アルゴリズム論
Theory of Algorithms

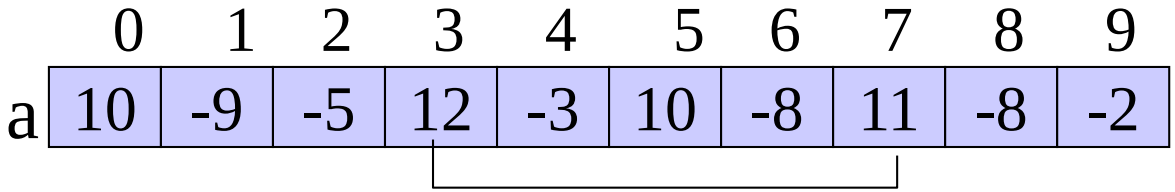
第2回講義
アルゴリズムの設計と解析の基礎(2)

アルゴリズム論
Theory of Algorithms

Lecture #2

Foundation of Design and Analysis of Algorithms(2)

問題P4: n個のデータが配列a[]に蓄えられているとする．区間[p,q]における平均値をave[p,q]とするとき，この平均値を最大にする区間を求めよ.



ave[3,7] = 22/5=4.4　　ave[3,5]=19/3=6.33

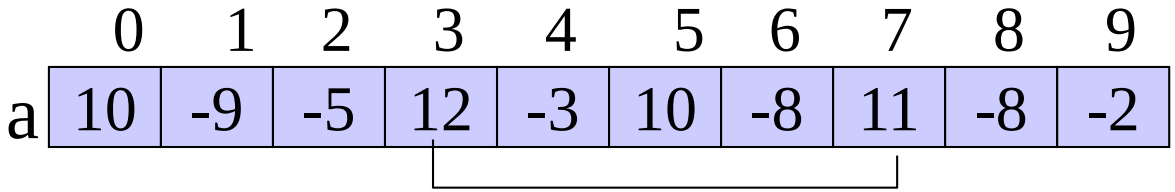
実は，区間に制限がなければ，この問題は実に簡単.
配列内の最大値をa[i]とすると，平均値を最大にする区間は[i,i]ということになる.

質問：上記の事実を証明せよ.

では，区間の長さを2以上に限定した場合はどうだろう？

今度は単純な方法では解は求まらない.

Problem P4: Suppose n data are stored in an array a[]. By ave[p,q] we denote the average of elements in an interval [p,q]. Then, find an interval maximizing the average.



ave[3,7] = 22/5=4.4 ave[3,5]=19/3=6.33

If there is no constraint in intervals, this problem is in fact quite easy. If a[i] is the maximum in the array, then the interval maximizing the average is obviously [i,i].

Exercise: Question: Prove the above fact.

Then, what about if the length of an interval must be at least two?

In this case there is no simple algorithm.

（腕力法）

すべての区間について平均値を求め，その最大値を求める

アルゴリズムP4−A0:

```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++)
  for(q=p+1; q<n; q++){
    sum = 0;
    for(i=p; i<=q; i++)
      sum = sum + a[i];
    ave = sum/(q-p+1);
    if(ave > maxave) maxave = ave;
  }
return maxave;
```

プログラムの構造が3重ループであるから，$O(n^3)$時間かかる.

（Brute-force algorithm）

computes average for every interval to find the largest value.

Algorithm P4-A0:

```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++)
  for(q=p+1; q<n; q++){
    sum = 0;
    for(i=p; i<=q; i++)
      sum = sum + a[i];
    ave = sum/(q-p+1);
    if(ave > maxave) maxave = ave;
  }
return maxave;
```

Triple-loop structure => $O(n^3)$ time.

（改良１）

区間和を求める計算と組み合わせると無駄が省ける

アルゴリズムP3-A1:

```
maxsum=a[0];
for(p=0; p<n-1; p++){
    sum=0;
    for(q=p; q<n; q++){
       sum = sum + a[q];
       if( sum > maxsum) maxsum = sum;
    }
return maxsum;
```

上記のアルゴリズムを区間平均用に変形すればよい.
ただし，線形時間のアルゴリズムを応用することはできない.
（何が違うか？）

(Improvement 1）

Application of algorithm for largest sum interval leads efficiency.

Algorithm P3-A1:

```
maxsum=a[0];
for(p=0; p<n-1; p++){
   sum=0;
   for(q=p; q<n; q++){
      sum = sum + a[q];
      if( sum > maxsum) maxsum = sum;
   }
return maxsum;
```

It suffices to convert the above algorithm for average in intervals.
Note that we cannot apply the linear-time algorithm.
(What is different?)

アルゴリズムP4-A1:

```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++){
    sum=a[p];
    for(q=p+1; q<n; q++){
       sum = sum + a[q];
       ave = sum/(q-p+1);
       if( ave > maxave) maxave = ave;
    }
}
return maxave;
```

区間平均の場合には，区間の長さが2以上でなければならないことに注意．

計算時間

今度は2重ループの構造なので，$O(n^2)$時間．

区間和問題のように線形時間アルゴリズムは存在するか？

Algorithm P4-A1:

```
maxave=(a[0]+a[1])/2;
for(p=0; p<n-1; p++){
    sum=a[p];
    for(q=p+1; q<n; q++){
       sum = sum + a[q];
       ave = sum/(q-p+1);
       if( ave > maxave) maxave = ave;
    }
}
return maxave;
```

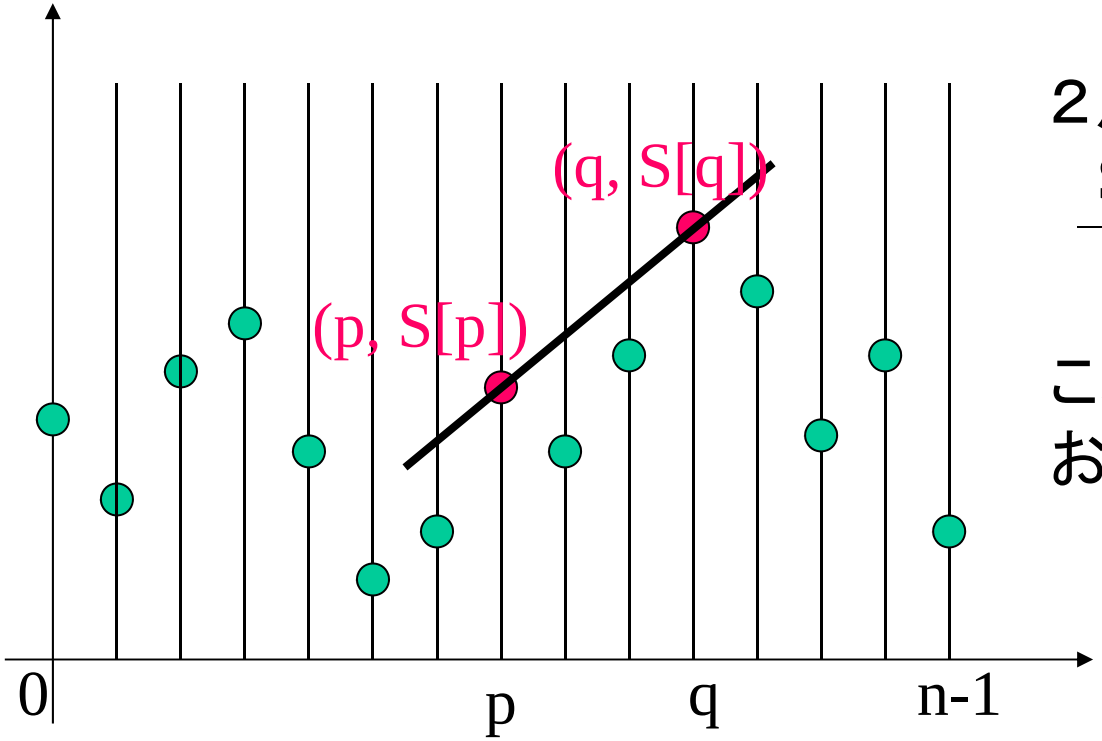
For interval average, note that the length of interval must be at least 2.

Computation time

double loop structure => $O(n^2)$ time.

Is there a linear algorithm like in the largest sum interval problem?

区間和を最大にする問題P3と同様に，配列の0番目からの和
S[p] = a[0]+a[1]+...+a[p]をすべてのpについて求めて，(p, S[p])で
定まる点を平面上にプロットしてみよう.



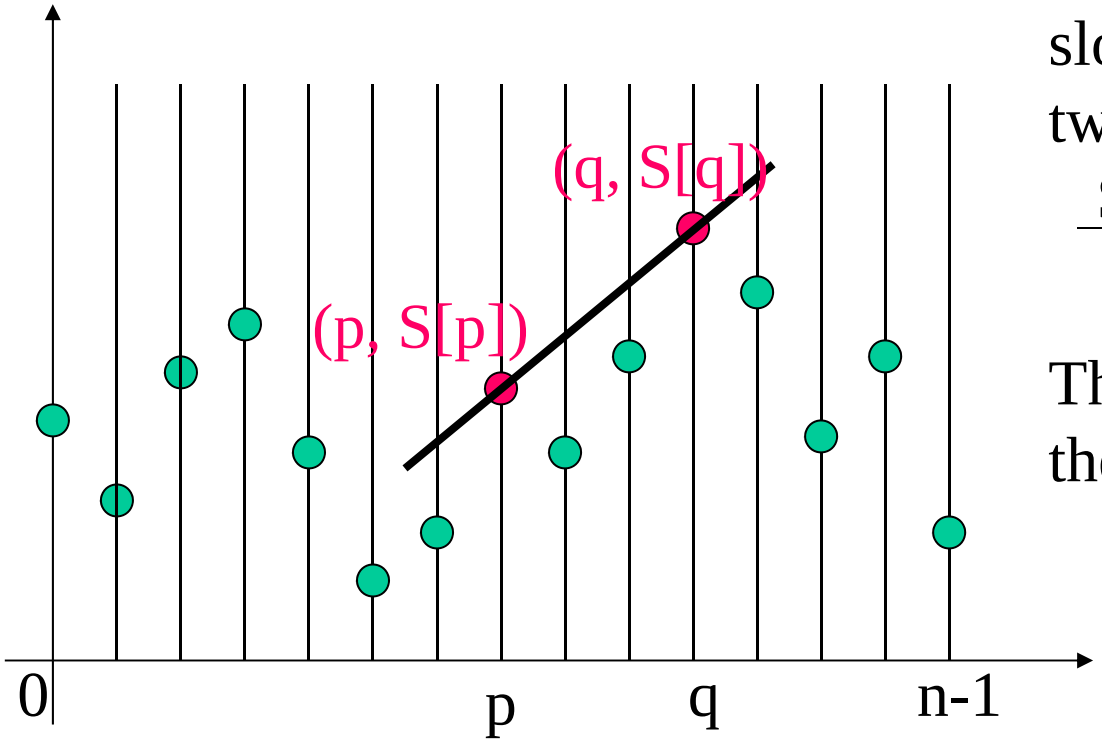
２点を通る直線の傾き

$$\frac{S[q]-S[p]}{q-p}$$

これは区間[p+1,q]に
おける平均値に等しい

したがって，上記のn点の内，どの２点を通る直線を引けば傾きが
最大になるかを求める問題となる．ただし，区間の長さを2以上に
制限しているので，横座標は2以上離れていなければならない.

As in the largest sum interval problem P3, compute the sum S[p] = a[0]+a[1]+...+a[p] for each p and plot it at (p, S[p]) in the plane.



