Chapter 5
Representative Complexity Classes

5.1. Representative time complexity classes

$\mathcal{P} \equiv \bigcup_{p:\text{polynomial}} \mathrm{TIME}(p(l))$

$\mathcal{E} \equiv \bigcup_{c>1} \mathrm{TIME}(2^{cl})$

$\mathcal{EXP} \equiv \bigcup_{p:\text{polynomial}} \mathrm{TIME}(2^{p(l)})$

$\mathcal{C}$ set: set in the complexity class $\mathcal{C}$.
$\mathcal{C}$ problem: problem of recognizing a $\mathcal{C}$ set.

Problems not in $\mathcal{P}$ are intractable from the practical viewpoint…

第5章 代表的な計算量クラス

5.1. 代表的な時間計算量クラス

$\mathcal{P} \equiv \bigcup_{p:\text{多項式}} \mathrm{TIME}(p(l))$

$\mathcal{E} \equiv \bigcup_{c>1} \mathrm{TIME}(2^{cl})$

$\mathcal{EXP} \equiv \bigcup_{p:\text{多項式}} \mathrm{TIME}(2^{p(l)})$

$\mathcal{C}$集合：計算量クラス$\mathcal{C}$に入る集合.
$\mathcal{C}$問題：$\mathcal{C}$集合の認識問題

ある問題が$\mathcal{P}$に入っていないなら、現実的には手に負えない…

Ex.5.1: Polynomial makes no serious difference in the classes $\mathcal{P}$, $\mathcal{E}$, $\mathcal{EXP}$.

$\mathcal{P}$: polynomial × polynomial → polynomial
$\mathcal{E}$: linear power of 2 × polynomial → linear power of 2
$\mathcal{EXP}$: poly. power of 2 × poly. → poly. power of 2

Ex.5.2: Complexity class of PRIME
Ex.4.7 → PRIME $\in \mathrm{TIME}(2^l)$
Thus, PRIME $\in \mathcal{E}$

$O(l^6)$ time algorithm puts it into $\mathcal{P}$!!

Def.5.1: $\mathcal{T}$: set of time limits

$\bigcup_{t\in\mathcal{T}} \mathrm{TIME}(t)$: $\mathcal{T}$ time complexity class
→It is denoted by $\mathrm{TIME}(\mathcal{T})$.

Theorem5.1 (1) $\mathcal{P} = \bigcup_{c>0} \mathrm{TIME}(l^c)$, (2) $\mathcal{EXP} = \bigcup_{c>0} \mathrm{TIME}(2^{l^c})$

例5.1: クラス$\mathcal{P}$, $\mathcal{E}$, $\mathcal{EXP}$では，多項式時間程度の違いは問題ではない.

$\mathcal{P}$: 多項式 × 多項式→多項式
$\mathcal{E}$: 2の線形乗 × 多項式 → 2の線形乗
$\mathcal{EXP}$: 2の多項式乗 × 多項式 → 2の多項式乗

例5.2: PRIMEの計算量クラス
例4.7 → PRIME $\in \mathrm{TIME}(2^l)$
故に，PRIME $\in \mathcal{E}$

余談: 2002年に $O(l^6)$ のアルゴリズムが考案されたので、今では$\mathcal{P}$

定義5.1. $\mathcal{T}$: 制限時間の集合

$\bigcup_{t\in\mathcal{T}} \mathrm{TIME}(t)$: $\mathcal{T}$時間計算量クラス
→これを$\mathrm{TIME}(\mathcal{T})$と表す.

定理5.1: (1) $\mathcal{P} = \bigcup_{c>0} \mathrm{TIME}(l^c)$, (2) $\mathcal{EXP} = \bigcup_{c>0} \mathrm{TIME}(2^{l^c})$

Theorem 5.1: (1) $\mathcal{P} = \bigcup_{c>0} \mathrm{TIME}(l^c)$, (2) $\mathcal{EXP} = \bigcup_{c>0} \mathrm{TIME}(2^{l^c})$

Proof: The proof of (2) is omitted.
T_1: set of polynomials of the form of l^c.
T_2: set of all polynomials
→ since $T_1 \subseteq T_2$, $\mathrm{TIME}(T_1) \subseteq \mathrm{TIME}(T_2)$
p: arbitrary polynomial (p is any element of T_2)
if the maximum degree of a polynomial p is k, $p(l) = O(l^k)$
From Theorem 4.3,
$\mathrm{TIME}(p(l)) \subseteq \mathrm{TIME}(l^k) \subseteq \mathrm{TIME}(T_1)$
Therefore, $\mathrm{TIME}(T_1) = \mathrm{TIME}(T_2)$

Q.E.D.

