# Complexity of the stamp folding problem

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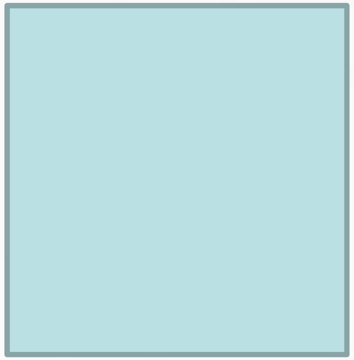
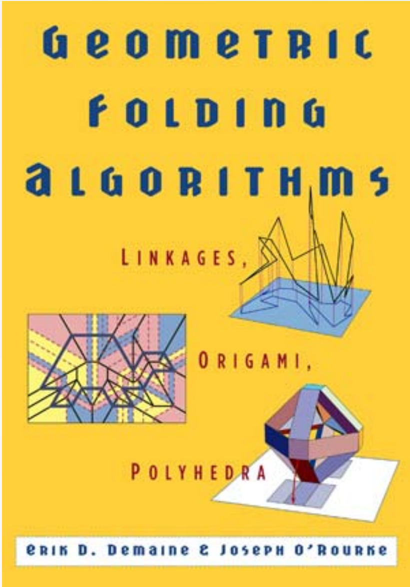
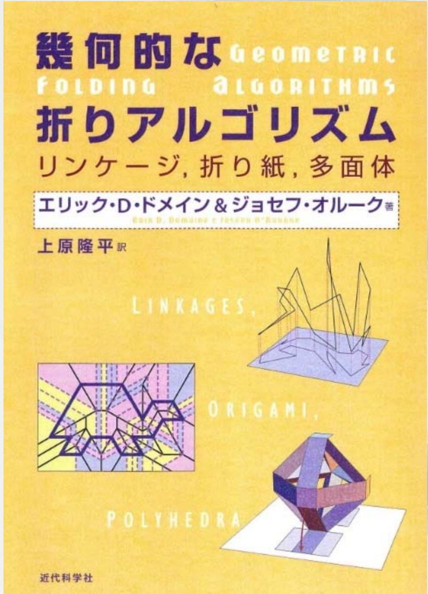
# Introduction