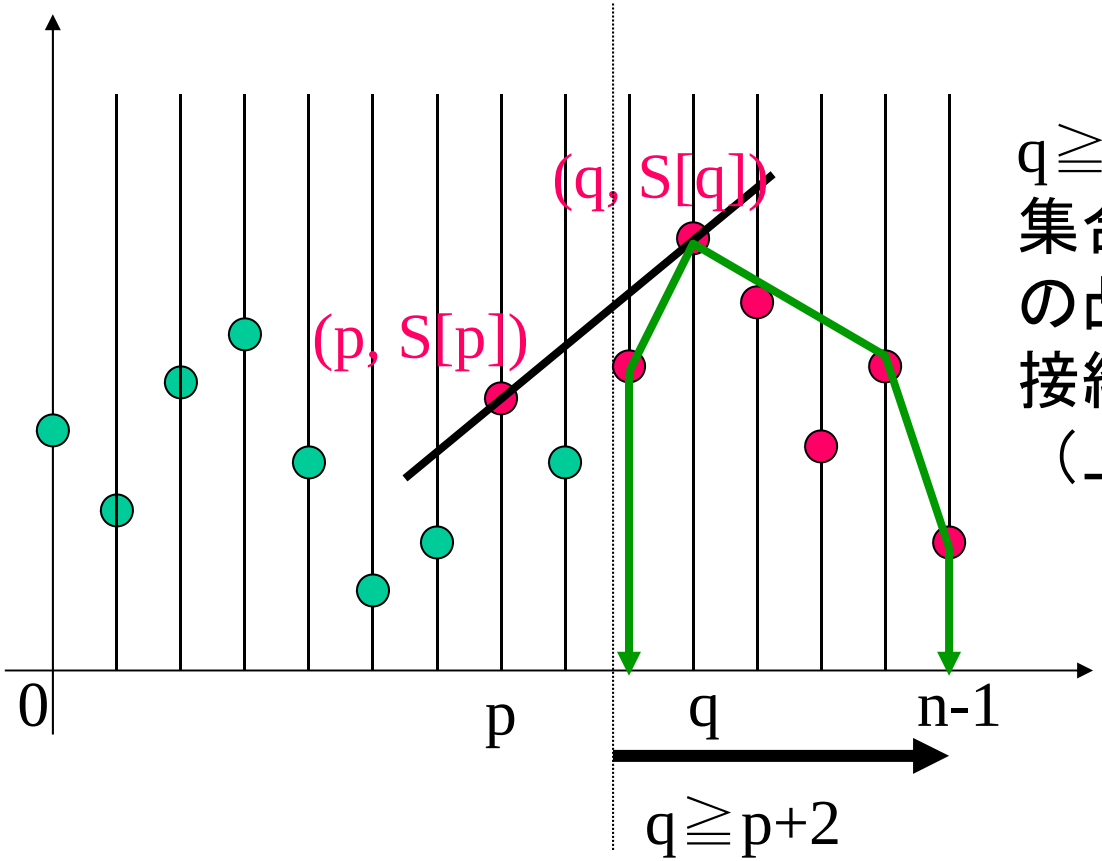
slope of line through the two points

$$\frac{S[q]-S[p]}{q-p}$$

This is the average in the interval [p+1,q].

Thus, the problem is to find two points among n points so that their associated line has a largest slope. Here, since the length of each interval must be at least 2, their x-coordinates must differ at least by 2.

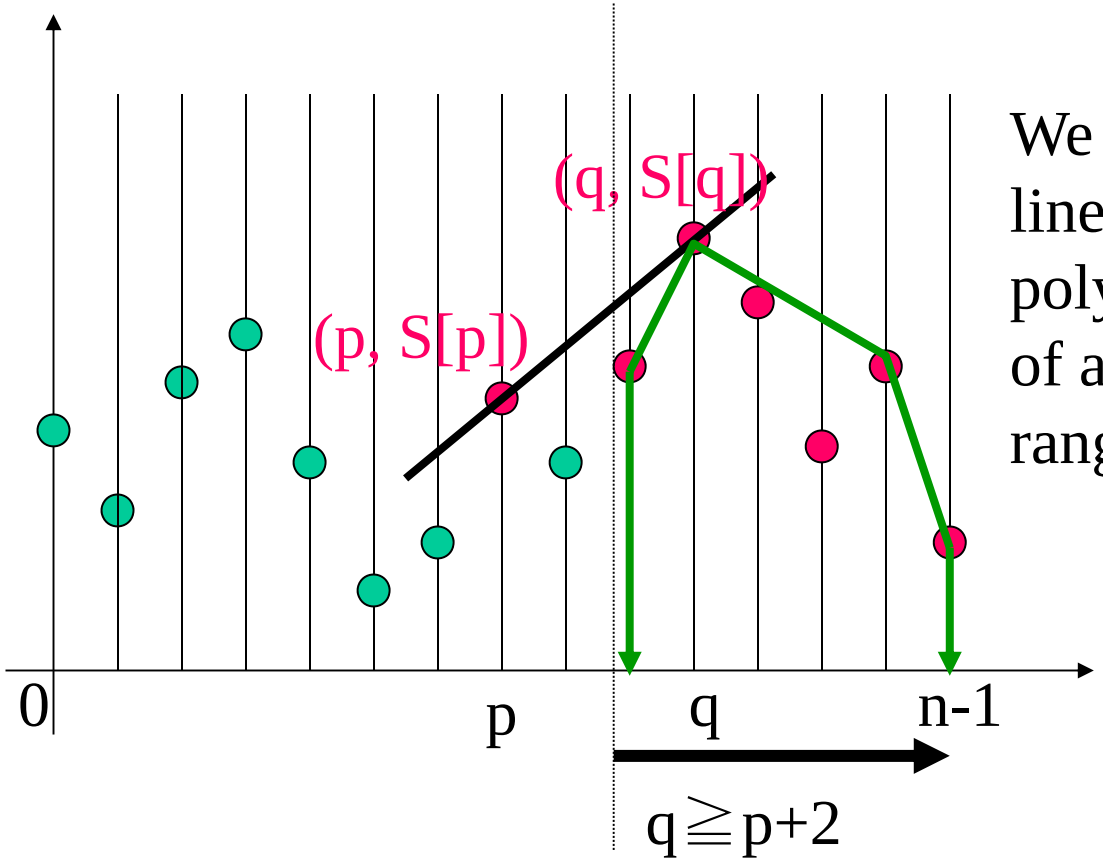
区間の左端pを固定したとき，
どの点(q, S[q])を選べば傾きが最大になるか？



q≧p+2の範囲の点
集合を包含する最小
の凸多角形への
接線を求めればよい.
（上部凸包）

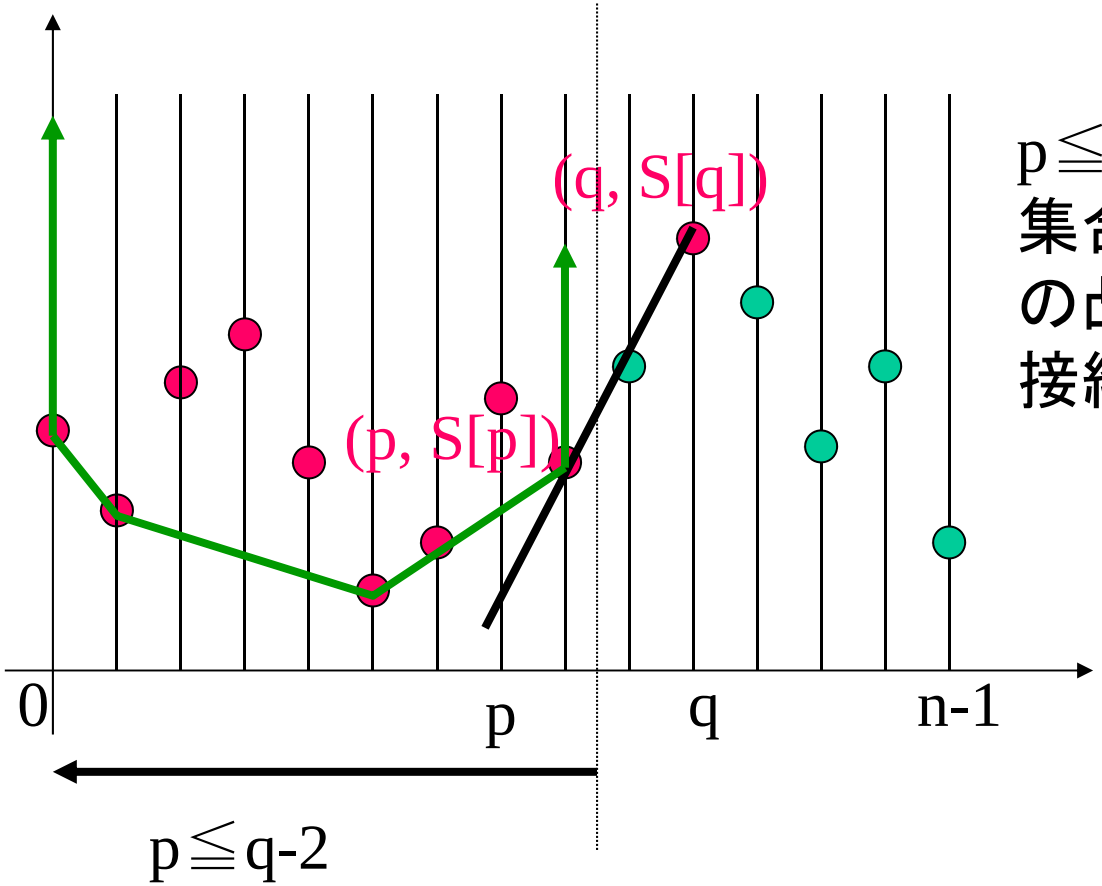
Fixing the left endpoint p,

which point (q, S[q]) should be selected to maximize the slope?



We should find a tangent line to the smallest convex polygon (upper convex hull) of a set of point in the range $q \geqq p+2$.

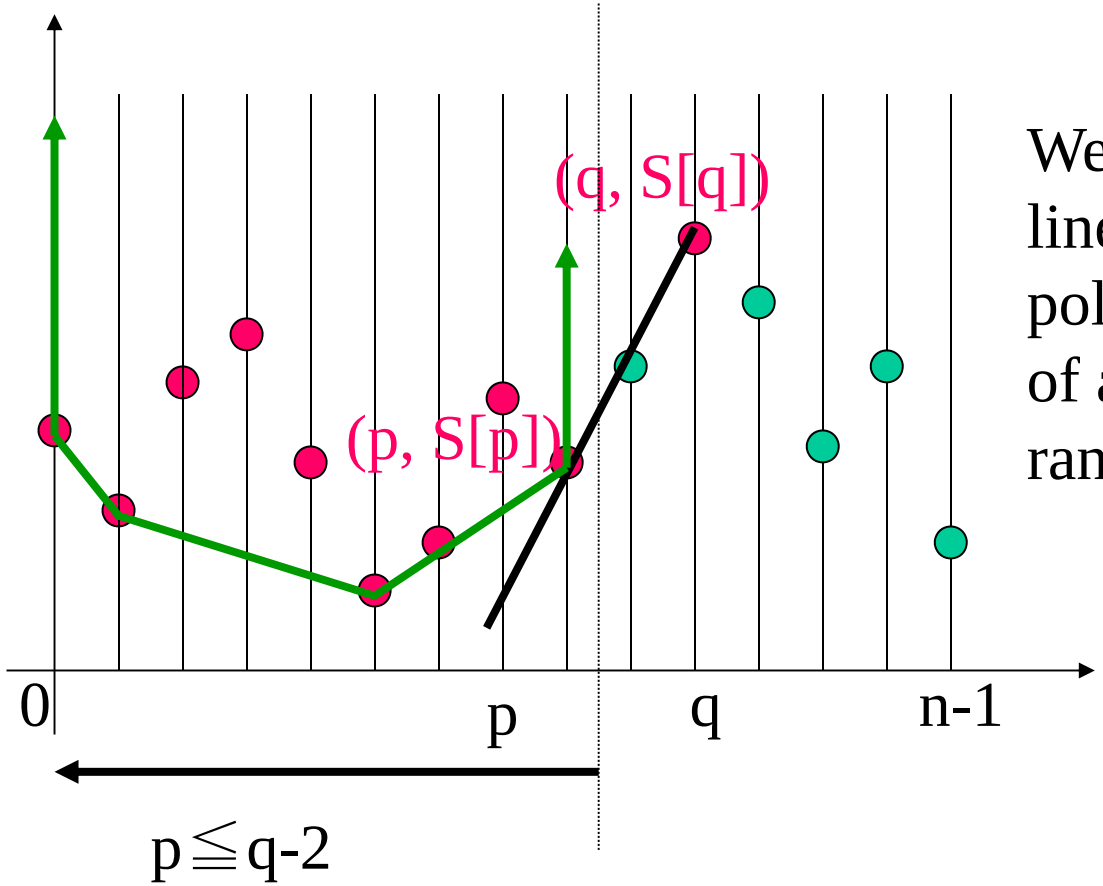
区間の右端qを固定したとき，
　　どの点(p, S[p])を選べば傾きが最大になるか？