Theorem 4.3:
For any times t_1, t_2,
$t_1 = O(t_2)$ implies $\mathrm{TIME}(t_1) \subseteq \mathrm{TIME}(t_2)$

定理5.1: (1) $\mathcal{P} = \bigcup_{c>0} \mathrm{TIME}(l^c)$, (2) $\mathcal{EXP} = \bigcup_{c>0} \mathrm{TIME}(2^{l^c})$

証明: (2)の証明は省略.
T_1: l^cという形の多項式の集合.
T_2: 多項式の全体
→ $T_1 \subseteq T_2$なので，$\mathrm{TIME}(T_1) \subseteq \mathrm{TIME}(T_2)$
p: 任意の多項式 (pはT_2の任意の要素)
多項式pの最大次数をkとすると，$p(l) = O(l^k)$
定理4.3より，
$\mathrm{TIME}(p(l)) \subseteq \mathrm{TIME}(l^k) \subseteq \mathrm{TIME}(T_1)$
したがって，$\mathrm{TIME}(T_1) = \mathrm{TIME}(T_2)$

証明終

定理4.3:
すべての制限時間 t_1, t_2 に対し、
$t_1 = O(t_2)$ ならば $\mathrm{TIME}(t_1) \subseteq \mathrm{TIME}(t_2)$

Ex.5.3. Problem of evaluating propositional expression(PROP-EVAL)

Input: $<F, <a_1, a_2, \dots, a_n>>$

F is an extended prop. expression

$(a_1, a_2, \dots, a_n)$ is a truth assignment to F

Question: $F(a_1, a_2, \dots, a_n) = 1$?

(x,y)	$x \to y$ $(\neg x \vee y)$	$x \leftrightarrow y$ $((x\to y)\wedge(y\to x))$
(0,0)	1	1
(0,1)	1	0
(1,0)	0	0
(1,1)	1	1

例5.3. 命題論理式評価問題(PROP-EVAL)

入力: $<F, <a_1, a_2, \dots, a_n>>$

Fは拡張命題論理式 $\wedge \vee \neg \to \leftrightarrow$

$(a_1, a_2, \dots, a_n)$は F に対する真理値割り当て

質問: $F(a_1, a_2, \dots, a_n) = 1$?

(x,y)	$x \to y$ $(\neg x \vee y)$	$x \leftrightarrow y$ $((x\to y)\wedge(y\to x))$
(0,0)	1	1
(0,1)	1	0
(1,0)	0	0
(1,1)	1	1

Ex.5.3. Problem of evaluating propositional expression(PROP-EVAL)

Input: $<F, <a_1, a_2, \dots, a_n>>$

F is an extended prop. expression

$(a_1, a_2, \dots, a_n)$ is a truth assignment to F

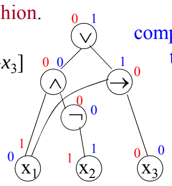
Question: $F(a_1, a_2, \dots, a_n) = 1$?

Construct a computation tree from a code $\ulcorner F \urcorner$ of ext. prop. expression

It is built in time $O(|\ulcorner F \urcorner|^3)$.

If computation tree is available, we can easily obtain the value $F(a_1, a_2, \dots, a_n)$ in a bottom-up fashion.

computation tree



Ex.: $F(x_1, x_2, x_3) = [x_1 \wedge \neg x_2] \vee [x_1 \to x_3]$

$F(0,1,0)=1$

$F(1,1,0)=0$

Hence PROP-EVAL $\in \mathcal{P}$

例5.3. 命題論理式評価問題(PROP-EVAL)

入力: $<F, <a_1, a_2, \dots, a_n>>$

Fは拡張命題論理式 $\wedge \vee \neg \to \leftrightarrow$

$(a_1, a_2, \dots, a_n)$はFに対する真理値割り当て

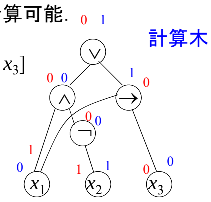
質問: $F(a_1, a_2, \dots, a_n) = 1$?

拡張命題論理式 F がコード化されたもの $\ulcorner F \urcorner$ から計算木を作る.

計算木は$O(|\ulcorner F \urcorner|^3)$時間で構成できる.

計算木が得られていれば, ボトムアップ式で $F(a_1, a_2, \dots, a_n)$ の値は容易に計算可能.

計算木



例: $F(x_1, x_2, x_3) = [x_1 \wedge \neg x_2] \vee [x_1 \to x_3]$

$F(0,1,0)=1$

$F(1,1,0)=0$

よって PROP-EVAL $\in \mathcal{P}$

Ex. 5.3. 2-Satisfiability (2SAT)

Input: $<F>$ F is 2-conjunctive normal form

Question: Is there any assignment such that $F(a_1, a_2, \dots, a_n) = 1$?