- Computational Origami
- A sheet of paper





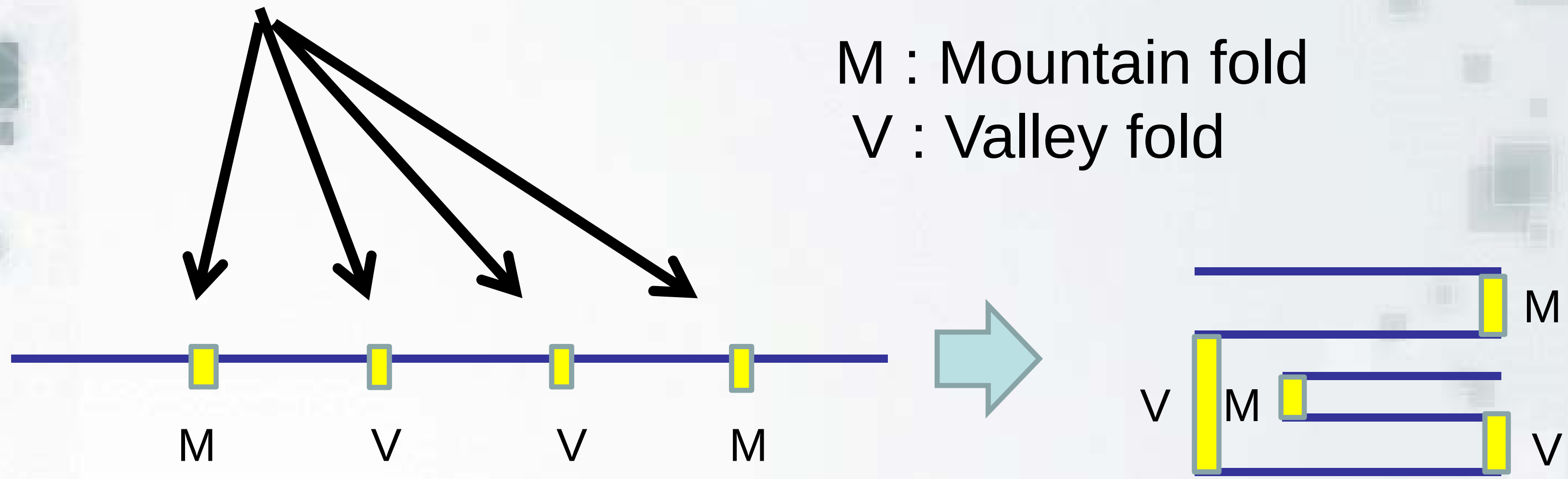

- In 2D, it is NP-hard to determine if a sheet of paper can be folded flat for a given crease pattern. [Bern and Hayes, 1996]

# Introduction

- 1D paper (or long strip paper)