p≦q-2の範囲の点
集合を包含する最小
の凸多角形への
接線を求めればよい.
（下部凸包）

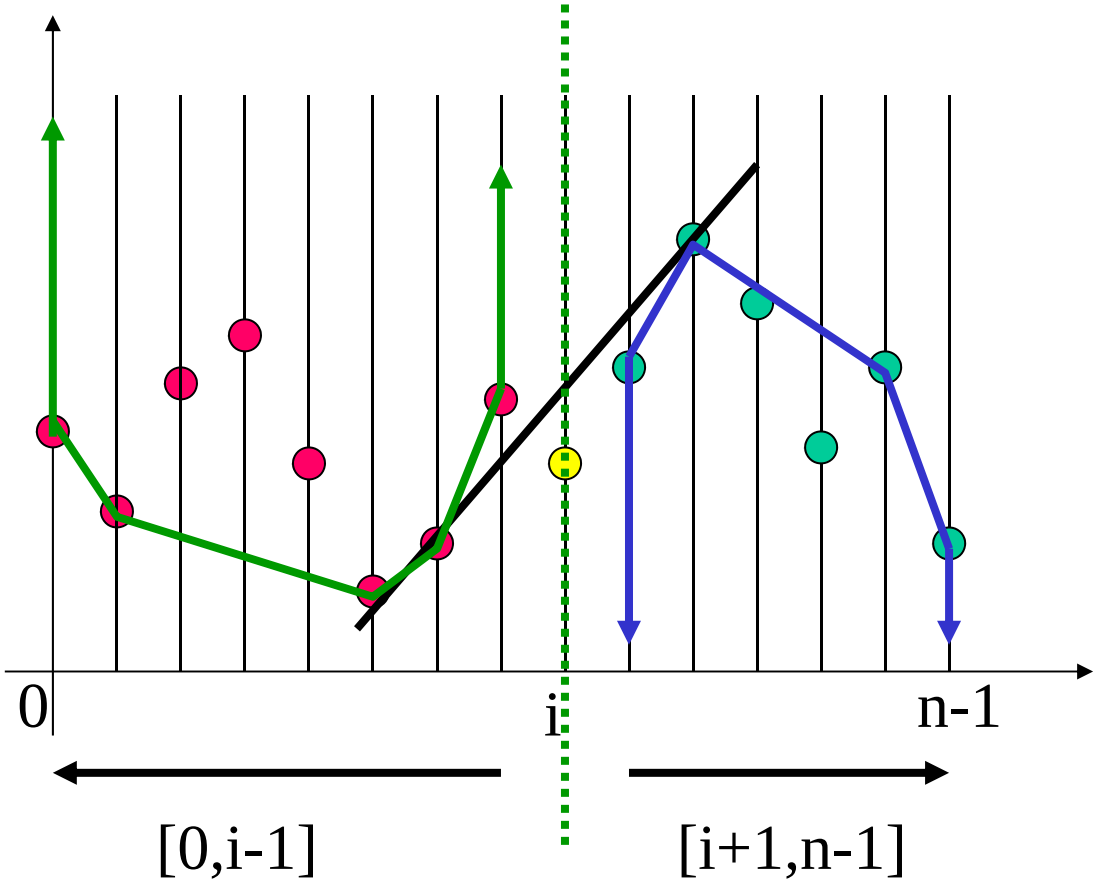
Fixing the right endpoint p,

which point (p, S[p]) should be selected to maximize the slope?

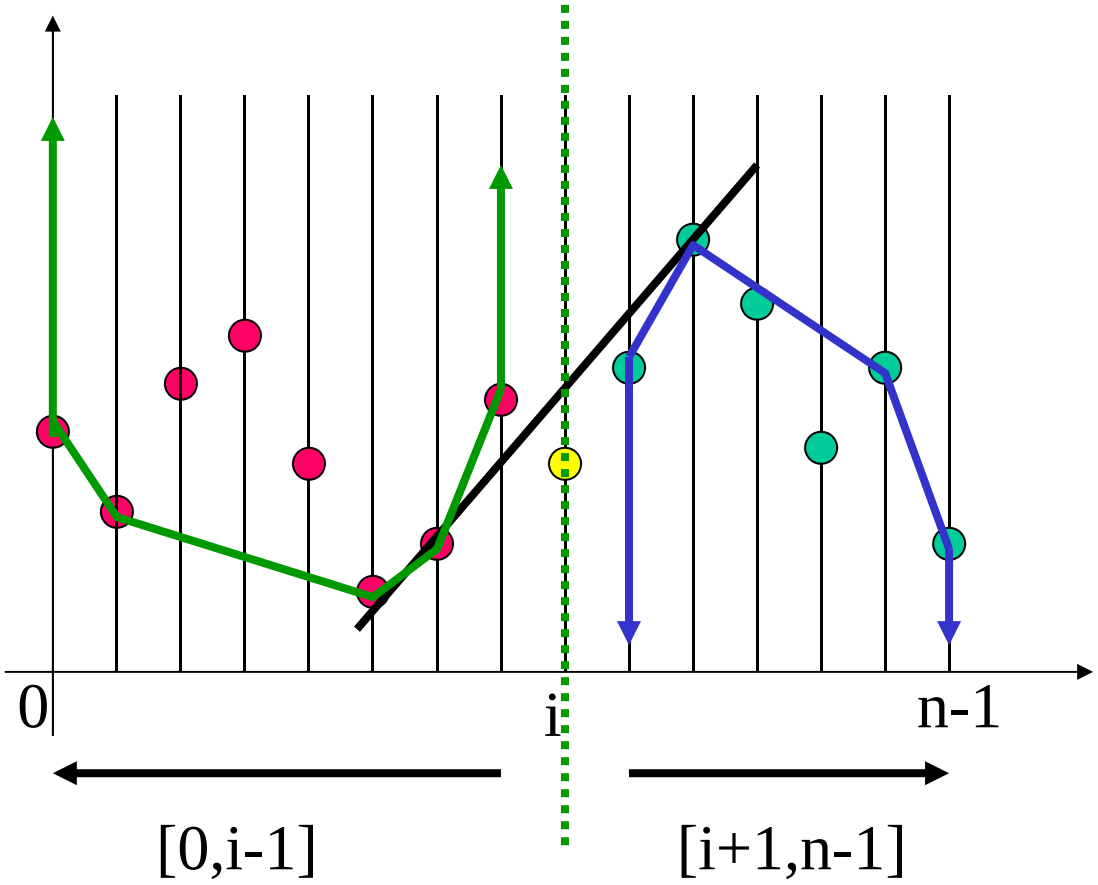


We should find a tangent line to the smallest convex polygon (lower convex hull) of a set of point in the range p≦q-2.

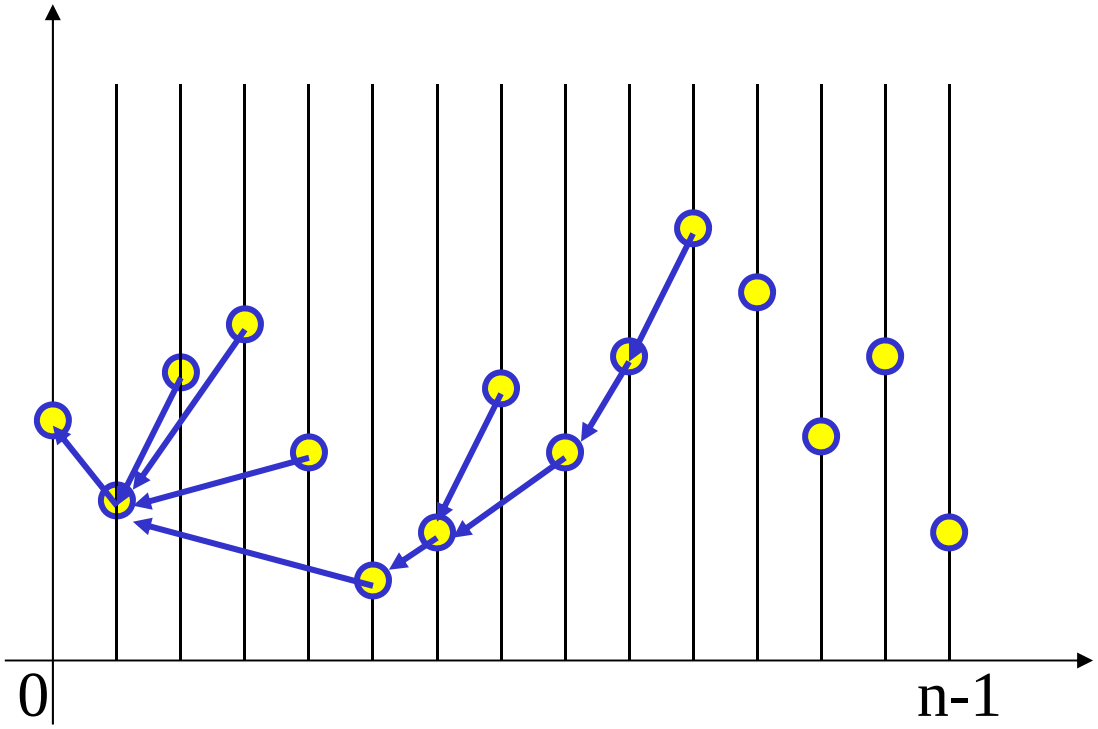
結局，各iについて，区間[0,i-1]の点の下部凸包LCH(0,i-1)と区間[i+1,n-1]の点の上部凸包UCH(i+1,n-1)を求めておき，両者の共通接線を求めればよい.



After all we should compute for each i the lower convex hull LCH(0,i-1) for an interval [0,i-1] and the upper convex hull UCH(i+1, n-1) for an interval [i+1, n-1] and then find their common tangent.