Conjunctive Normal Form (CNF)

$F = (\bullet \vee \bullet \vee \dots \vee \bullet) \wedge (\bullet \vee \dots \vee \bullet) \wedge \dots \wedge (\dots)$

- described by $\wedge$ of $\vee$ of literals.

k SAT

- Each closure contains k literals

exactly/at most

- We can define 3SAT, 4SAT similarly.
- SAT consists of any CNF.
- ExSAT consists of any extended propositional expression.

例5.3. 命題論理式充足性問題:2和積形(2SAT)

入力: $<F>$ Fは2和積形命題論理式

質問: $F(a_1, a_2, \dots, a_n) = 1$を満たす割り当てがあるか?

和積形:

$F = (\bullet \vee \bullet \vee \dots \vee \bullet) \wedge (\bullet \vee \dots \vee \bullet) \wedge \dots \wedge (\dots)$

- リテラルの論理和の論理積で表現されたもの

k和積形(k SAT)

- 和積形の各論理和が k 個のリテラルを含む

ちょうど/たかだか

- 3SAT, 4SAT も同様に定義できる。
- SAT: 各論理和のリテラルの個数に制限がないもの
- ExSAT: 入力が拡張命題論理式($\to$や $\leftrightarrow$も許す)

17/18

Ex. 5.4: Graph reachability problem (ST-CON)
Input: <G,s,t> : an undirectd graph G, $1 \leqq s,t \leqq n(=|G|)$
Question: Does G have a path from s to t?

- Cycle is a path that shares two endpoints.
- Euler cycle is a cycle that visits all edges once.
- Hamiltonian cycle is a cycle that visits all vertices once.

Ex. 5.4: Euler cycle problem (DEULER)
Input: <G>: a directed graph G
Question: Does G have an Euler cycle?

Ex. 5.5 Hamiltonian cycle problem (DHAM)
Input: <G>: a directed graph G
Question: Does G have a Hamiltonian cycle?

17/18

例5.4: 到達可能性問題(ST-CON)
入力: <G,s,t> : 無向グラフG, $1 \leqq s,t \leqq n(=|G|)$
質問: G上で s から t への道があるか?

- 閉路とは、始点と終点が同じである路
- オイラー閉路とは、すべての辺を一度づつ通る閉路
- ハミルトン閉路とは、すべての頂点を一度づつ通る閉路

例5.4: 一筆書き閉路問題(DEULER)
入力: <G>: 有向グラフG
質問: Gはオイラー閉路をもつか?

例5.5: ハミルトン閉路問題(DHAM)
入力: <G>: 有向グラフG
質問: Gはハミルトン閉路をもつか?

18/18

It is known that:

- The following problems are in $\mathcal{P}$:
  - ✓ PROP-EVAL, 2SAT, ST-CON, DEULER
- The following problems are in $\mathcal{E}$, but…
  - ✓ 3SAT, DHAM

The class $\mathcal{NP}$ between $\mathcal{P}$ and $\mathcal{E}$?

18/18

以下の事実が知られている:

- 以下の問題は $\mathcal{P}$ に属する:
  - ✓ PROP-EVAL, 2SAT, ST-CON, DEULER
- 以下の問題は $\mathcal{E}$ に属する、が、、、
  - ✓ 3SAT, DHAM

$\mathcal{P}$と$\mathcal{E}$の間(?)のクラス$\mathcal{NP}$

1/12

5.2. Class $\mathcal{NP}$

Def. 5.2: Suppose that we have a polynomial q and polynomial time computable predicate R for a set L such that

$$\text{for each } x \in \Sigma^*, x \in L \leftrightarrow \exists w \in \Sigma^* : |w| \leq q(|x|)[R(x,w)] \tag{5.1}$$

i.e., $L = \{x : \exists w \in \Sigma^* [|w| \leq q(|x|) \wedge R(x,w)]\}$

Then, L is called an $\mathcal{NP}$ set, and the problem of recognizing L is called an $\mathcal{NP}$ problem.
Also, the whole set of $\mathcal{NP}$ sets is called the class $\mathcal{NP}$.

