6.3(2) ($\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$,
Here, $\mathcal{EXP} \not\subset \mathcal{NP} \rightarrow D \notin \mathcal{NP}$)
$\mathcal{NP} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \not\subset \mathcal{NP}$ $(\because \mathcal{NP} \subseteq \mathcal{EXP}) \rightarrow D \notin \mathcal{NP}$
Contraposition of Theorem 6.3(3) ($\mathcal{C} \notin$ co-$\mathcal{NP} \rightarrow A \not\subset$ co-$\mathcal{NP}$,
here, $\mathcal{EXP} \not\subset$ co-$\mathcal{NP} \rightarrow D \notin$ co-$\mathcal{NP}$)
co-$\mathcal{NP} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \not\subset$ co-$\mathcal{NP} \rightarrow D \notin$ co-$\mathcal{NP}$
But, by Theorem 5.7, since we know $\mathcal{P} \subsetneq \mathcal{EXP}$, we have $D \notin \mathcal{P}$.

$\mathcal{EXP}$-complete sets are not computable in polynomial time.

定理6.4. A: 任意の$\mathcal{C}$-完全集合
すべての集合Bに対し,
(1) $A \leq_m^P B$ → Bは$\mathcal{C}$-困難.
(2) $A \leq_m^P B \wedge B \in \mathcal{C}$ → Bは$\mathcal{C}$-完全.

証明:
定義6.2より, $\forall L \in \mathcal{C}[L \leq_m^P A]$
定理6.2より, $L \leq_m^P A \wedge A \leq_m^P B \rightarrow L \leq_m^P B$
したがって, $\forall L \in \mathcal{C}[L \leq_m^P B]$

すなわち, Bは$\mathcal{C}$-困難.

Theorem 6.4. A: any $\mathcal{C}$-complete set

For any set B we have

(1) $A \le_m^P B \rightarrow B$ is $\mathcal{C}$-hard.

(2) $A \le_m^P B \wedge B \in \mathcal{C} \rightarrow B$ is $\mathcal{C}$-complete.

Proof:

By Def. 6.2 $\quad \forall L \in \mathcal{C}[L \le_m^P A]$

By Theorem 6.2, $\quad L \le_m^P A \wedge A \le_m^P B \rightarrow L \le_m^P B$

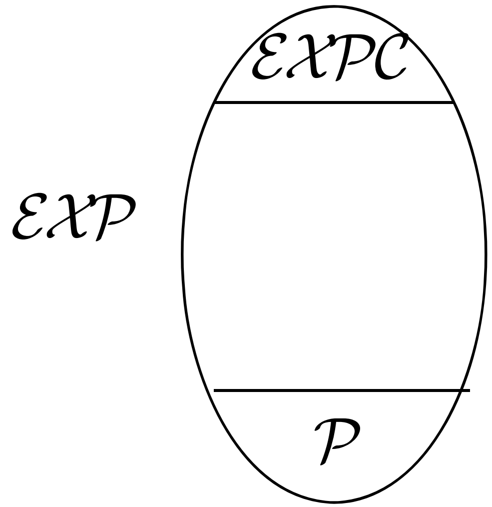
Therefore, $\quad \forall L \in \mathcal{C}[L \le_m^P B]$

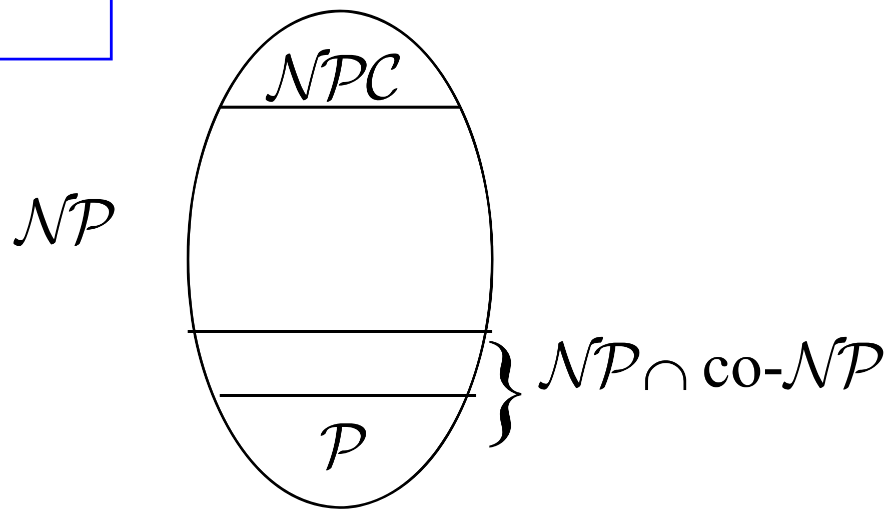
That is, B is $\mathcal{C}$-hard.