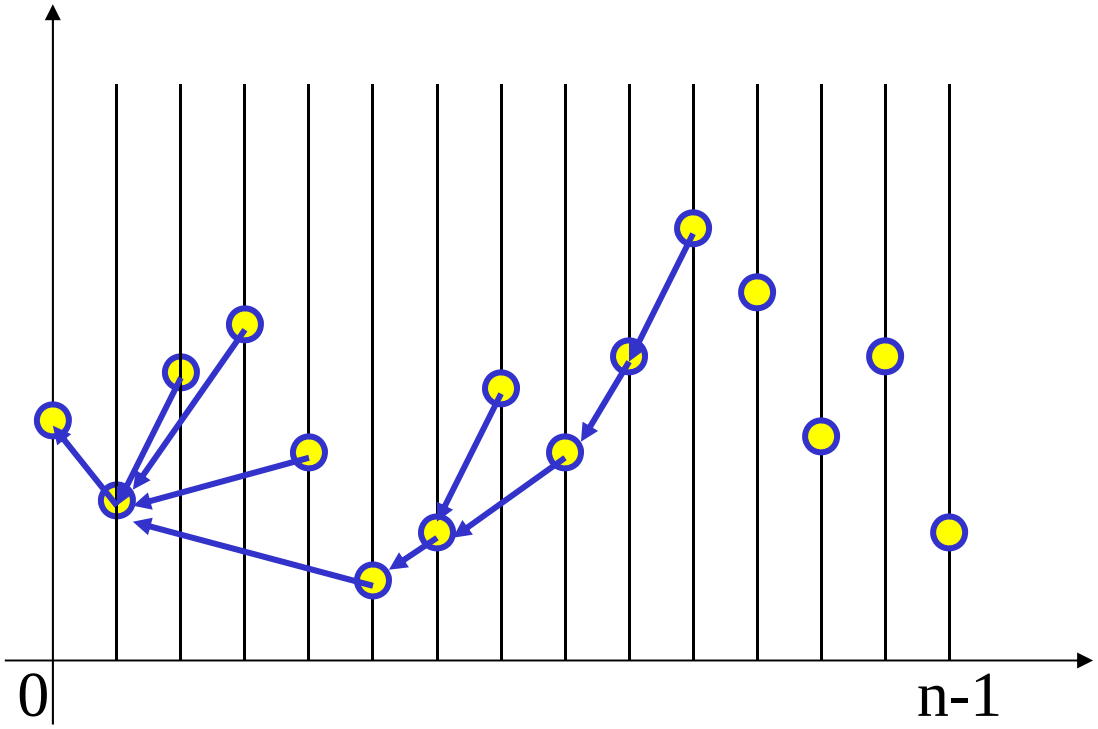


区間[0,i-1]の点の下部凸包LCH(0,i-1)の計算
iを順に増やしながらLCH(0,i)を順に更新

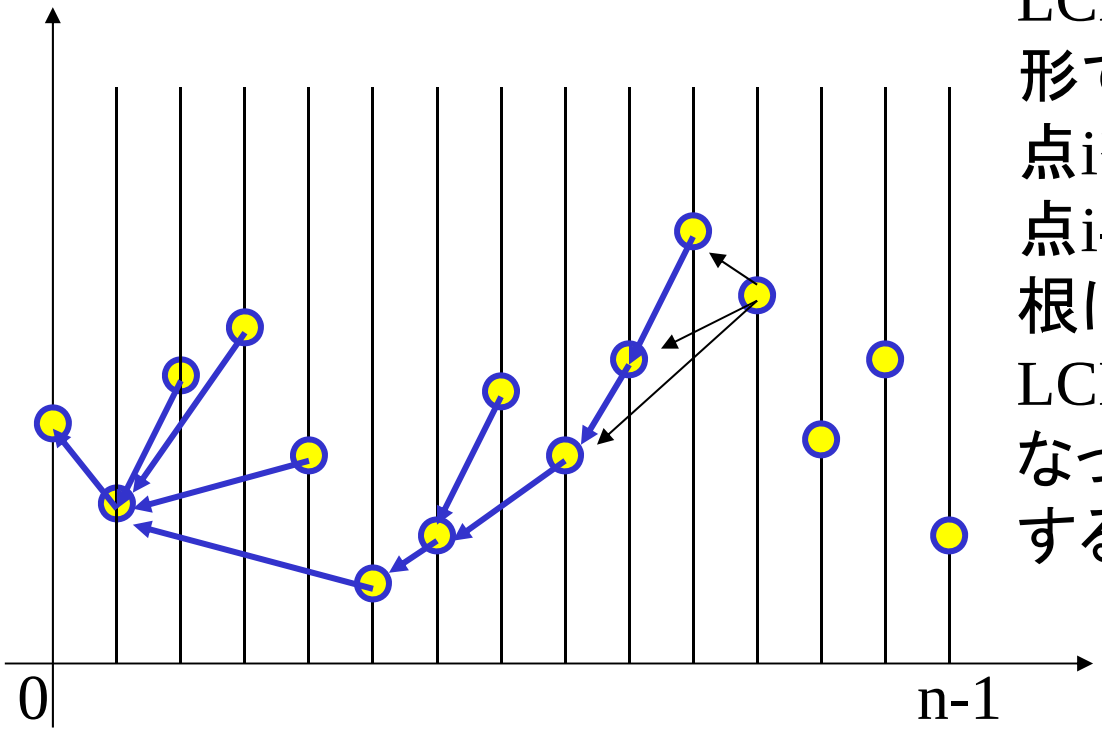


Compute the lower convex hull LCH(0,i-1) for an interval [0,i-1].

Update LCH(0,i) while increasing i.



区間[0,i-1]の点の下部凸包LCH(0,i-1)の計算
　iを順に増やしながらLCH(0,i)を順に更新

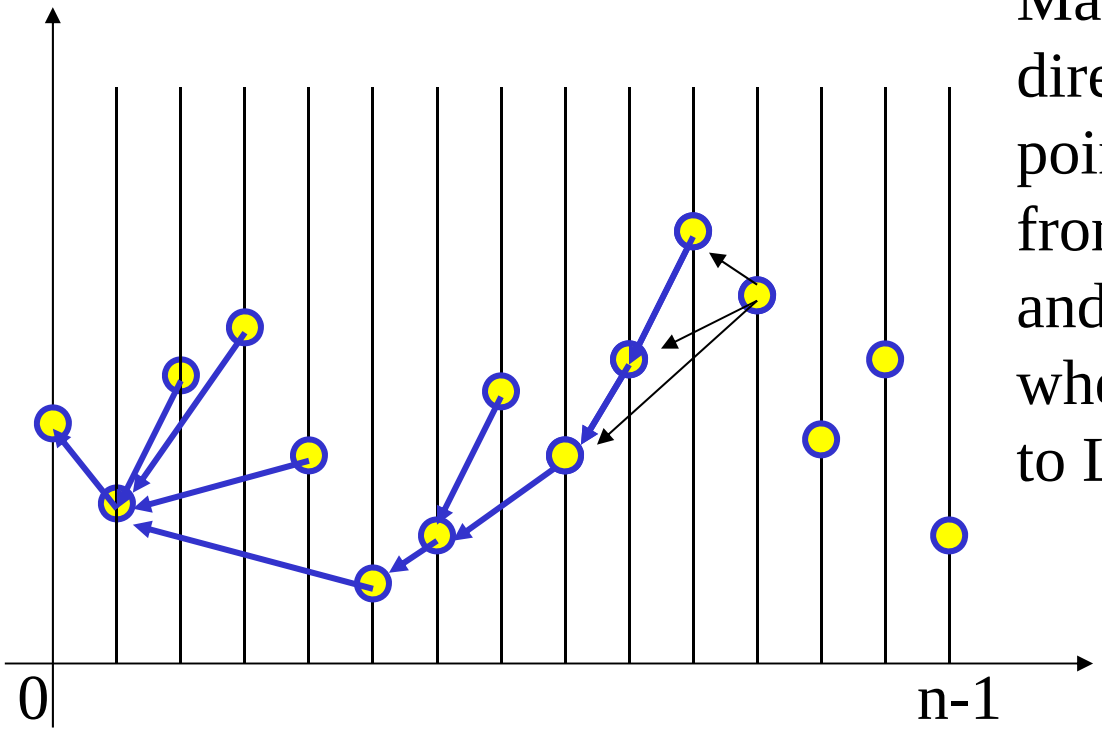


LCH(0,i-1)を有向木の形で管理しておく.
点iを加えるときは,
点i-1から始めて，木を根にむけて辿り,
LCH(0,i-1)への接線になったところで辺を確定する.

演習問題：上記の計算は線形時間O(n)でできることを示せ.

Compute the lower convex hull LCH(0,i-1) for an interval [0,i-1].

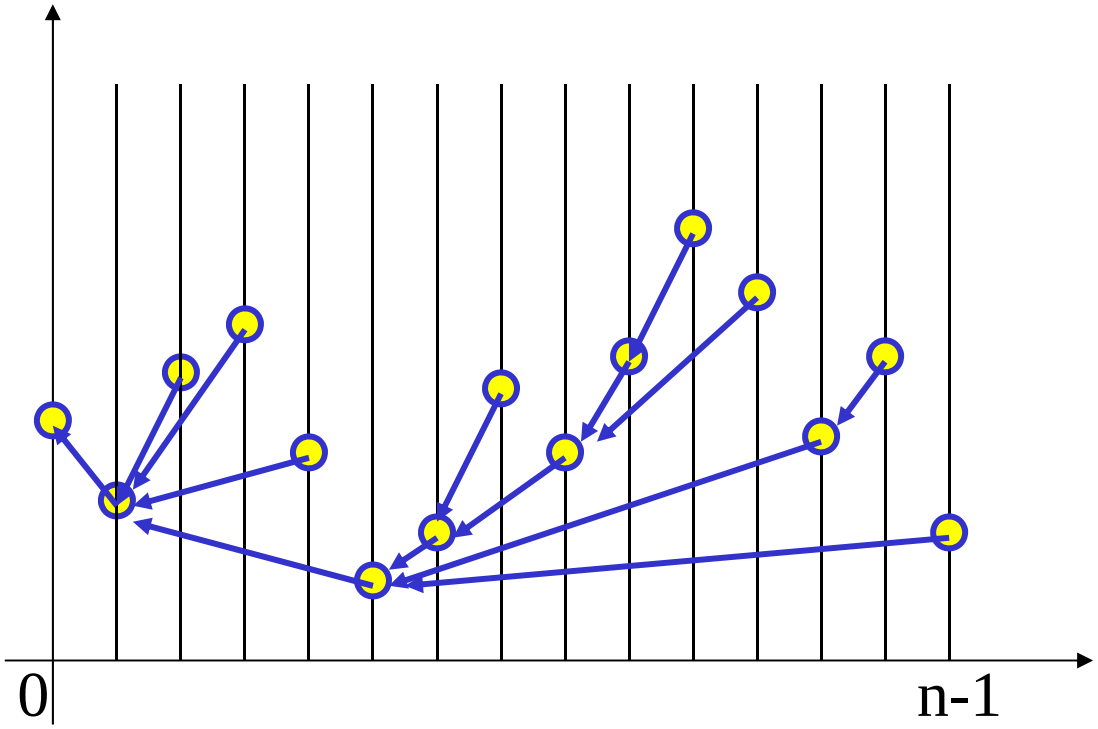
Update LCH(0,i) while increasing i.