Note: For each $x \in \Sigma^*$, $w_x \in \Sigma^*$ satisfying the predicate $|w| \leq q(|x|) \wedge R(x,w)$ is called (polynomial) *witness* of x.
Hereafter, we use notation $\exists w \in \Sigma^* : |w| \leq q(|x|) \quad \Rightarrow \quad \exists_q w$

"Given a witness of polynomial length in the input size, we can determine in polynomial time whether it satisfies the condition of a given problem."

c.f.: $\mathcal{NP}$=**N**ondeterministic **P**olynomial

1/12

5.2. クラス$\mathcal{NP}$

定義5.2: 集合 L に対して次の条件を満たす多項式 q と多項式時間計算可能述語 R が存在したとする.

$$\text{各} x \in \Sigma^* \text{で} x \in L \leftrightarrow \exists w \in \Sigma^* : |w| \leq q(|x|)[R(x,w)] \tag{5.1}$$

つまり, $L = \{x : \exists w \in \Sigma^* [|w| \leq q(|x|) \wedge R(x,w)]\}$

このとき, Lを$\mathcal{NP}$集合といい, Lの認識問題を$\mathcal{NP}$問題という.
また, $\mathcal{NP}$集合の全体をクラス$\mathcal{NP}$という.

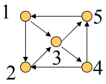
補注: 各 $x \in \Sigma^*$に対して, 論理式 $|w| \leq q(|x|) \wedge R(x,w)$ を満たす $w_x \in \Sigma^*$をxの(多項式長の)証拠という.
以下では, $\exists w \in \Sigma^* : |w| \leq q(|x|) \quad \Rightarrow \quad \exists_q w$ と略記.

「入力サイズの多項式長の証拠が与えられたとき, これが問題の条件を満たすかどうかを多項式時間で判定できる.」

補足: $\mathcal{NP}$=**N**ondeterministic **P**olynomial

Ex.5.7: Hamilton Cycle Problem (DHAM) ∈ $\mathcal{NP}$

Assume graph vertices are numbered 1∼n.
Trace on a Hamilton cycle➔ permutation of 1∼n $< l_1, l_2, \dots, l_n >$
This permutation is a witness of polynomial length.

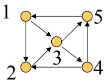
Ex. :


candidates of witness ← (*c.f.*)There are $n!$∼n^n many
<1,2,3,4,5> ➔Hamilton cycle ➔witness
<1,2,3,5,4> ➔not Hamilton cycle
<1,4,3,2,5> ➔not Hamilton cycle

$R_D(x, w)$ ↔[x is a code of a graph G(with n vertices)]
∧[w is a permutation of 1∼n: $< l_1, l_2, \dots, l_n >$]
∧[w represents a Hamilton cycle in G]
For each $x \in \Sigma^*$ we have
if x is a code of a graph G:
$x \in \mathrm{DHAM} \leftrightarrow \exists w_G (=< l_1, \dots, l_n >)[R_D(x, w_G)]$
if x is not a code of any graph: $\forall w[\neg R_D(x, w)]$

例5.7: ハミルトン閉路問題 (DHAM) ∈ $\mathcal{NP}$

グラフの頂点は1∼nと番号づけされていると仮定.
ハミルトン閉路の辿り方➔ 1∼n の順列$< l_1, l_2, \dots, l_n >$
この順列が多項式長の証拠

例:


証拠の候補 ← (注)全部で $n!$∼n^n 通りある
<1,2,3,4,5> ➔ハミルトン閉路 ➔証拠
<1,2,3,5,4> ➔ハミルトン閉路でない
<1,4,3,2,5> ➔ハミルトン閉路でない

$R_D(x, w)$ ↔[xはあるグラフG(n頂点)のコード]
∧[wは1∼nの順列$< l_1, l_2, \dots, l_n >$]
∧[wはGのハミルトン閉路を表している]
すべての $x \in \Sigma^*$について次の関係が成り立つ.
xがあるグラフGのコードになっているとき:
$x \in \mathrm{DHAM} \leftrightarrow \exists w_G (=< l_1, \dots, l_n >)[R_D(x, w_G)]$
xがグラフのコードになっていないとき: $\forall w[\neg R_D(x, w)]$

Ex.5.8: Satisfiability Problem of Prop. Express. (3SAT, SAT, ExSAT)
Goal : ExSAT ∈ $\mathcal{NP}$

$F(x_1, \dots, x_n)$: arbitrary extended prop. logic. expression
F is satisfiable ↔ $\exists a_1, \dots, a_n$: each a_i is 0 or 1 $[F(a_1, \dots, a_n) = 1]$
length of a witness q_E
Truth assignment to F is denoted by $< a_1, \dots, a_n >$.
➔ its length is $3(n+n+1)=6n+3 \leq 6|\lceil F \rceil| + 3$
$q_E(l) = 6l+3$
predicate R_E
$R_E(x, w)$ ↔ [x is a code of an extended prop. express. F (n variables)]
∧ [w is an assignment to F : $< a_1, a_2, \dots, a_n >$]
∧ $[F(a_1, \dots, a_n) = 1]$
Using a computation tree, the value of $F(a_1, \dots, a_n)$ is computed in polynomial time. Thus, R_E is also computable in polynomial time.