- Creases at unit intervals

M : Mountain fold
V : Valley fold

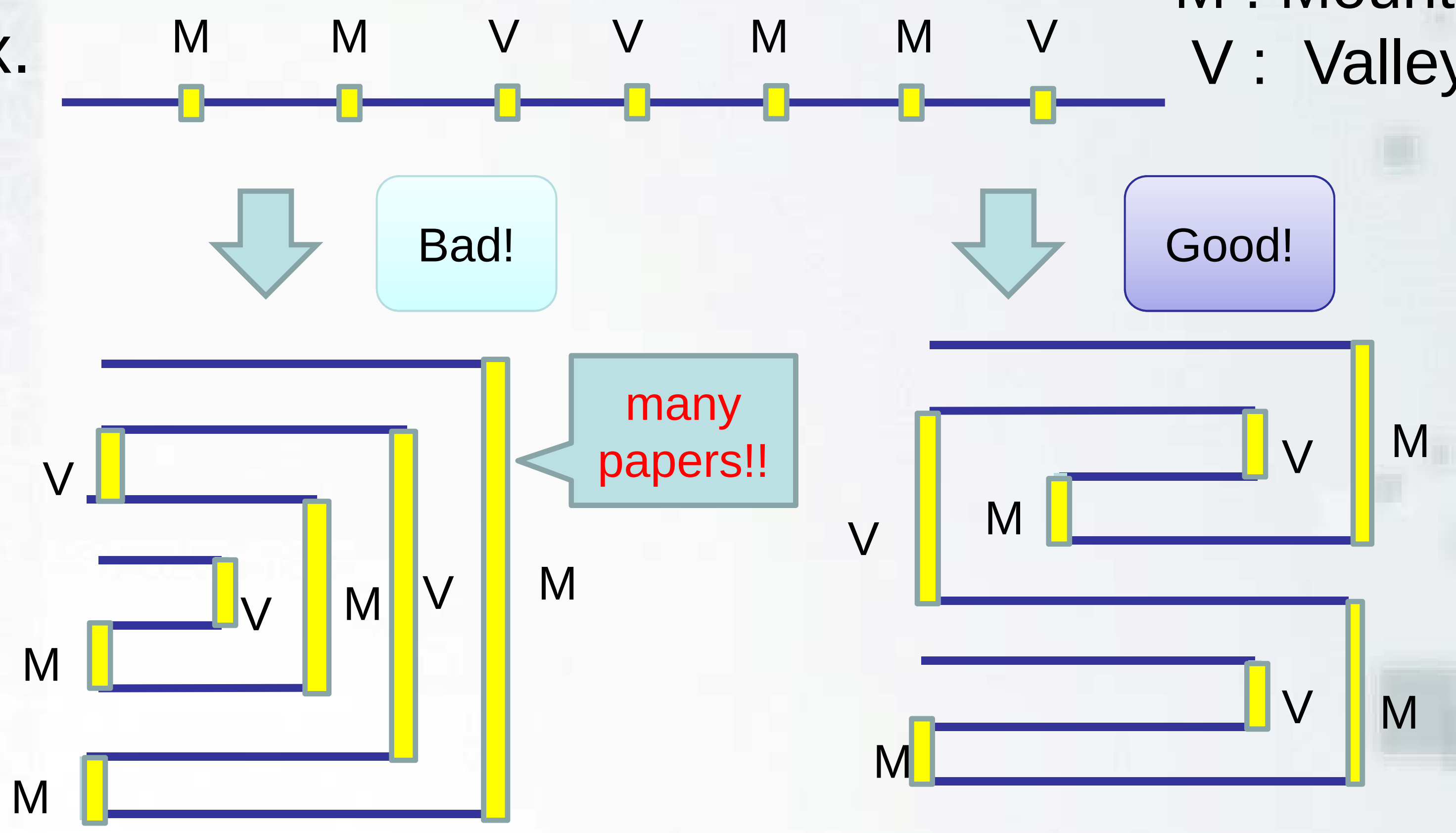


# Introduction

- Given M/V pattern, fold it into unit length

Ex.

M : Mountain
V : Valley



# New minimization problems

- Folding with least **crease width**

**Input**: Paper of length $n+1$ and $s \in \{M, V\}^n$

**Output**: folded paper according to $s$

**Goal**: Find a *best* folded state with small *crease width*

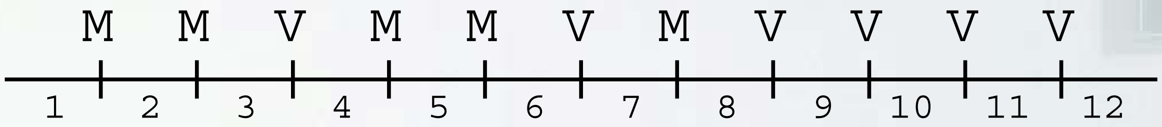
- At each crease, the number of papers between the papers hinged at the crease is *crease width.*
- Two minimization problems;
  - minimize maximum
  - minimize total (=average)

No!!

It seems simple, … so easy??

# Simple non-trivial example

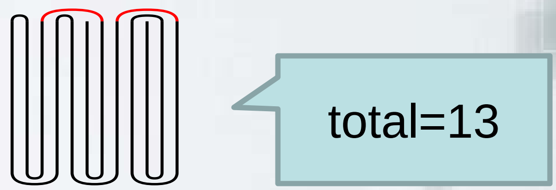
**Input**: MMVMMVMVVVV



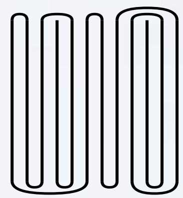

The number of feasible folded states: **100**

**Goal**: Find a *best* folded state with small *c.w.*

- The **unique** solution having MinMax value 3
  [5|4|3|6|7|1|2|8|10|12|11|9]




- The **unique** solution having MinTotal value 11
  [5|4|3|1|2|6|7|8|10|12|11|9]



# Stamp folding problem

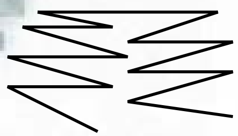
- Folding with least **crease width**

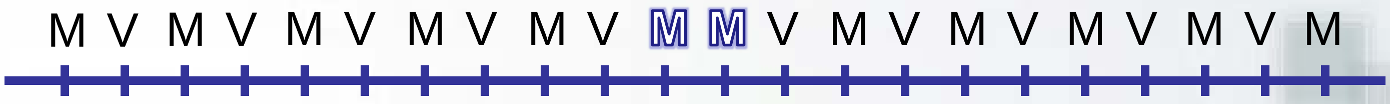
**Input**: Paper of length $n+1$ and $s \in \{M, V\}^n$

**Output**: folded paper according to $s$

**Goal**: Find a *best* folded state with small *crease width*

- Two criteria; MinMax and MinTotal
- A few facts;
  - solutions of MinMax and MinTotal are different depending on a crease pattern.
  - there is a pattern having exponential combinations





# Stamp folding problem

Known result:

- If the crease pattern is given uniformly at random, the expected number of folding ways is exponential [Uehara, 2010].
  - so simple search does not work efficiently.