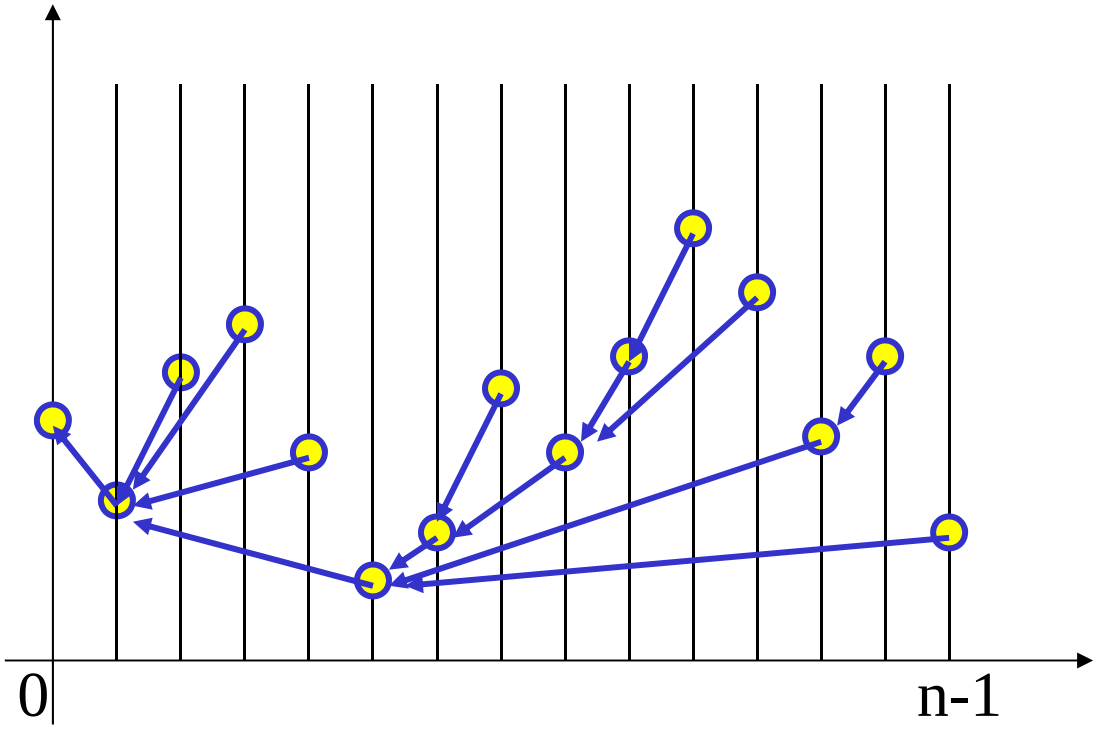
Maintain LCH(0,i-1) as a directed tree. To insert a point i, traverse the tree from point i-1 to the root and determine the edge when it becomes a tangent to LCH(0, i-1).

Exercise : Show that the above computation is done in O(n) time.

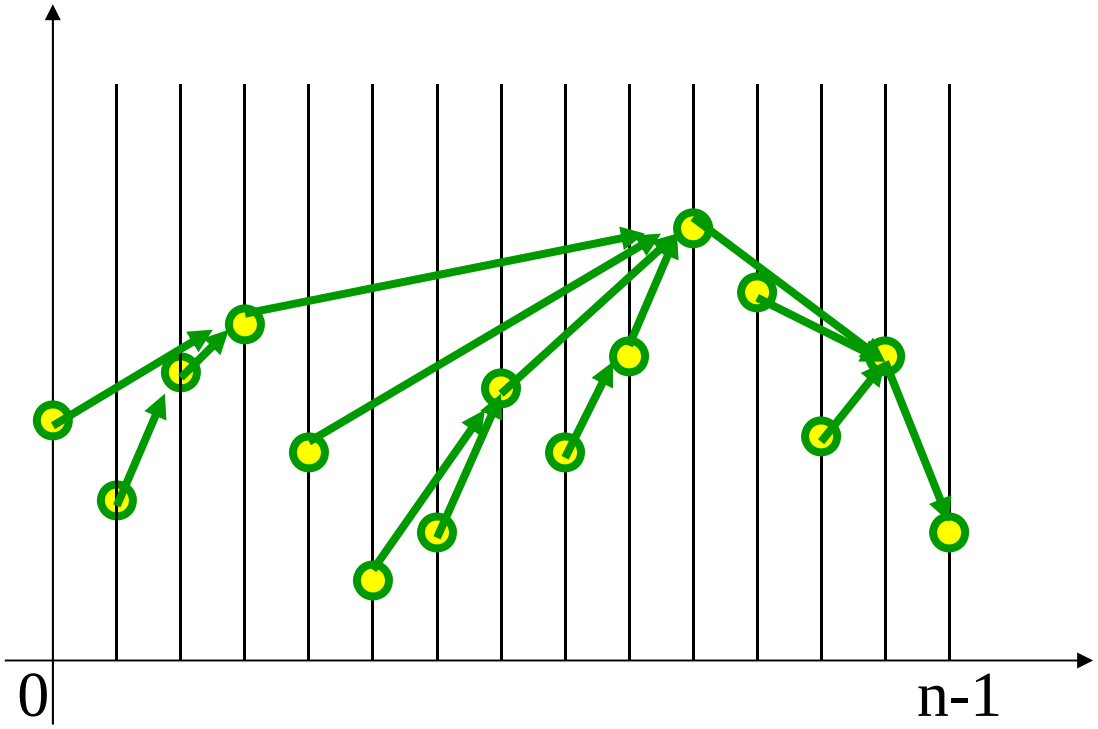
区間[0,i-1]の点の下部凸包LCH(0,i-1)の計算



Compute the lower convex hull LCH(0,i-1) for an interval [0,i-1].

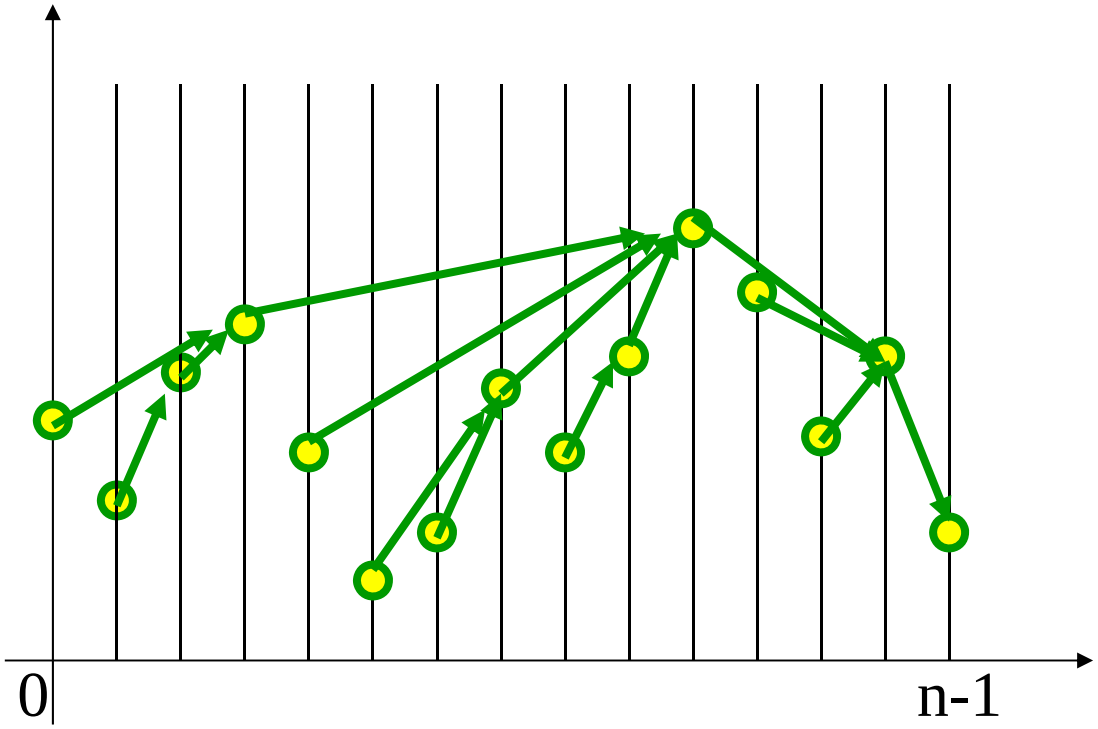


区間[i+1,n-1]の点の上部凸包UCH(i+1, n-1)の計算
iを順に減らしながらUCH(i+1,n-1)を順に更新



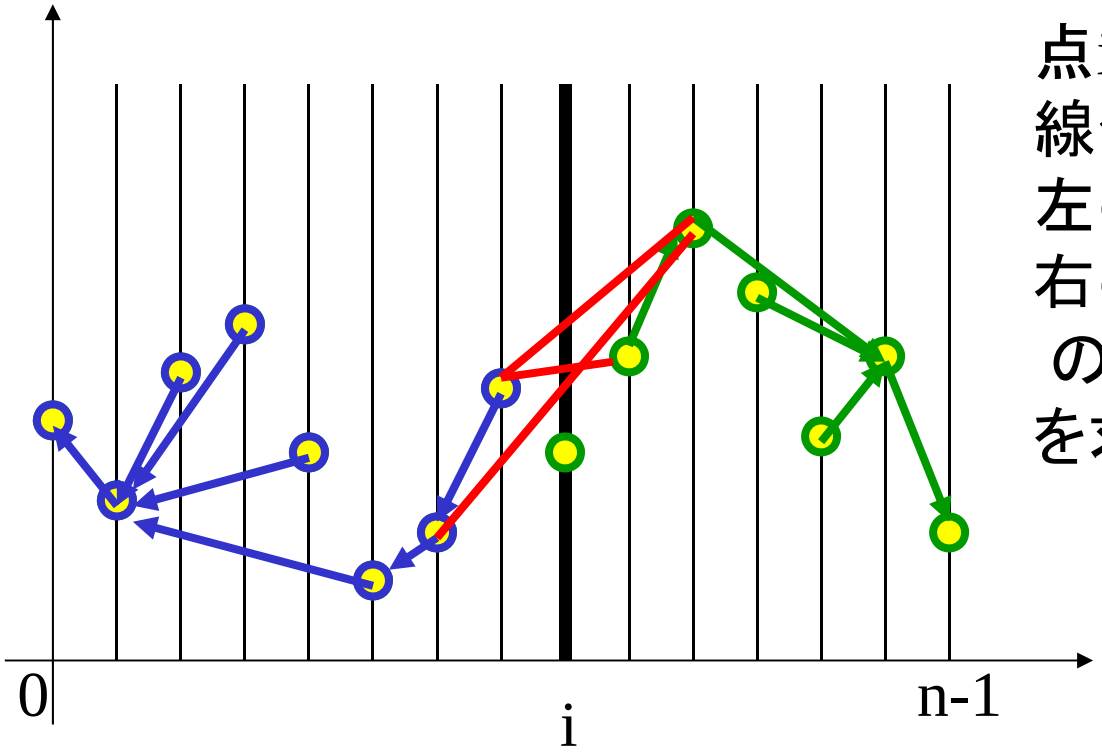
Compute the upper convex hull UCH(i+1,n-1) for an interval [i+1,n-1].

Update UCH(i+1,n-1) while decreasing i.



iに関して区間平均の最大値を求める.

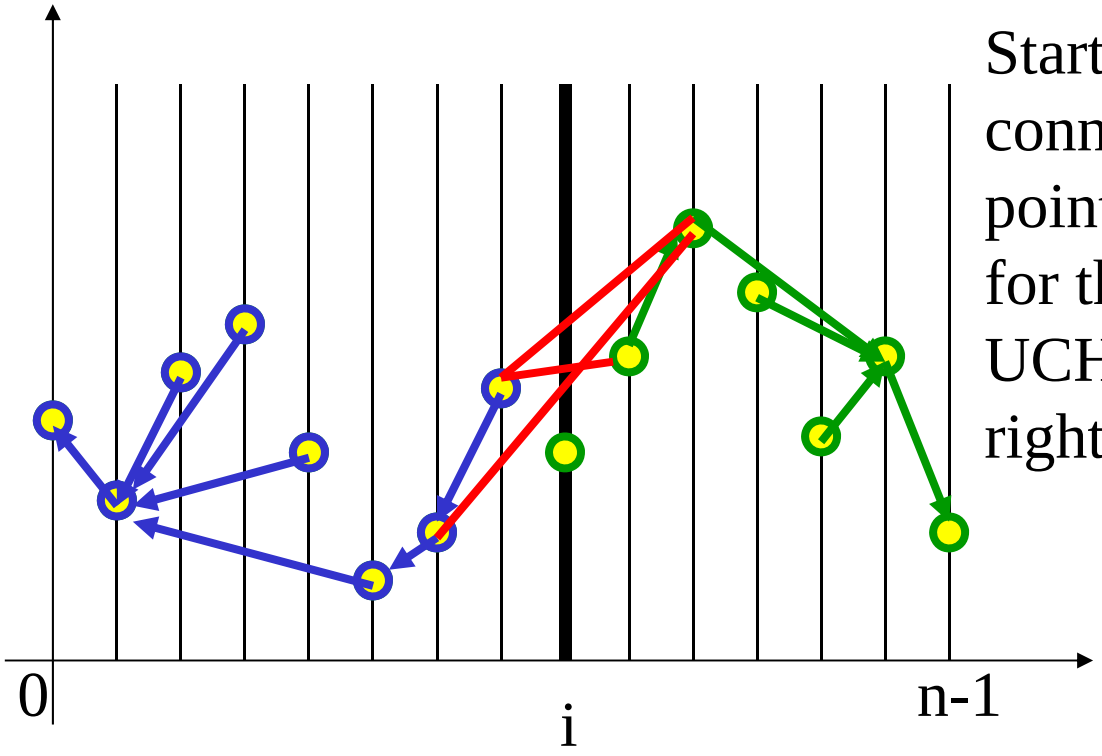
[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.



