計算量クラス間の定義を概観すると…

クラス$\mathcal{P}$の定義(5章)
集合Lがクラス$\mathcal{P}$に入る ⇔
以下を満たす多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow R(x)$

クラス$\mathcal{NP}$の定義(定義5.2)
集合Lがクラス$\mathcal{NP}$に入る ⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \exists w \in \Sigma^*: |w| \leq q(|x|)[R(x,w)]$

クラスco-$\mathcal{NP}$の定義(定理5.5)
集合Lがクラスco-$\mathcal{NP}$に入る ⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \forall w \in \Sigma^*: |w| \leq q(|x|)[R(x,w)]$

Observation of the definitions of the classes…

Def: Class $\mathcal{P}$ (Chapter 5)
Set L is in the class $\mathcal{P}$ ⇔
There exists a poly-time computable predicate R such that
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow R(x)$

Def: Class $\mathcal{NP}$ (Def 5.2)
Set L is in the class $\mathcal{NP}$ ⇔
There exists a poly q and a poly-time computable pred. R s.t.
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \exists w \in \Sigma^*: |w| \leq q(|x|)[R(x,w)]$

Def: Class co-$\mathcal{NP}$ (Theorem 5.5)
Set L is in the class co-$\mathcal{NP}$ ⇔
There exists a poly q and a poly-time computable pred. R s.t.
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \forall w \in \Sigma^*: |w| \leq q(|x|)[R(x,w)]$

第6章 多項式時間計算可能性の分析

6.1. 多項式時間還元可能性

定義6.1:
AとBを任意の集合とする.
(1) 関数 h: $A \rightarrow B$: 多項式時間還元(polynomial-time reduction)
⇔
(a) h はΣ^*からΣ^*への全域的関数
(b) $x \in \Sigma^*[x \in A \leftrightarrow h(x) \in B]$
(c) h は多項式時間計算可能.

(2) AからBへの多項式時間還元が存在するとき,
AはBへ多項式時間還元可能という(polynomial time reducible).
このとき, 次のように書く:

$$A \leq_m^P B$$

Chapter 6. Analysis on Polynomial-Time Computability

6.1. Polynomial-time Reducibility

Def.6.1:
Let A and B be arbitrary sets.
(1) function h: $A \rightarrow B$: polynomial-time reduction
⇔
(a) h is a total function from Σ^* onto Σ^*
(b) $x \in \Sigma^*[x \in A \leftrightarrow h(x) \in B]$
(c) h is polynomial-time computable.

(2) When there is a polynomial-time reduction from A to B,
we say A is polynomial-time reducible to B.
Then, we denote by

$$A \leq_m^P B$$

$A \leq_m^P B$ 多項式時間の範囲内では, Aの難しさ ≤ Bの難しさ

定理6.1. $A \leq_m^P B$ のとき,
(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

補注:クラス$\mathcal{E}$は例外. 一般には, $B \in \mathcal{E} \rightarrow A \in \mathcal{E}$とはならない.

例6.2: ONE≡{1}と定義するとき, クラス$\mathcal{P}$のすべての集合Lについて $L \leq_m^P$ ONE
が成り立つ.

$$h(x) \equiv \begin{cases} 1, & x \in L\text{のとき,} \\ 0, & \text{その他のとき} \end{cases}$$

と定義すると, (1) hはΣ^*からΣ^*への全域的関数.
(2) $x \in \Sigma^*[x \in L \leftrightarrow h(x) \in \mathrm{ONE}]$
(3) hは多項式時間計算可能($L \in \mathcal{P} \rightarrow x \in L$の判定も多項式時間内)

$A \leq_m^P B$ within polynomial time, hardness of A ≤ that of B

Theorem 6.1 $A \leq_m^P B$ leads to,
(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

Note: class $\mathcal{E}$ is exceptional. Generally, $B \in \mathcal{E} \rightarrow A \in \mathcal{E}$ is not true.