- Computational complexity of the stamp folding problem was open.

# Main results

- MinMax : NP-complete

- MinTotal :

  restricted case can be solved

  in polynomial time.

  (If MinTotal≦$k$ for small <u>constant $k$</u>,

  it can be solved in poly-time.)

# MinMax is NP-complete

Proof: Polynomial time reduction from 3-Partition.

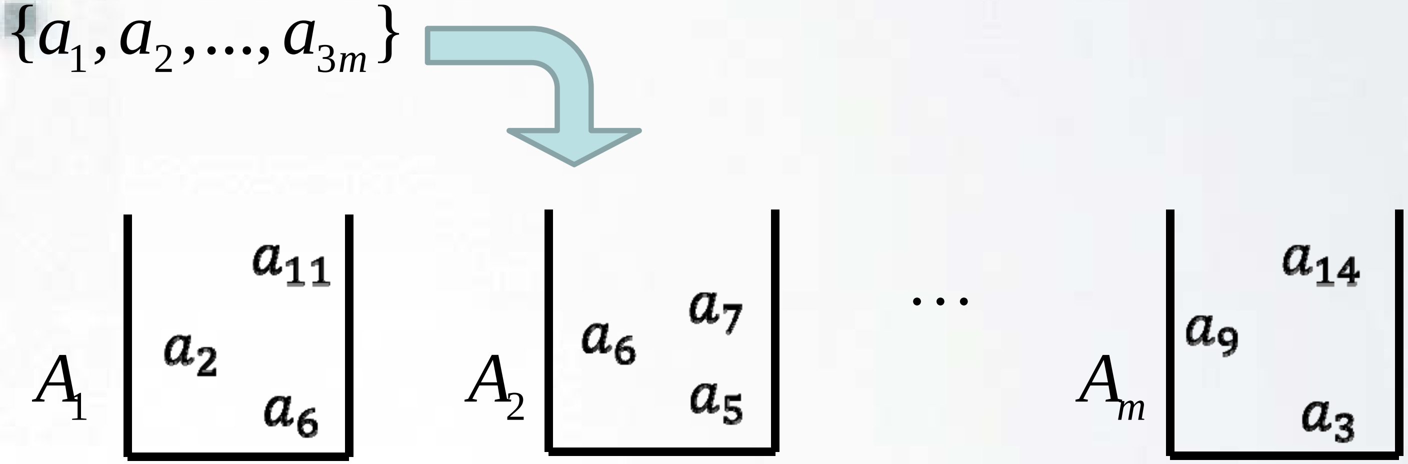
<u>3-Partition</u>:

$(B/4 < a_j < B/2)$

Input: Set of integers $A = \{a_1, a_2, \ldots, a_{3m}\}$ and integer $B$

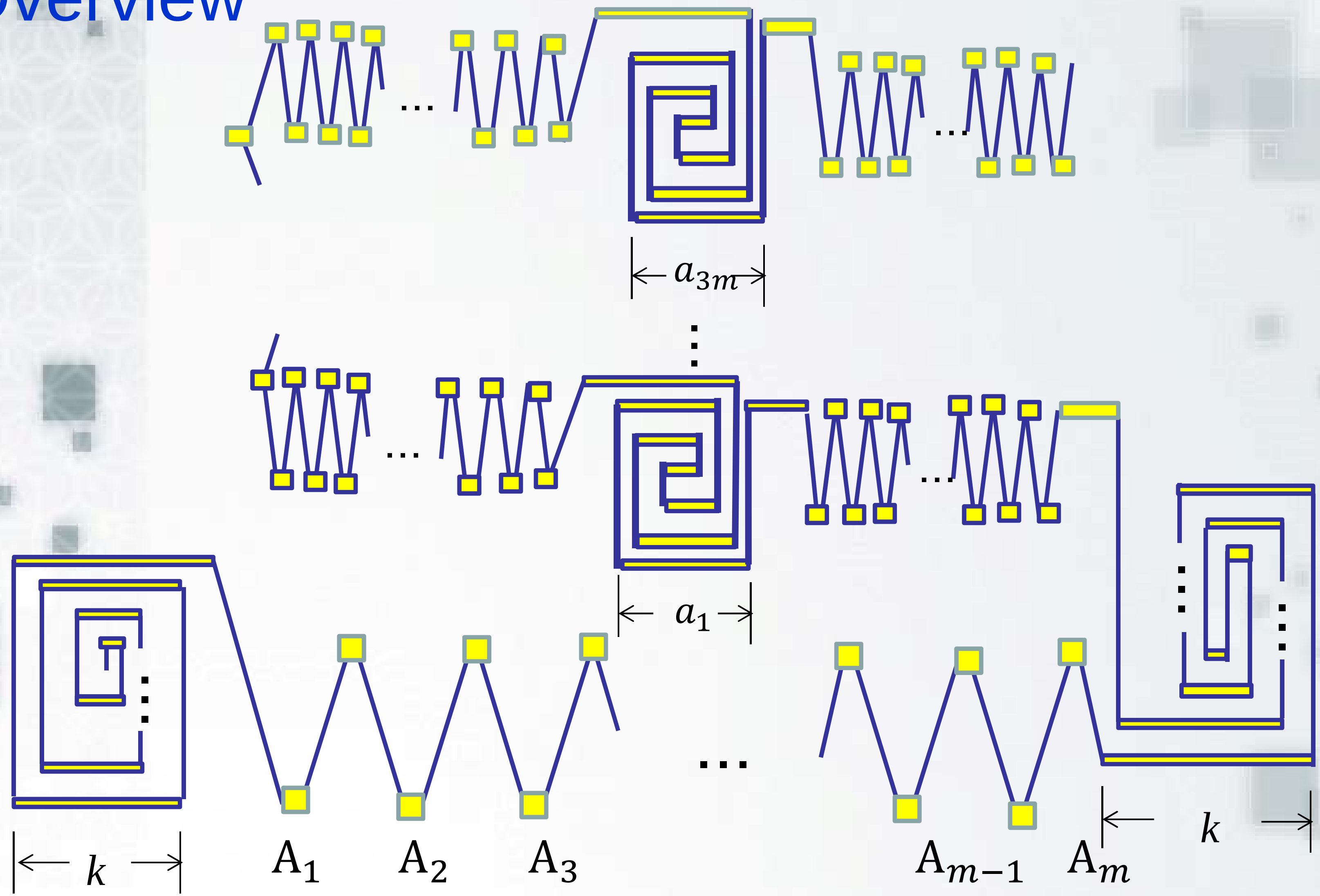
Question: Is there a partition of $A$ to $A_1, \ldots, A_m$ such that $|A_i| = 3$ and $\sum_{a_j \in A_i} a_j = B$