点i-1と点i+1を結ぶ線分から始めて,
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り，共通接線を求める.

Find an interval maximizing the average for each i.

Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].



Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

具体的な手続きは次の通り

```
p=i-1; q=i+1;
do{
  change=false;
  while((p,q,UCH(i+1,n-1)におけるqの親)が反時計回り){
    q=UCH(i+1,n-1)におけるqの親; change=true;
  }
  while((LCH(0,i-1)におけるpの親,p,q,)が時計回り){
    p=LCH(0,i-1)におけるpの親; change=true;
  }
}
```

上記の手続きはO(n)時間でできる.
これをn回繰り返すと全体ではO(n^2)になってしまう.
線形時間に改善できるか？
あるいは，本当にO(n^2)時間かかるのか？

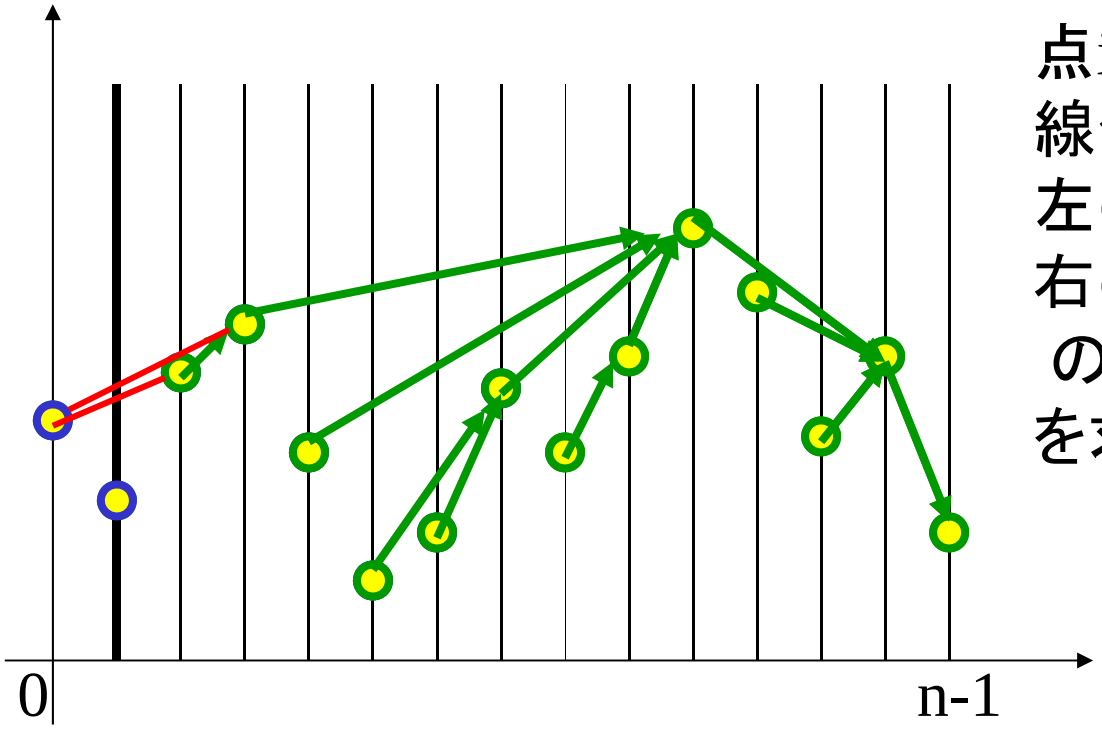
The concrete procedure is as follows:

```
p=i-1; q=i+1;
do{
  change=false;                    ccw: counter-clockwise
  while((p,q,parent of q in UCH(i+1,n-1)) is ccw){
    q=parent of q in UCH(i+1,n-1); change=true;
  }
  while((parent of p in LCH(0,i-1),p,q,) is cw){
    p=parent of p in LCH(0,i-1); change=true;
  }
}
```

The above procedure is done in O(n) time.
If we iterate it n times, the overall time becomes $O(n^2)$.
Can we improve it into linear time?
Or, does it really take $O(n^2)$ time?

iに関して区間平均の最大値を求める.

[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.

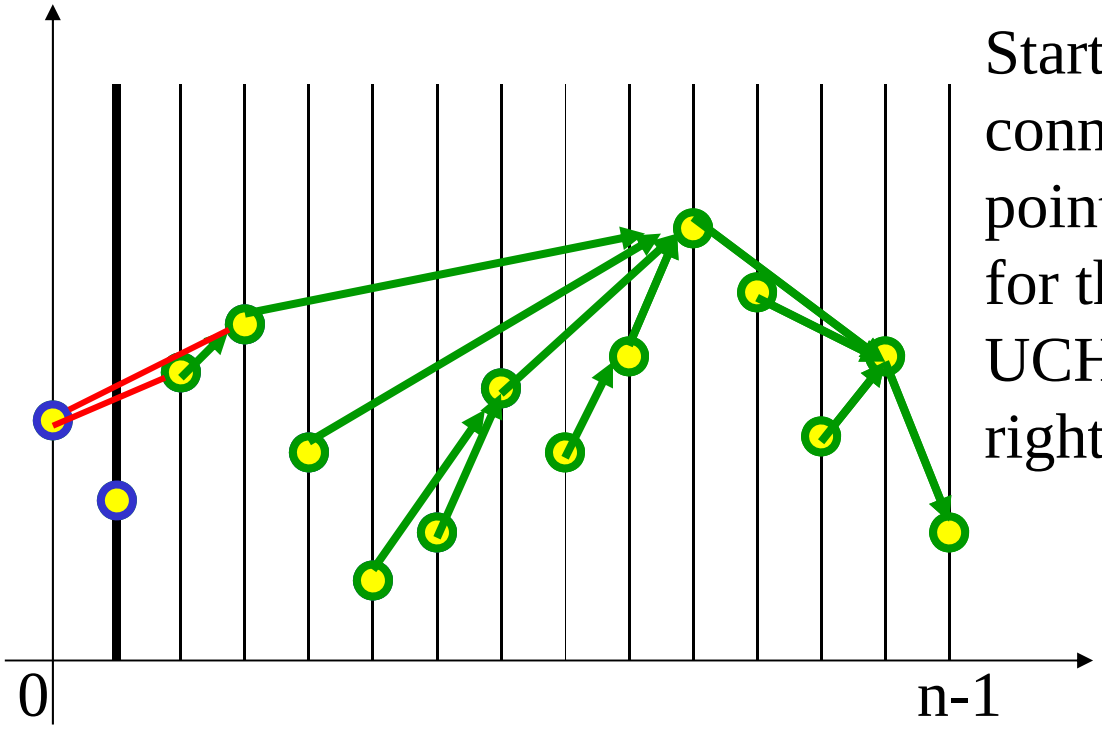


点i-1と点i+1を結ぶ線分から始めて,
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り, 共通接線を求める.

i=1

Find an interval maximizing the average for each i.

Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].

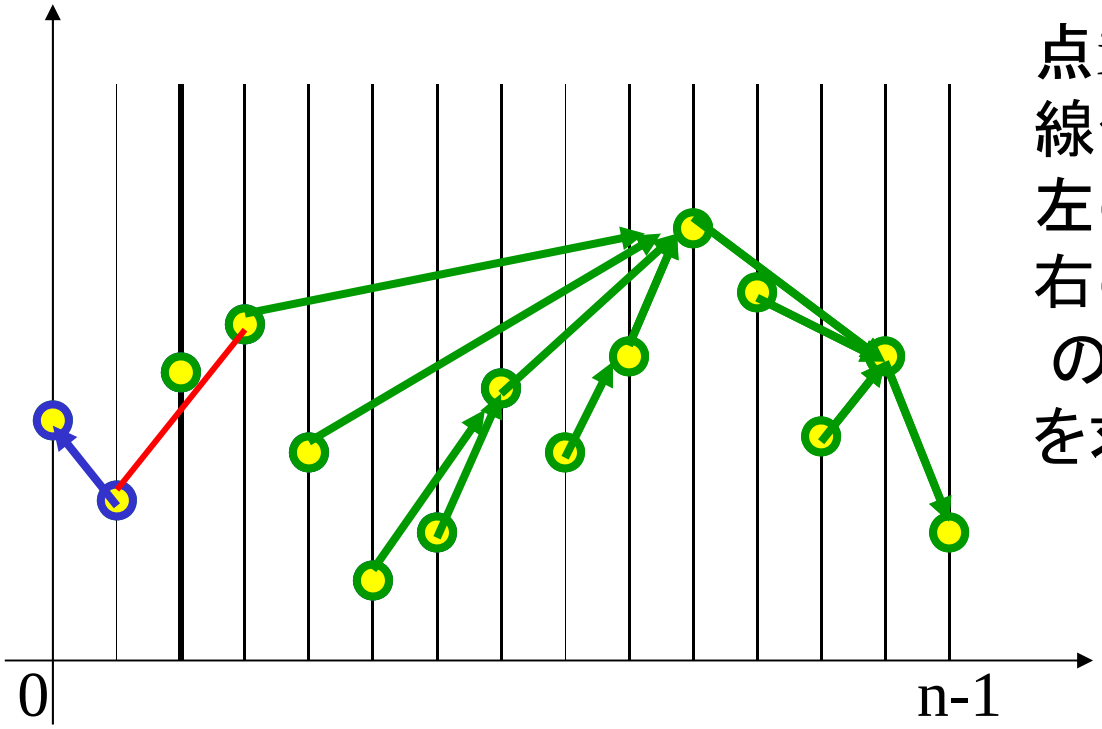


i=1

Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

iに関して区間平均の最大値を求める.

[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.