例5.8: 命題論理式充足性問題(3SAT, SAT, ExSATなど)
目標 : ExSAT ∈ $\mathcal{NP}$

$F(x_1, \dots, x_n)$: 任意の拡張命題論理式
Fが充足可能 ↔ $\exists a_1, \dots, a_n$: 各a_iは1か0 $[F(a_1, \dots, a_n) = 1]$
証拠の長さq_E
Fへの真偽値の割り当てを$< a_1, \dots, a_n >$で表す.
➔ 長さは $3(n+n+1)=6n+3 \leq 6|\lceil F \rceil| + 3$
$q_E(l) = 6l+3$
述語R_E
$R_E(x, w)$ ↔ [xはある拡張命題論理式F (n変数)のコード]
∧[wはFへの割り当て$< a_1, a_2, \dots, a_n >$]
∧$[F(a_1, \dots, a_n) = 1]$

計算木を用いると$F(a_1, \dots, a_n)$ の値は多項式時間で計算可能.
よって, R_Eも多項式時間で計算可能.

What does it mean by being an $\mathcal{NP}$ set?
Using q and R satisfying the predicate characterizing an $\mathcal{NP}$ set, we can determine "$x \in L$?" in the following way.

```
for each w ∈ Σ^{≤q(|x|)} do
  if R(x, w) then accept end-if
end-for;
reject;
```

If we enumerate and check all possible strings of length at most $q(|x|)$, then we can accept or reject them. Here note that there are 2 to the $q(|x|)$ (exponentially many) such strings.

We may think that those sets recognizable as above are $\mathcal{NP}$ sets.

$\mathcal{NP}$集合であることの意味は何か?
(5.1)を満たすq, Rを用いると, $x \in L$? を次のように判定できる.

```
for each w ∈ Σ^{≤q(|x|)} do
  if R(x, w) then accept end-if
end-for;
reject;
```

長さが$q(|x|)$以下の文字列をすべて列挙して調べれば, acceptかrejectか判定できる. ただ, そのような文字列は 2の$q(|x|)$乗個(指数関数)存在することに注意.

上記の計算方式で認識できる集合を$\mathcal{NP}$集合と考えてよい.

Classes related to $\mathcal{NP}$

Def.5.3.
A set L is called a **co-$\mathcal{NP}$** set if its complement $\overline{L}$ belongs to $\mathcal{NP}$.
The whole family of co-$\mathcal{NP}$ sets is called the class **co-$\mathcal{NP}$**.

Note: It is nonsense to define co-$\mathcal{P}$ since it is equal to $\mathcal{P}$.

Theorem 5.5. For every set L, the following conditions are equivalent.
(a) $L \in$ co-$\mathcal{NP}$
(b) The set L can be represented as
$$L = \{x : \forall w \in \Sigma^* : |w| \le q(|x|)[Q(x,w)]\}$$
by using some polynomial q and polynomial-time computable predicate Q.

$\mathcal{NP}$に関連したクラス

定義5.3. 集合Lは，その補集合$\overline{L}$が$\mathcal{NP}$に属しているとき，**co-$\mathcal{NP}$集合**という．また，co-$\mathcal{NP}$集合の全体を**クラスco-$\mathcal{NP}$**という．

補注: co-$\mathcal{P}$ を定義しても $\mathcal{P}$ と同じなので無意味．

定理5.5. すべての集合 L に対し，次の条件は同値．
(a) $L \in$ co-$\mathcal{NP}$
(b) 集合 L を，適当な多項式 q と多項式時間計算可能述語Qを用いて，
$$L = \{x : \forall w \in \Sigma^* : |w| \le q(|x|)[Q(x,w)]\}$$
と表せる．

Ex.5.9: Primality testing

$$\lceil n \rceil \notin \mathrm{PRIME} \leftrightarrow \exists m : 1 < m < n[\ n \bmod m = 0]$$

Therefore, for $q_p(n) = n$,

$$R_p(x,w) \leftrightarrow [x \notin \mathrm{N}] \vee [[w \in \mathrm{N}] \wedge [1 < m < n] \wedge [n \bmod m = 0]]$$

(where n and m are natural numbers represented by x and w.
N is a set of all natural numbers in the binary form)

This definition leads to
for every $x \in \Sigma^*$ we have $x \notin \mathrm{PRIME} \leftrightarrow \exists q_p w[R_p(x,w)]$
This is a witness to $x \notin$PRIME
Thus, $\overline{\mathrm{PRIME}} \in \mathcal{NP}$, i.e., PRIME $\in$ co-$\mathcal{NP}$
In fact, using $Q(x, w) \leftrightarrow \neg R_p(x, w)$, PRIME can be expressed as
PRIME = $\{x: \forall q_p w\ [Q_p(x, w)]\}$

We can also show that PRIME $\in \mathcal{NP}$, but its proof is more complex.