Ex.6.2: If we define ONE≡{1}, for each set L in $\mathcal{P}$ we have
$L \leq_m^P$ ONE

If we define $h(x) \equiv \begin{cases} 1, & \text{if } x \in L, \\ 0, & \text{otherwise} \end{cases}$

(1) h is a total function from Σ^* onto Σ^*.
(2) $x \in \Sigma^*[x \in L \leftrightarrow h(x) \in \mathrm{ONE}]$
(3) h is polynomial-time computable(so is computation $L \in \mathcal{P} \rightarrow x \in L$)

定理6.2: A, B, C: 任意の集合

(1) $A \le_m^P A$

(2) $A \le_m^P B \wedge B \le_m^P C \rightarrow A \le_m^P C$

定義: $A \equiv_m^P B \leftrightarrow A \le_m^P B \wedge B \le_m^P A$

$\equiv_m^P$ は同値関係

Theorem 6.2: A, B, C: arbitrary sets

(1) $A \le_m^P A$

(2) $A \le_m^P B \wedge B \le_m^P C \rightarrow A \le_m^P C$

Def: $A \equiv_m^P B \leftrightarrow A \le_m^P B \wedge B \le_m^P A$

$\equiv_m^P$ is an equivalence relation.

命題論理式の充足可能性問題の間の関係

2SAT （命題論理式充足性問題：二和積形式）
3SAT （命題論理式充足性問題：三和積形式）
SAT （命題論理式充足性問題）
ExSAT （拡張命題論理式充足性問題）

2SAT $\le_m^P$ 3SAT

- 高々k個…自明
- ちょうどk個…
  - 同じリテラルを使ってよいなら簡単。
  - だめなら…考えてみよう！

同様に,

3SAT $\le_m^P$ SAT $\le_m^P$ ExSAT

2SAT $\le_m^P$ 3SAT $\le_m^P$ SAT $\le_m^P$ ExSAT (6.1)

ここで

ExSAT $\le_m^P$ 3SAT

であることを示せると,

3SAT $\equiv_m^P$ SAT $\equiv_m^P$ ExSAT

となる.

Relation among satisfiability problems of propositional expressions

2SAT (propositional satisfiability problem)
3SAT
SAT
ExSAT (extended propositional satisfiability problem)

2SAT $\le_m^P$ 3SAT

- at most k…trivial
- exactly k…
  - easy if you can repeat the same literal.
  - the other case … good exercise!

Similarly,

3SAT $\le_m^P$ SAT $\le_m^P$ ExSAT

2SAT $\le_m^P$ 3SAT $\le_m^P$ SAT $\le_m^P$ ExSAT (6.1)

Here, if we can show

ExSAT $\le_m^P$ 3SAT

then we have

3SAT $\equiv_m^P$ SAT $\equiv_m^P$ ExSAT

例6.3: ExSATから3SATへの還元

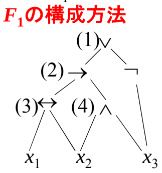
$$E_1(x_1,x_2,x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1,x_2,x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]] \wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

このとき, [E_1が充足可能] $\leftrightarrow$ [F_1が充足可能] (6.2)

F_1は三和積形式に直しやすい形になっている.

F_1の構成方法



(1) $V_1 \equiv V_2 \vee \neg x_3$

(2) $V_2 \equiv [V_3 \rightarrow V_4]$

(3) $V_3 \equiv [x_1 \leftrightarrow x_2]$

(4) $V_4 \equiv x_2 \wedge x_3$

$\boldsymbol{F_1}$を構成するために, $V_i \rightarrow U_i$とし, V_iの定義式を $\wedge$で結ぶ

Ex. 6.3: Reduction from ExSAT to 3SAT

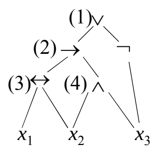
$$E_1(x_1,x_2,x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1,x_2,x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]] \wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

Then, [E_1 is satisfiable] $\leftrightarrow$ [F_1 is satisfiable] (6.2)

F_1 is easier to be converted to 3SAT form.