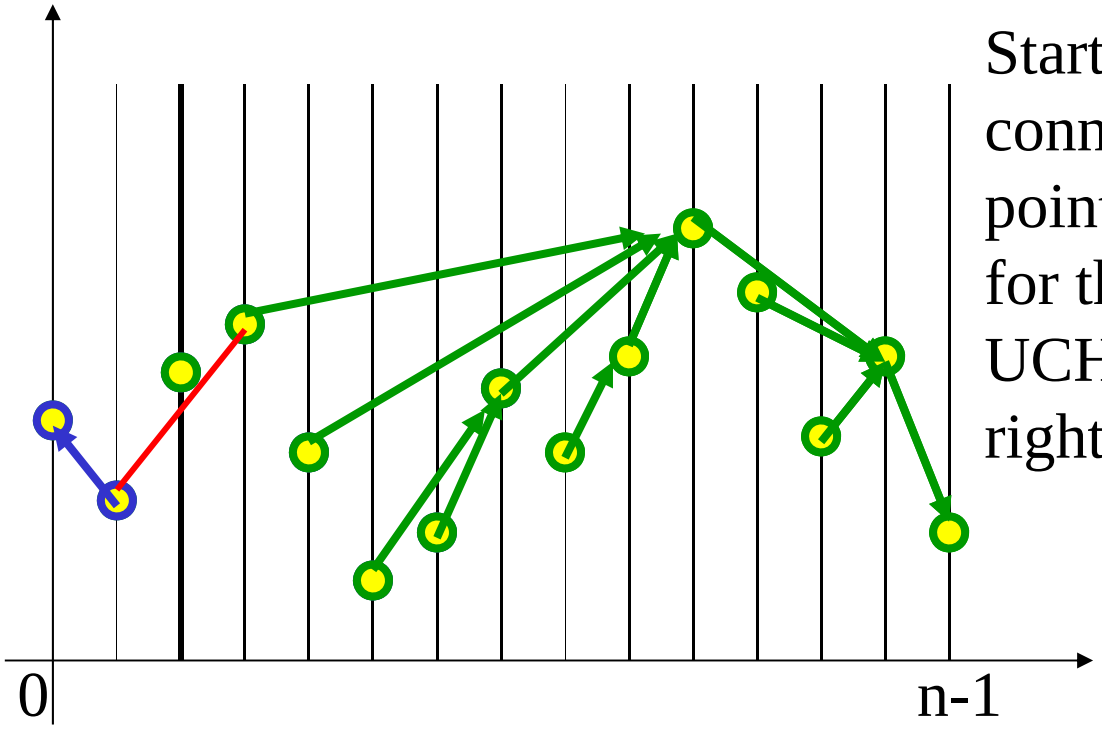


点i-1と点i+1を結ぶ線分から始めて,
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り，共通接線を求める.

i=2

Find an interval maximizing the average for each i.

Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].

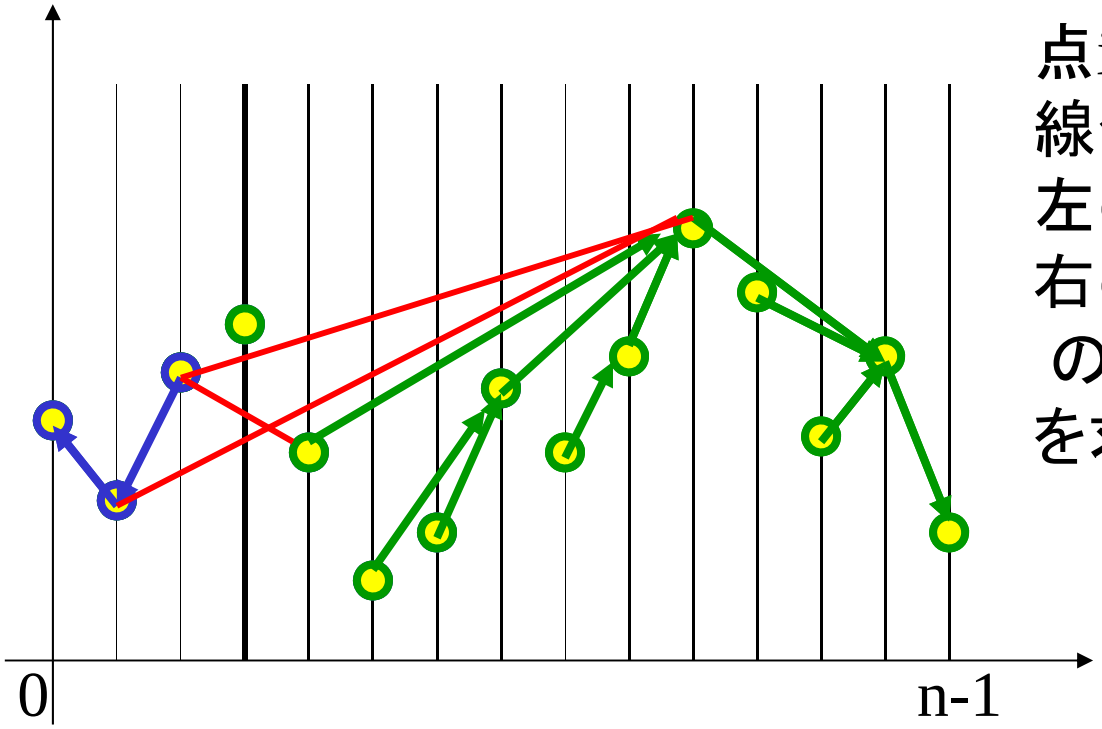


Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

i=2

iに関して区間平均の最大値を求める.

[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.

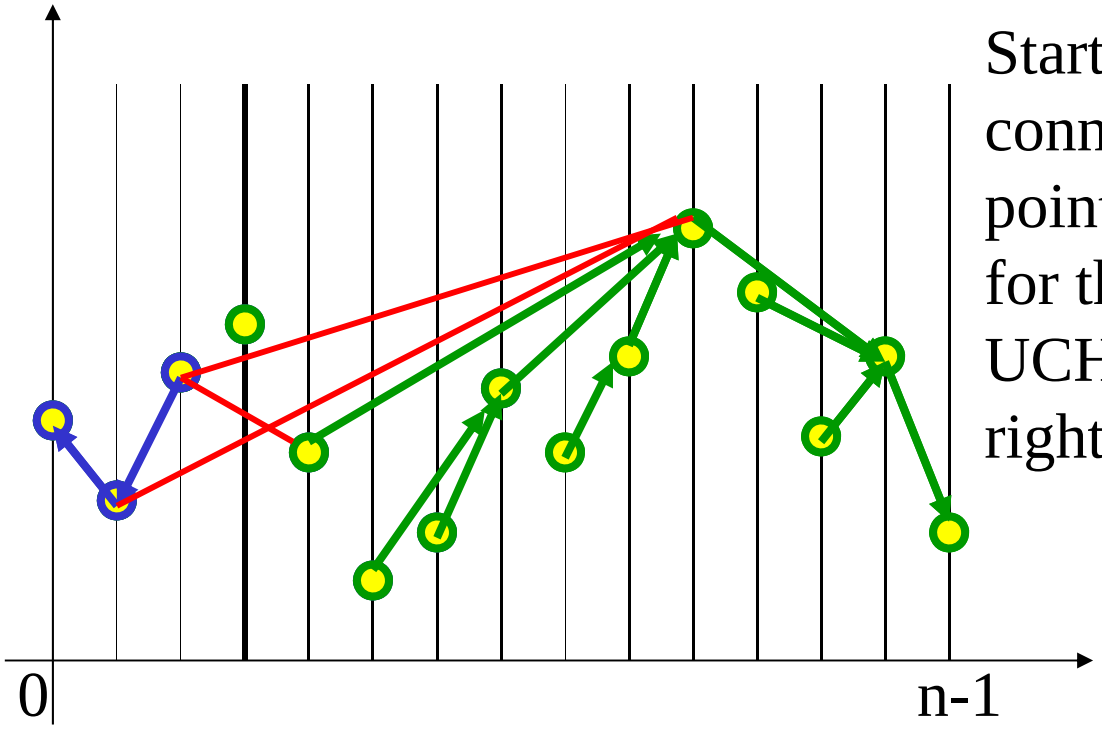


点i-1と点i+1を結ぶ線分から始めて,
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り, 共通接線を求める.

i=3

Find an interval maximizing the average for each i.

Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].

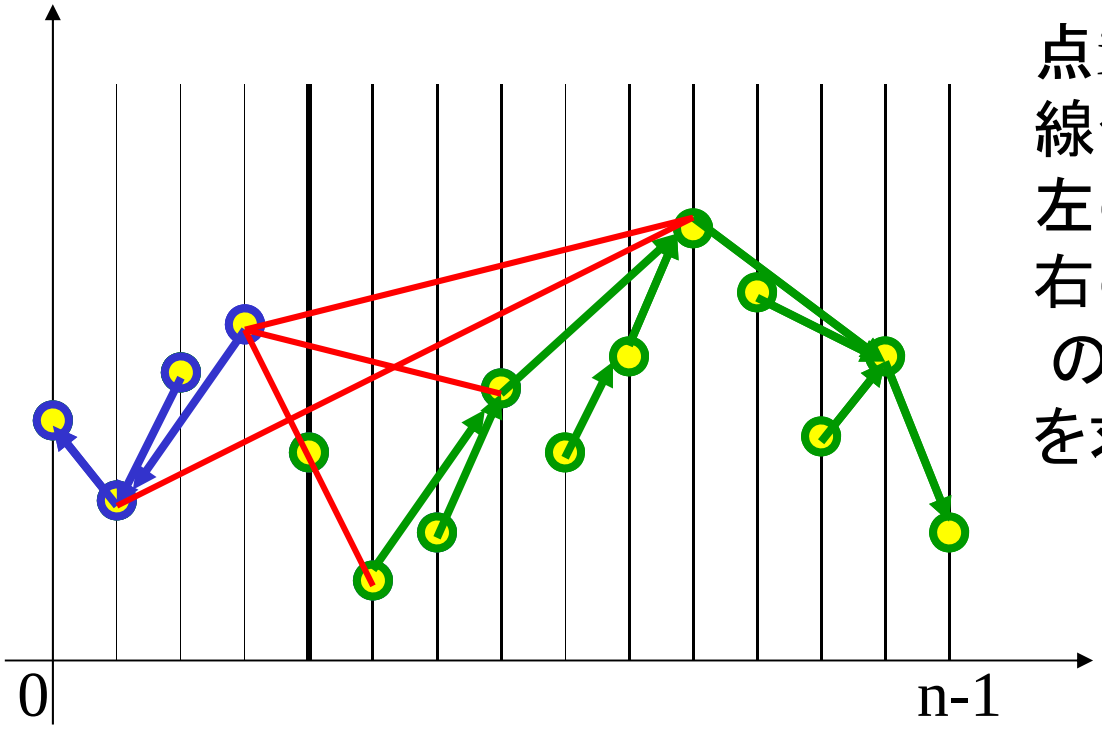


Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

i=3

iに関して区間平均の最大値を求める.

[0,i-1]の点と[i+1,n-1]の点を結ぶ傾き最大の線分を求める.

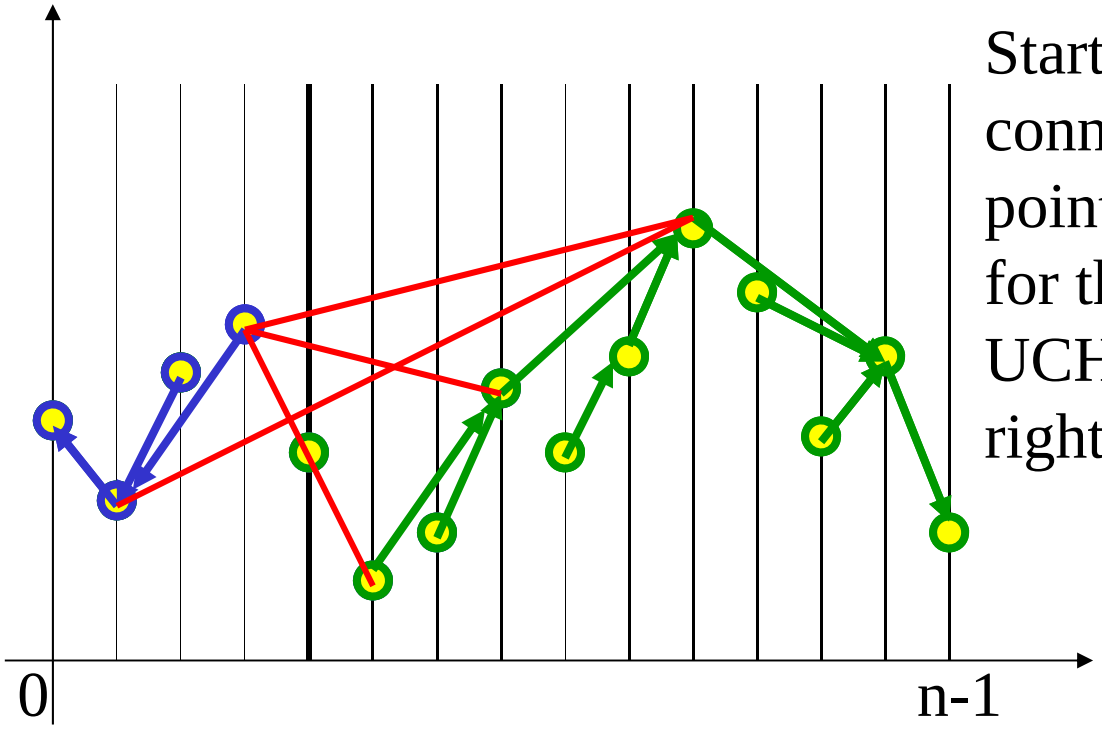


点i-1と点i+1を結ぶ線分から始めて,
左の点はLCH(0,i-1)
右の点はUCH(i+1,n-1)
の辺を辿り, 共通接線を求める.

i=4

Find an interval maximizing the average for each i.