例5.9: 素数判定問題

$$\lceil n \rceil \notin \mathrm{PRIME} \leftrightarrow \exists m{:}1 < m < n[\ n \bmod m = 0]$$

したがって，$q_p(n) = n$とし，

$$R_p(x,w) \leftrightarrow [x \notin \mathrm{N}] \vee [[w \in \mathrm{N}] \wedge [1 < m < n] \wedge [n \bmod m = 0]]$$

(ただし，n, mは各々x, wが表す自然数，
Nは自然数の2進表記全体）
と定義すると，
すべての$x \in \Sigma^*$に対し，$x \notin \mathrm{PRIME} \leftrightarrow \exists q_p w[R_p(x,w)]$
これは，x $\notin$PRIMEに対する証拠
よって，$\overline{\mathrm{PRIME}} \in \mathcal{NP}$, i.e., PRIME $\in$ co-$\mathcal{NP}$
実際，$Q(x, w) \leftrightarrow \neg R_p(x, w)$ とすると
PRIME = $\{x: \forall q_p w\ [Q_p(x, w)]\}$
と表せる．

PRIME $\in \mathcal{NP}$ も示せるが，その証明はもっと複雑．

Examples of $\mathcal{NP}$ problems

・**Composite Number Testing Problem**(COMPOSITE)
input: natural number n
question: Is n composite? (Is it not prime?)

・**Knapsack Problem**(KNAP)
input: n+1 tuple of natural numbers $< a_1, a_2, \dots, a_n, b>$
question: Is there a set of indices $S \subseteq \{1, \dots, n\}$ s.t. $\sum_{i \in S} a_i = b$?

・**Bin Packing Problem**(BIN)
input: n+2 tuple of natural numbers $< a_1, a_2, \dots, a_n, b, k>$
question: Is there a partition of a set of indices $U = \{1, \dots, n\}$ into $U_1, \dots, U_k$ such that $\sum_{i \in U_j} a_i \le b$ for each j?

・**Vertex Cover Problem**(VC)
input: pair of undirected graph G and natural number k $<G, k>$
question: Is there a vertex cover of k vertices over G?

Vertex Cover S contains at least one of u and v for each edge (u,v).

$\mathcal{NP}$問題の例

・**合成数判定問題**(COMPOSITE)
入力: 自然数n
質問: nは合成数か？(素数でないか？)

・**ナップサック問題**(KNAP)
入力: 自然数の組$< a_1, a_2, \dots, a_n, b>$
質問: $\sum_{i \in S} a_i = b$ となる添字の集合 $S \subseteq \{1, \dots, n\}$があるか？

・**箱詰め問題**(BIN)
入力: 自然数の組$< a_1, a_2, \dots, a_n, b, k>$
質問: 添字の集合$U = \{1, \dots, n\}$を$U_1, \dots, U_k$のk個に分割し，各jで $\sum_{i \in U_j} a_i \le b$ とすることは可能か？

・**頂点被覆問題**(VC)
入力: 無向グラフGと自然数kの組$<G, k>$
質問: Gにk頂点の頂点被覆が存在するか？

頂点被覆S: どの辺(u,v)もu,vの一方はSに含まれる

5.3. Relation in the Complexity Class

Theorem 5.6: $\mathcal{P} \subseteq \mathcal{E} \subseteq \mathcal{EXP}$.

Obvious from the definition.

Theorem 5.7: $\mathcal{P} \subsetneq \mathcal{E} \subsetneq \mathcal{EXP}$.

Hierarchy Thm. (Thm. 4.4):
For any times t_1, t_2 ,
$\forall c > 0, \overset{\infty}{\forall} n[ct_1(n)^2 \le t_2(n)$
$\to \mathrm{TIME}(t_1) \underset{\neq}{\subset} \mathrm{TIME}(t_2)$

Proof:

(1) $\mathcal{P} \subsetneq \mathcal{E}$.

For $t_1(n)=2^n$, $t_2(n)=2^{3n}$, from the hierarchy theorem we have
$\mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}(2^{3n})$
On the other hand, since $\mathcal{P} \subseteq \mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}(2^{3n}) \subseteq \mathcal{E}$
$\mathcal{P} \subsetneq \mathcal{E}$.

(2) is similar.

Q.E.D.

5.3. 計算量クラス間の関係

定理5.6: $\mathcal{P} \subseteq \mathcal{E} \subseteq \mathcal{EXP}$.

定義より，明らか.

定理5.7: $\mathcal{P} \subsetneq \mathcal{E} \subsetneq \mathcal{EXP}$.