How to construct F_1



(1) $V_1 \equiv V_2 \vee \neg x_3$

(2) $V_2 \equiv [V_3 \rightarrow V_4]$

(3) $V_3 \equiv [x_1 \leftrightarrow x_2]$

(4) $V_4 \equiv x_2 \wedge x_3$

To construct $\boldsymbol{F_1}$ we let $V_i \rightarrow U_i$, and connect expressions of V_i by $\wedge$

F_1 の構成方法より，
(1)各U_i の値を$V_i(x_1, x_2, x_3)$としない限り，F_1は真にはならない．
(2)各U_iの値を$V_i(x_1, x_2, x_3)$としたとき，$F_1 = E_1$

上の性質が成り立つことは，帰納法を用いるなどして証明可能．
証明は省略．

三和積形式への変換

$a \to b = \neg a \vee b$
$a \leftrightarrow b = (a \to b) \wedge (b \to a) = [\neg a \vee b] \wedge [\neg b \vee a]$であることを用いる．

$$\begin{aligned} U_1 \leftrightarrow [U_2 \vee \neg x_3] &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg[U_2 \vee \neg x_2]] \\ &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]] \\ &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2] \\ &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2] \end{aligned}$$

他も同様．
よって，すべて三和積形式に変形できることがわかる．

From the construction of F_1
(1) F_1 is never true unless each U_i is $V_i(x_1, x_2, x_3)$.
(2) If each U_i is $V_i(x_1, x_2, x_3)$, we have $F_1 = E_1$

The above properties are proved by using induction.
proof is omitted.

Conversion to 3SAT form

$a \to b = \neg a \vee b$
$a \leftrightarrow b = (a \to b) \wedge (b \to a) = [\neg a \vee b] \wedge [\neg b \vee a]$: useful relations

$$\begin{aligned} U_1 \leftrightarrow [U_2 \vee \neg x_3] &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg[U_2 \vee \neg x_2]] \\ &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]] \\ &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2] \\ &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2] \end{aligned}$$

Others are similar.
Thus, every 3SAT form is converted.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

定義6.2: 計算量クラス$\mathcal{C}$に対し，集合Aが次の条件を満たすとき，それを($\le_m^P$の下で)$\mathcal{C}$-完全という．
(a) $\forall L \in \mathcal{C}\ [L \le_m^P A]$
(b) $A \in \mathcal{C}$
補注：条件(a)を満たす集合は$\mathcal{C}$-困難．

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Def.6.2: For a class $\mathcal{C}$, if a set A satisfies the following conditions, then it is called $\mathcal{C}$-complete (under $\le_m^P$)
(a) $\forall L \in \mathcal{C}\ [L \le_m^P A]$
(b) $A \in \mathcal{C}$
Note: Sets satisfying the condition (a) are called $\mathcal{C}$-hard.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

例6.5. クラス$\mathcal{NP}$の完全集合の例
3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VCなど
クラス$\mathcal{EXP}$の完全集合
EVAL-IN-E, HALT-IN-Eなど

EVAL-IN-E:
入力: $< a, x, \bar{t} >$
a: 1入力プログラムのコード, $x \in \Sigma^*, \bar{t} \ge 0$
出力: $eval\text{-}in\text{-}time(a, x, \bar{2}^t) = accept$?

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Ex.6.5. Examples of $\mathcal{NP}$-complete sets
3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VC, etc
$\mathcal{EXP}$-complete sets
EVAL-IN-E, HALT-IN-E, etc.

EVAL-IN-E:
Input: $< a, x, \bar{t} >$
a: the code of a program with 1 input, $x \in \Sigma^*, \bar{t} \ge 0$
Output: $eval\text{-}in\text{-}time(a, x, \bar{2}^t) = accept$?