$A = \{a_1, a_2, \ldots, a_{3m}\}$



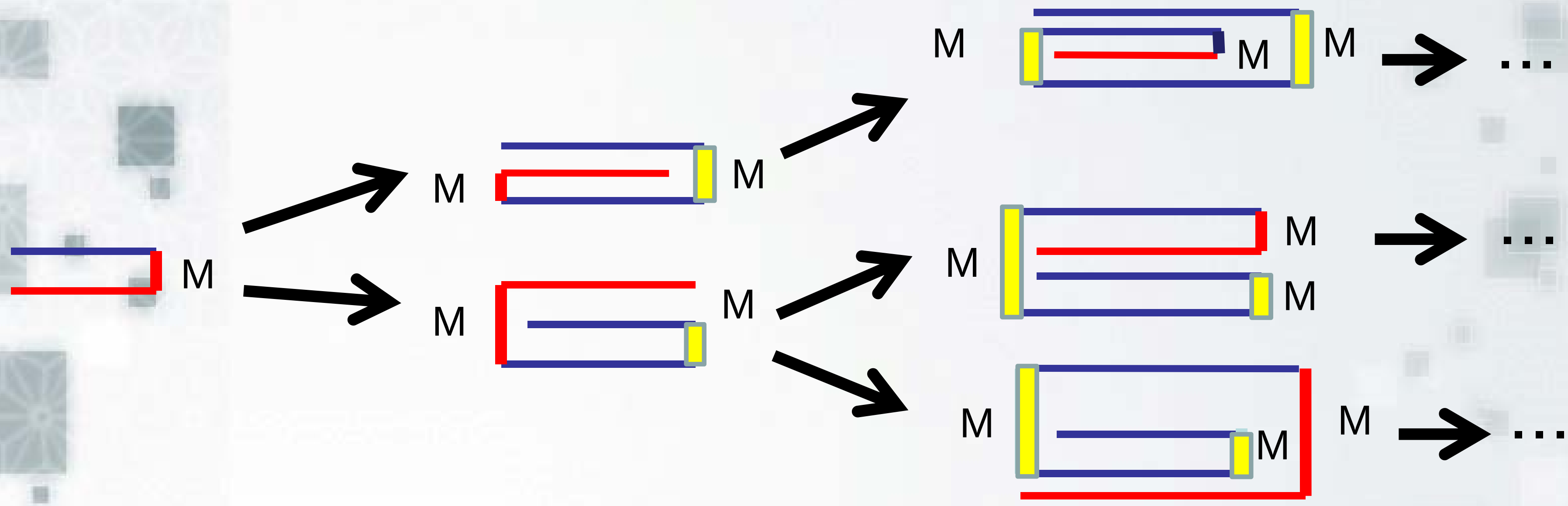
# Construction for MinMax

## Overview



# (Poly-time) Algorithm for MinTotal

- Enumerate all folding ways with respect to the string *s* up to total crease width *k*.
- Each folded state is generated incrementally.




- Check the total crease width at each increment.

# Running time

- The algorithm for given fixed total crease width $k$ runs in $O(n^{2+k})$ time.

  - at each crease, the sequence of c.w. is
    - $c_1, c_2, \ldots, c_i, \ldots$ with $\sum_{i=1,2,\ldots} c_i \le k$
  - that is a partition of $\leqq k$

# Summary

- MinMax : NP-complete
- MinTotal :
  - polynomial time algorithm for given fixed total crease width $k$
  - running time is $O(n^{2+k})$

# Summary

Poly-time algorithm under some reasonable assumption?

- MinMax : NP-complete

Computational Complexity (NP-complete?)

- MinTotal :
  - polynomial time algorithm for given fixed total crease width $k$
  - running time is $O(n^{2+k})$

★ the algorithm indeed runs in $O(2^k n^3)$ (by Yoshio Okamoto)

Fixed Parameter Tractable!!

# 最新情報

- 折り目を等間隔でないものにした，より一般的なモデルにおける同様の結果が以下の国際会議で発表：
  - Erik D. Demaine, David Eppstein, Adam Hesterberg, Hiro Ito, Anna Lubiw, Ryuhei Uehara and Yushi Uno: Folding a Paper Strip to Minimize Thickness, *The 9th Workshop on Algorithms and Computation (WALCOM 2015)*, Lecture Notes in Computer Science, 2015/02/26-02/28, Dhaka, Bangladesh.
  - 折り目の「厚さ」の定義にいろいろと考えられるけれど，本質的には crease width と同様の結果が得られた.