階層定理(定理4.4):
任意の制限時間 t_1, t_2 に対し、
$\forall c > 0, \overset{\infty}{\forall} n[ct_1(n)^2 \le t_2(n)$
$\to \mathrm{TIME}(t_1) \underset{\neq}{\subset} \mathrm{TIME}(t_2)$

証明:

(1) $\mathcal{P} \subsetneq \mathcal{E}$.

$t_1(n)=2^n$, $t_2(n)=2^{3n}$とすると，階層定理より，
$\mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}(2^{3n})$
一方，$\mathcal{P} \subseteq \mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}(2^{3n}) \subseteq \mathcal{E}$だから，
$\mathcal{P} \subsetneq \mathcal{E}$.

(2)も同様.

証明終

Theorem 5.8.
(1) $\mathcal{P} \subseteq \mathcal{NP}$, $\mathcal{P} \subseteq$ co-$\mathcal{NP}$ (thus, $\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$)
(2) $\mathcal{NP} \subseteq \mathcal{EXP}$, co-$\mathcal{NP} \subseteq \mathcal{EXP}$ (thus, $\mathcal{NP} \cup$ co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

Proof:

(1) $\mathcal{P} \subseteq \mathcal{NP}$ ($\mathcal{P} \subseteq$ co-$\mathcal{NP}$ is similar)

L: arbitrary $\mathcal{P}$ set

➔ L is recognizable in polynomial time
Thus, we have the following description using a polynomial-time computable predicate P:
$\forall x \in \Sigma^*: [x \in L \leftrightarrow P(x)]$ or $P = \{x: P(x)\}$

We define $R(x, w) = P(x)$ (neglecting the second argument)
→ for any polynomial q,
$L = \{x: \exists_q w[\ R(x,w)]\}$
Thus, from the definition of $\mathcal{NP}$, $L \in \mathcal{NP}$ i.e., $\mathcal{P} \subseteq \mathcal{NP}$.

定理5.8.
(1) $\mathcal{P} \subseteq \mathcal{NP}$, $\mathcal{P} \subseteq$ co-$\mathcal{NP}$ (よって，$\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$)
(2) $\mathcal{NP} \subseteq \mathcal{EXP}$, co-$\mathcal{NP} \subseteq \mathcal{EXP}$ (よって，$\mathcal{NP} \cup$ co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

証明: (1) $\mathcal{P} \subseteq \mathcal{NP}$ ($\mathcal{P} \subseteq$ co-$\mathcal{NP}$ も同様)

L: 任意の$\mathcal{P}$集合

➔ Lは多項式時間で認識可能
よって，多項式時間計算可能述語Pを用いて次のように書ける.
$\forall x \in \Sigma^*: [x \in L \leftrightarrow P(x)]$ or $P = \{x: P(x)\}$
$R(x, w) = P(x)$と定義 （第2引数は無視）
→ 任意の多項式qについて，
$L = \{x: \exists_q w[\ R(x,w)]\}$
よって，$\mathcal{NP}$の定義より，$L \in \mathcal{NP}$ i.e., $\mathcal{P} \subseteq \mathcal{NP}$.

(2) $\mathcal{NP} \subseteq \mathcal{EXP}$ (co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

L: any $\mathcal{NP}$ set

➔ There is some polynomial q and polynomial-time computable predicate R such that
$L = \{x : \exists_q w[R(x, w)]\} = \{x : \exists_q w[|w| \le q(|x|) \wedge R(x, w)]\}$

program recognizing L using q and R

```
prog L(input x);
begin
    for each w ∈ Σ^{≤q(|x|)} do
        if R(x, w) then accept end-if
    end-for;
    reject
end.
```

time complexity of the program for an input of length l:

Since R is polynomial-time computable, for some polynomial q
time of $R=p(|x| + |w|) \le p(l + q(l)) \longleftarrow$ polynomial of l

In total, $\{p(l+q(l)) + cq(l)\}2^{q(l)} + d = O(2^{\ l+q(l)})$

Hence, $L \in \mathcal{EXP}$ → $\mathcal{NP} \subseteq \mathcal{EXP}$ Q.E.D.