定理6.3. 任意の$\mathcal{C}$-困難集合(含:$\mathcal{C}$-完全集合)Aに対し,
(1) $A \in \mathcal{P} \to \mathcal{C} \subseteq \mathcal{P}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{P} \to A \notin \mathcal{P}$
(2) $A \in \mathcal{NP} \to \mathcal{C} \subseteq \mathcal{NP}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{NP} \to A \notin \mathcal{NP}$
(3) $A \in$ co-$\mathcal{NP} \to \mathcal{C} \subseteq$ co-$\mathcal{NP}$ 対偶は $\mathcal{C} \not\subseteq$ co-$\mathcal{NP} \to A \notin$ co-$\mathcal{NP}$
(4) $A \in \mathcal{EXP} \to \mathcal{C} \subseteq \mathcal{EXP}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{EXP} \to A \notin \mathcal{EXP}$

証明:
(1) Bを任意の$\mathcal{C}$集合とすると,Aは$\mathcal{C}$-困難だから,
$B \le_m^P A$ 一方,$A \in \mathcal{P}$の仮定より,$B \in \mathcal{P}$ (定理6.1)
(2), (3), (4)も同様

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,
(1) $A \in \mathcal{P} \to \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subseteq \mathcal{P} \to A \notin \mathcal{P}$
(2) $A \in \mathcal{NP} \to \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subseteq \mathcal{NP} \to A \notin \mathcal{NP}$
(3) $A \in$ co-$\mathcal{NP} \to \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subseteq$ co-$\mathcal{NP} \to A \notin$ co-$\mathcal{NP}$
(4) $A \in \mathcal{EXP} \to \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subseteq \mathcal{EXP} \to A \notin \mathcal{EXP}$

Proof: CP: contraposition
(1) Let B be any $\mathcal{C}$-set. Then, since A is $\mathcal{C}$-hard,
$B \le_m^P A$ and by the assumption $A \in \mathcal{P}$ we have $B \in \mathcal{P}$ (Th. 6.1)
(2), (3), (4) are similar.

定理6.3. 任意の$\mathcal{C}$-困難集合(含:$\mathcal{C}$-完全集合)Aに対し,
(1) $A \in \mathcal{P} \to \mathcal{C} \subseteq \mathcal{P}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{P} \to A \notin \mathcal{P}$
(2) $A \in \mathcal{NP} \to \mathcal{C} \subseteq \mathcal{NP}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{NP} \to A \notin \mathcal{NP}$
(3) $A \in$ co-$\mathcal{NP} \to \mathcal{C} \subseteq$ co-$\mathcal{NP}$ 対偶は $\mathcal{C} \not\subseteq$ co-$\mathcal{NP} \to A \notin$ co-$\mathcal{NP}$
(4) $A \in \mathcal{EXP} \to \mathcal{C} \subseteq \mathcal{EXP}$ 対偶は $\mathcal{C} \not\subseteq \mathcal{EXP} \to A \notin \mathcal{EXP}$

定理5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \to \mathcal{NP} =$ co-$\mathcal{NP}$

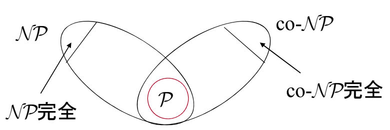
例6.6. 定理6.3の意味(クラス$\mathcal{NP}$)
Aを$\mathcal{NP}$-完全集合とする.
定理6.3(1)の対偶より,
$\mathcal{NP} \neq \mathcal{P} \to A \notin \mathcal{P}$
定理6.3(3)の対偶と定理5.9(1)の対偶より,
$A \notin$ co-$\mathcal{NP}$
つまり,$\mathcal{NP}$-完全集合は$\mathcal{P} \neq \mathcal{NP}$である限り,
多項式時間では認識できない.