Find a largest-slope line connecting points in [0,i-1] and [i+1,n-1].



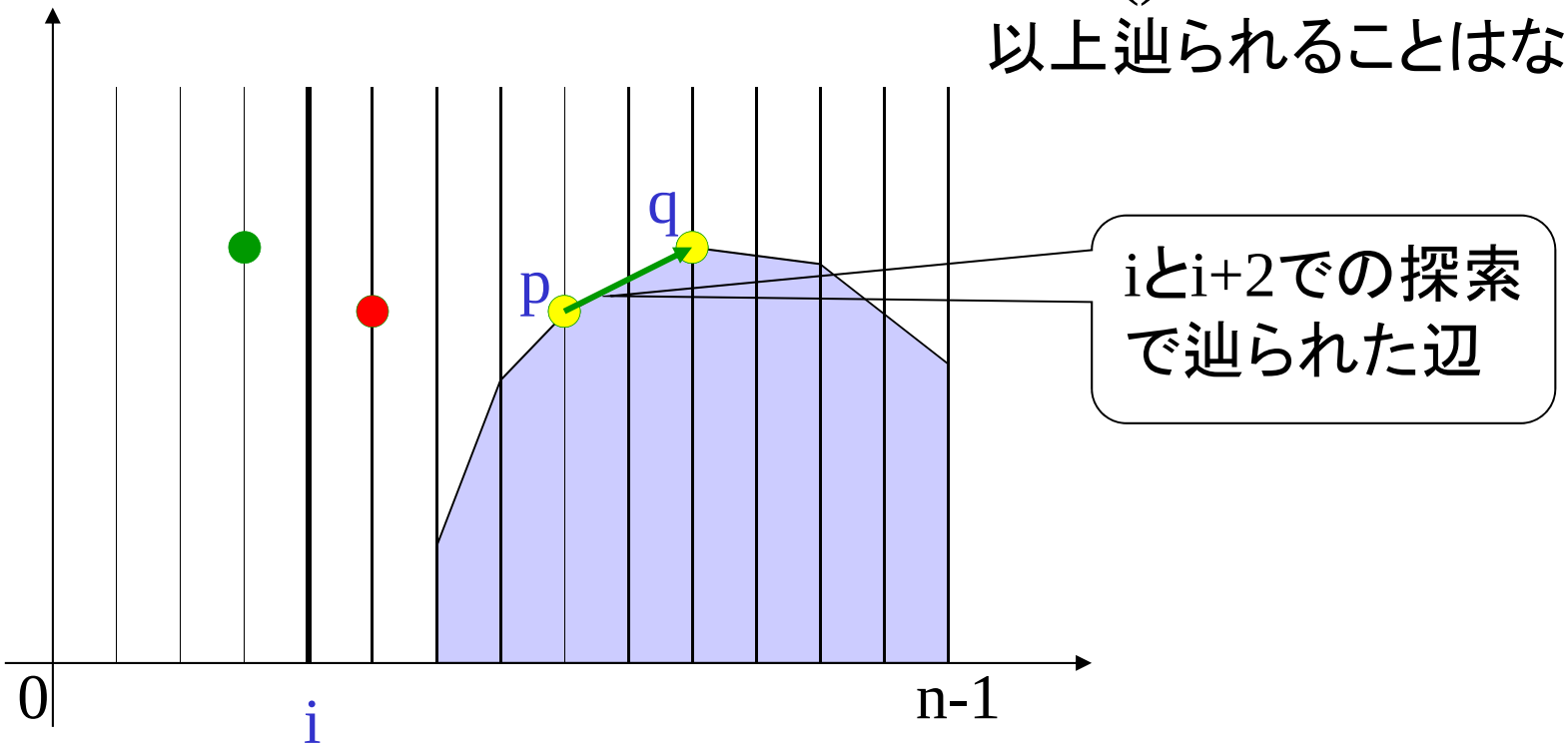
i=4

Starting from the line connecting point i-1 and point i+1, trace LCH(0,i-1) for the left points and UCH(i+1,n-1) for the right points.

この方法では同じ辺を何度も辿ってしまうことがあり，
計算時間がO(n)であることを保証できない．
$O(n^2)$時間かかってしまう例はあるか？

線形時間の保証

UCH()のどの辺も3度
以上辿られることはない．

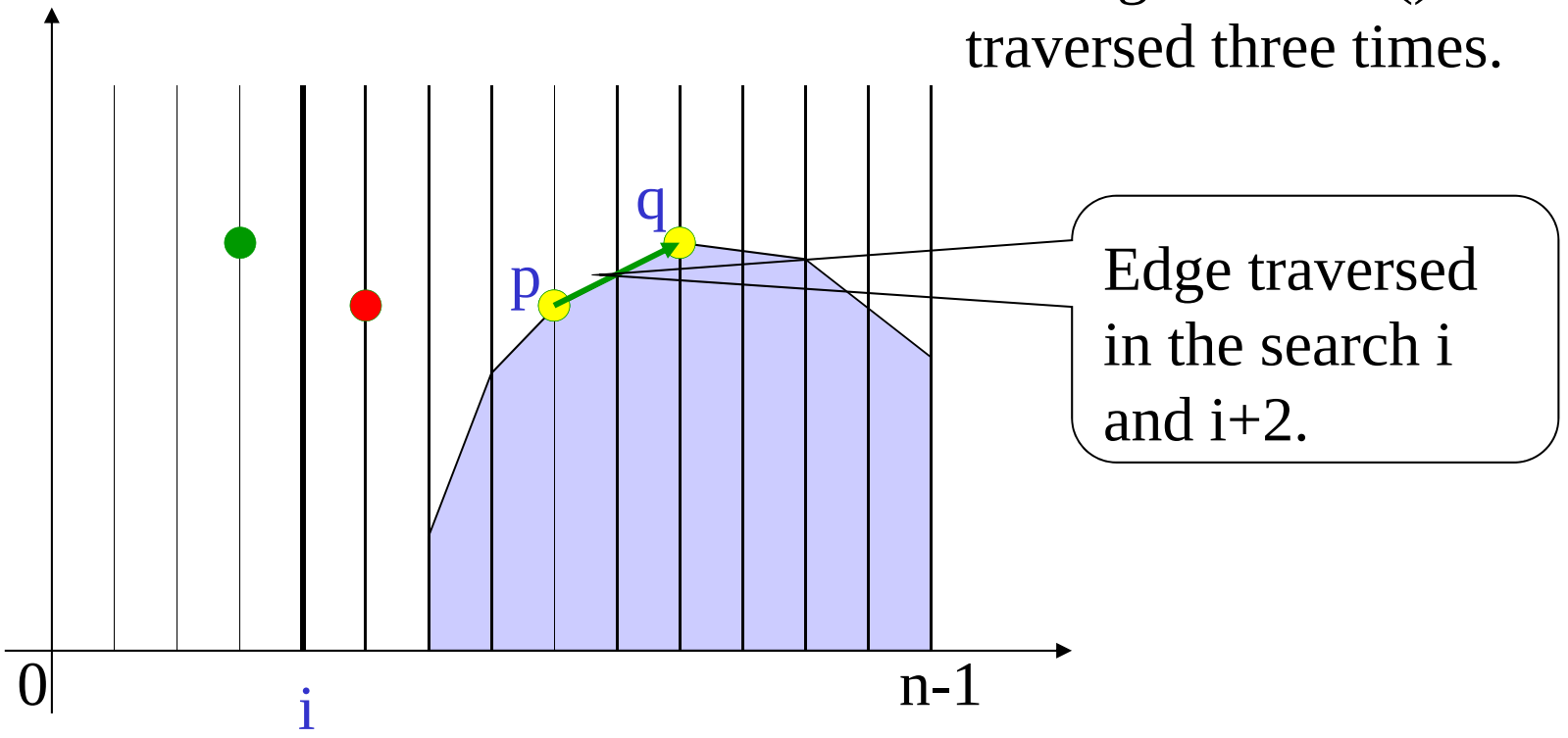


2点i-1, i+1とも直線pqより上になければならない．しかし，そう
すると，iでの探索で辺pqを辿ることはないから矛盾（何故か？）

The same edge can be traversed more than once in this algorithm and thus it cannot guarantee linear computation time.
Is there any example requiring $O(n^2)$ time?

Guarantee of linear time

No edge of UCH() is traversed three times.



The two points i-1 and i+1 must lie above the line pq. But, then the edge is never traversed in the search at i, a contradiction (Why?).

同様に，k≧2に対して，点i-1からの探索と点i+kからの探索でUCHの同じ辺が辿られることはない.

注意：点i-1からの探索と点iからの探索でUCHの同じ辺が辿られることはある.

結局，UCHのどの辺も3度以上辿られることはないことが証明された．LCHについても同様である.

Similarly, for $k \geqq 2$, the same edge in UCH is never traversed in the search from point i-1 and from i+k.

Remark : It happens that the same edge in UCH is traversed in the search from point i-1 and point i.

At last, it has been proven that no edge in UCH is traversed three times. Same for LCH.

最終的にアルゴリズムは次のようになる.

アルゴリズムP4-A2:
最初に, S[i]=a[0]+a[1]+...+a[i]を求める(i=0, 1, ... , n-1)
次に点集合(i, S[i]), i=0, ..., n-1を構成.
LCH(0, i-1)とUCH(i+1, n-1)を順に構成.
maxslope=(a[0]+a[1])/2;
for(i=1; i<n-1; i++){
  点i-1と点i+1を結ぶ線分から始めて,
  左の点はLCH(0,i-1), 右の点はUCH(i+1,n-1) の辺を辿り,
  共通接線(p, S[p])-(q, S[q])を求める.
  if(共通接線の傾き>maxslope) maxslope=共通接線の傾き;
}
maxslopeを出力;

演習問題:このアルゴリズムを実装せよ.

Finally we have the following algorithm.

Algorithm P4-A2:
First, we compute S[i]=a[0]+a[1]+...+a[i] (i=0, 1, ... , n-1).
Then, construct a set of points (i, S[i]), i=0, ..., n-1.
Construct LCH(0, i-1) and UCH(i+1, n-1) in order.
maxslope=(a[0]+a[1])/2;
for(i=1; i<n-1; i++){
  Starting from the line connecting point i-1 and point i+1,
  traverse edges in LCH(0,i-1) for the left point and in UCH(i+1,n-1)
  for the right point to find the common tangent (p, S[p])-(q, S[q]).
   if(slope of tangent line>maxslope) maxslope=slope of tangent line;
}
output maxslope;

Exercise: Implement this algorithm.