(2) $\mathcal{NP} \subseteq \mathcal{EXP}$ (co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

L: 任意の$\mathcal{NP}$集合

➔多項式qと多項式時間計算可能述語Rが存在して，
$L = \{x : \exists_q w[R(x, w)]\} = \{x : \exists_q w[|w| \le q(|x|) \wedge R(x, w)]\}$

qとRを用いて，Lを認識するプログラムを作る.

```
prog L(input x);
begin
    for each w ∈ Σ^{≤q(|x|)} do
        if R(x, w) then accept end-if
    end-for;
    reject
end.
```

長さlの入力に対するプログラムの時間計算量:

Rは多項式時間計算可能だったから，ある多項式pに対し，
Rの計算時間$=p(|x| + |w|) \le p(l + q(l)) \longleftarrow l$ の多項式

全体では，$\{p(l+q(l)) + cq(l)\}2^{q(l)} + d = O(2^{\ l+q(l)})$

よって，$L \in \mathcal{EXP}$ → $\mathcal{NP} \subseteq \mathcal{EXP}$ 証明終

Theorem 5.9
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$
(2) co-$\mathcal{NP} \subseteq \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$

Note: from (3) the proof for $\mathcal{NP} \neq$ co-$\mathcal{NP}$ is harder than that for $\mathcal{P} \neq \mathcal{NP}$.

Proof: (1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$ (proof of (2) is similar)
Since co-$\mathcal{NP} \subseteq \mathcal{NP}$ is shown if we prove $L \in \mathcal{NP}$ for any $L \in$ co-$\mathcal{NP}$
Combining it with the assumption $\mathcal{NP} \subseteq$ co-$\mathcal{NP}$, we have $\mathcal{NP} =$ co-$\mathcal{NP}$ and so

$L \in$ co-$\mathcal{NP} \rightarrow \overline{L} \in \mathcal{NP}$ (by Definition 5.3)
$\rightarrow \overline{L} \in$ co-$\mathcal{NP}$ ($\mathcal{NP} \subseteq$ co-$\mathcal{NP}$)
$\rightarrow L \in \mathcal{NP}$ (Definition 5.3 and $\overline{\overline{L}}=L$)

定理5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$
(2) co-$\mathcal{NP} \subseteq \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$.

補注: (3)より, $\mathcal{NP} \neq$co-$\mathcal{NP}$の証明は, $\mathcal{P} \neq \mathcal{NP}$の証明より難しい.

証明: (1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$ ((2)の証明も同様)
任意の$L \in$co-$\mathcal{NP}$に対して$L \in \mathcal{NP}$が示せれば, co-$\mathcal{NP} \subseteq \mathcal{NP}$が証明できるので, 仮定の$\mathcal{NP} \subseteq$ co-$\mathcal{NP}$と合わせて$\mathcal{NP} =$ co-$\mathcal{NP}$が言える.

$L \in$ co-$\mathcal{NP} \rightarrow \overline{L} \in \mathcal{NP}$ (定義5.3より)
$\rightarrow \overline{L} \in$ co-$\mathcal{NP}$ ($\mathcal{NP} \subseteq$ co-$\mathcal{NP}$より)
$\rightarrow L \in \mathcal{NP}$ (定義5.3と$\overline{\overline{L}}=L$より)

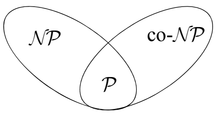
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$

Contraposition: $\mathcal{P} = \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

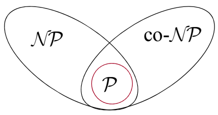
If we assume $\mathcal{P}=\mathcal{NP}$, for any L we have

$L \in \mathcal{NP} \leftrightarrow L \in \mathcal{P}$ ($\mathcal{P} = \mathcal{NP}$)
$\leftrightarrow \overline{L} \in \mathcal{P}$ (Exercise 5.5)
$\leftrightarrow \overline{L} \in \mathcal{NP}$ ($\mathcal{P} = \mathcal{NP}$)
$\leftrightarrow L\ (= \overline{\overline{L}}) \in$ co-$\mathcal{NP}$ (Definition 5.3)
$\therefore \mathcal{NP} =$ co-$\mathcal{NP}$ Q.E.D.

If $\mathcal{NP} \neq$co-$\mathcal{NP}$ is true,



or



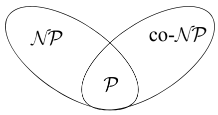
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$.

対偶: $\mathcal{P} = \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

$\mathcal{P}=\mathcal{NP}$と仮定すると, すべてのLに対し

$L \in \mathcal{NP} \leftrightarrow L \in \mathcal{P}$ ($\mathcal{P} = \mathcal{NP}$より)
$\leftrightarrow \overline{L} \in \mathcal{P}$ (演習問題5.5)
$\leftrightarrow \overline{L} \in \mathcal{NP}$ ($\mathcal{P} = \mathcal{NP}$より)
$\leftrightarrow L\ (= \overline{\overline{L}}) \in$ co-$\mathcal{NP}$ (定義5.3より)
$\therefore \mathcal{NP} =$ co-$\mathcal{NP}$ 証明終

$\mathcal{NP} \neq$ co-$\mathcal{NP}$が正しいと



or