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,
(1) $A \in \mathcal{P} \to \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subseteq \mathcal{P} \to A \notin \mathcal{P}$
(2) $A \in \mathcal{NP} \to \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subseteq \mathcal{NP} \to A \notin \mathcal{NP}$
(3) $A \in$ co-$\mathcal{NP} \to \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subseteq$ co-$\mathcal{NP} \to A \notin$ co-$\mathcal{NP}$
(4) $A \in \mathcal{EXP} \to \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subseteq \mathcal{EXP} \to A \notin \mathcal{EXP}$

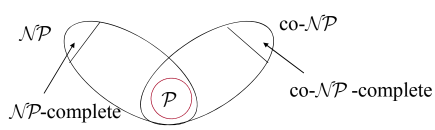
Theorem 5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \to \mathcal{NP} =$ co-$\mathcal{NP}$

Ex.6.6: Meaning of Theorem 6.3 (class $\mathcal{NP}$)
Let A be $\mathcal{NP}$-complete set.
By the contraposition of Theorem 6.3(1) we have
$\mathcal{NP} \neq \mathcal{P} \to A \notin \mathcal{P}$
By the contraposition of Theorem 6.3(3) and that of Theorem 5.9(1),
$A \notin$ co-$\mathcal{NP}$
That is, $\mathcal{NP}$-complete sets are $\mathcal{NP}$-sets that cannot be recognized in polynomial time unless $\mathcal{P} = \mathcal{NP}$.

$\mathcal{NP}$-完全集合は$\mathcal{P} \neq \mathcal{NP}$である限り,$\mathcal{NP} \cap$ co-$\mathcal{NP}$には入らない$\mathcal{NP}$集合である.



$\mathcal{NP}$-compete sets are $\mathcal{NP}$-sets that do not belong to $\mathcal{NP} \cap$ co-$\mathcal{NP}$ unless $\mathcal{P} = \mathcal{NP}$.



定理6.4. A: 任意の$\mathcal{C}$-完全集合
すべての集合Bに対し,
(1) $A \le_m^P B \rightarrow B$は$\mathcal{C}$-困難.
(2) $A \le_m^P B \wedge B \in \mathcal{C} \rightarrow B$は$\mathcal{C}$-完全.

証明:
定義6.2より, $\forall L \in \mathcal{C}[L \le_m^P A]$
定理6.2より, $L \le_m^P A \wedge A \le_m^P B \rightarrow L \le_m^P B$
したがって, $\forall L \in \mathcal{C}[L \le_m^P B]$

すなわち, Bは$\mathcal{C}$-困難.

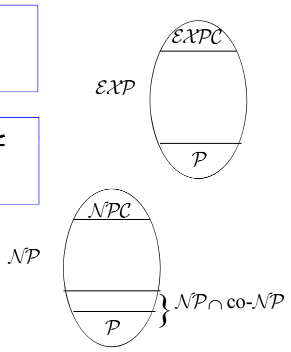
Theorem 6.4. A: any $\mathcal{C}$-complete set
For any set B we have
(1) $A \le_m^P B \rightarrow B$ is $\mathcal{C}$-hard.
(2) $A \le_m^P B \wedge B \in \mathcal{C} \rightarrow B$ is $\mathcal{C}$-complete.

Proof:
By Def. 6.2 $\forall L \in \mathcal{C}[L \le_m^P A]$
By Theorem 6.2, $L \le_m^P A \wedge A \le_m^P B \rightarrow L \le_m^P B$
Therefore, $\forall L \in \mathcal{C}[L \le_m^P B]$
That is, B is $\mathcal{C}$-hard.

$\mathcal{EXPC} \equiv \{L$: Lは$\mathcal{EXP}$-完全$\}$
$\mathcal{NPC} \equiv \{L$: Lは$\mathcal{NP}$-完全$\}$
とすると, 次の定理が成り立つ.

定理6.5.
(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$

定理6.6: $\mathcal{P} \neq \mathcal{NP}$**を仮定すると**
(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$



$\mathcal{EXPC} \equiv \{L$: L is $\mathcal{EXP}$-complete$\}$
$\mathcal{NPC} \equiv \{L$: L is $\mathcal{NP}$-complete$\}$
Then, we have the following theorems.

Theorem 6.5.
(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$

Theorem 6.6: Assuming $\mathcal{P} \neq \mathcal{NP}$
(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$