$\mathcal{EXPC}$ ≡{L: Lは$\mathcal{EXP}$-完全}
$\mathcal{NPC}$ ≡ {L: Lは$\mathcal{NP}$-完全}
とすると，次の定理が成り立つ．

定理6.5.
(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$

定理6.6: $\mathcal{P} \neq \mathcal{NP}$を仮定すると
(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$





$\mathcal{EXPC} \equiv \{L: L \text{ is } \mathcal{EXP}\text{-complete}\}$

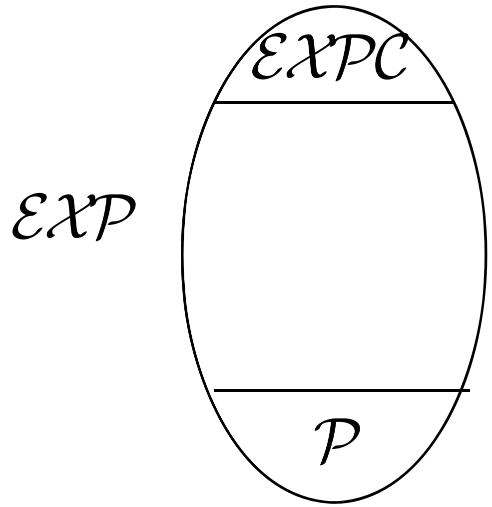
$\mathcal{NPC} \equiv \{L: L \text{ is } \mathcal{NP}\text{-complete}\}$

Then, we have the following theorems.

Theorem 6.5.

(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$

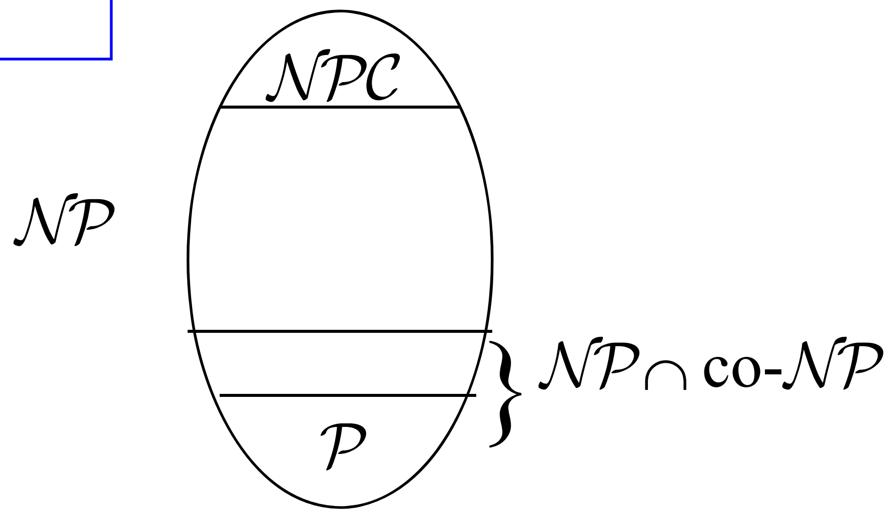
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$



Theorem 6.6: Assuming $\mathcal{P} \neq \mathcal{NP}$

(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$

(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$



残りの予定(Schedule)

- 4/23(Thu) Office Hour:
  - レポート(1)の解答と解説(Comments on report(1))
- 4/27(Mon): 最後の講義(Last class)
- 4/30(Thu): Office Hour:
  - レポート(2)の解答と解説(Comments on report(2))
  - 試験に対する希望調査（持ち込み/範囲）
  - その他
- 5/7(Thu): 中間試験(Mid term exam)
  - 4題40点満点