Folding a Paper Strip to Minimize Thickness

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Computational Geometry Session
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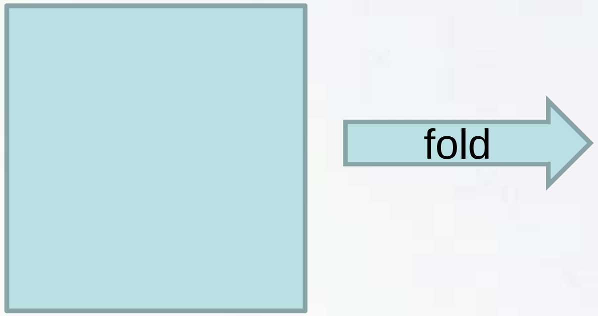
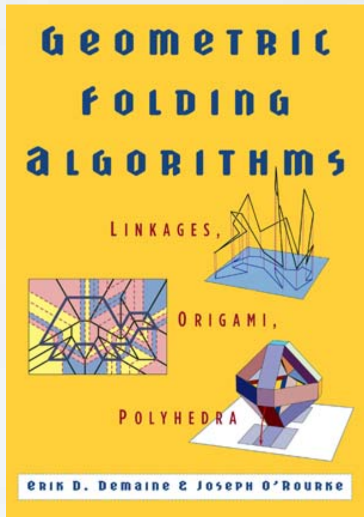
Introduction

- Origami: From a square sheet of paper



- “Computational Origami”
- In 2D, it is NP-hard to determine if a sheet of paper can be folded flat for a given crease pattern. [Bern and Hayes, 1996]



Introduction

- Computational Origami
  - Its “complexity/algorithm” are not well investigated from the viewpoint of theoretical computer science...
  - My motivation: reasonable “model” for *computation*
- Focus on quite simple case!
  - 1D: paper strip
  - Creases are at unit intervals
  - Repeat M/V…Pleat folding

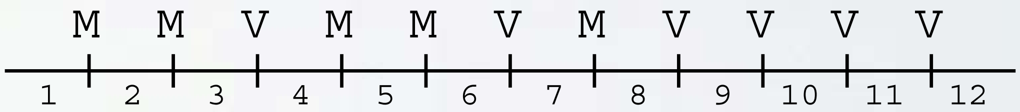
From this simple folding,

… more general folding Mountain/Valley pattern?

Already, not so simple…

Input: MMVMMVMVVVV

By exhaustive search



The number of feasible folded states：**100**

From the viewpoint of Theoretical Computer Science, we may consider

- Time complexity?
- Space complexity?

Computational Complexity of Origami

- From the viewpoint of theoretical computer science, two **Resources** of Origami?

1. Time: the number of folding operations
   - J. Cardinal, E. D. Demaine, M. L. Demaine, S. Imahori, T. Ito, M. Kiyomi, S. Langerman, R. Uehara, and T. Uno: Algorithmic Folding Complexity, *Graphs and Combinatorics*, Vol. 27, pp. 341-351, 2011.
2. Space…???
   - R. Uehara: Stretch Minimization Problem of a Strip Paper, *5th International Conference on Origami in Science, Mathematics and Education*, 2010/7/13-17.
   - T. Umesato, T. Saitoh, R. Uehara, H. Ito, and Y. Okamoto: Complexity of the stamp folding problem, *Theoretical Computer Science*, Vol. 497, pp. 13-19, 2012.

This talk is the next step of this work

Previous work…

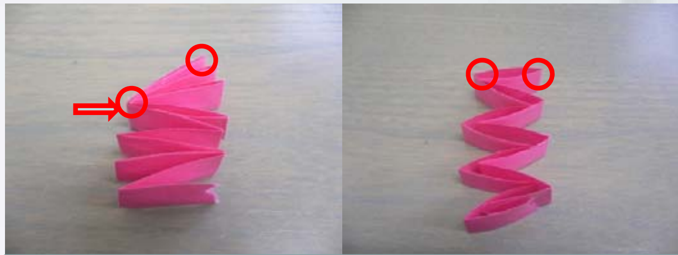
Input: paper strip of length n+1 and string s of length n over {M,V}

Output: flat folded state according to s

Goal: "Good" one with few stretch/stress

[Q] WHAT'S A GOOD FOLDED STATE?

Ex: MVMVMVMMVMVMVM



All right pleats are put into one crease on the left side. | Each crease has at most two paper layers

× Bad!! | ○Good!!

- Goodness = the number of paper layers at a crease

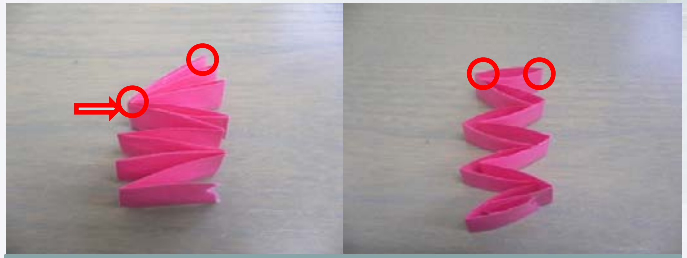
Previous work…

Input: paper strip of length $n+1$ and string s of length n over {M,V}

Output: flat folded state according to s

Goal: "Good" one with few stretch/stress

- Goodness = the number of paper layers at a crease
- Two optimization problems
  1. Minimize the maximum
  2. Minimize its total



All right pleats are put into one crease on the left side. | Each crease has at most two paper layers

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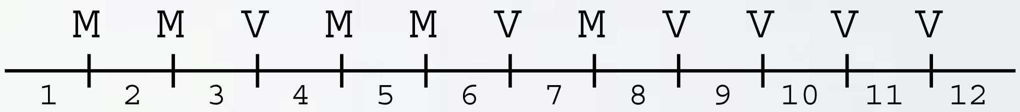
Two problems differ

<u>Input</u>: MMVMMVMVVVV

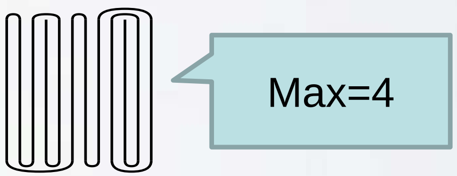


The number of feasible folded states : **<u>100</u>**

Solutions: We have unique different solution for each problem for this pattern:

- Minimum max. value=3
  [5|4|3|6|7|1|2|8|10|12|11|9]

- Minimum total value=11
  [5|4|3|1|2|6|7|8|10|12|11|9]



Previous work in [Umesato, et.al TCS, 2012]

Input: paper strip of length $n+1$ and string s of length n over {M,V}

Output: flat folded state according to s

Goal: "Good" one with few stretch/stress

- Goodness = the number of paper layers at a crease
- Two optimal problems
  1. Minimize the maximum
     - NP-complete
  2. Minimize its total
     - Open, but we give a FPT algorithm w.r.t. the total number.

Now we turn to…

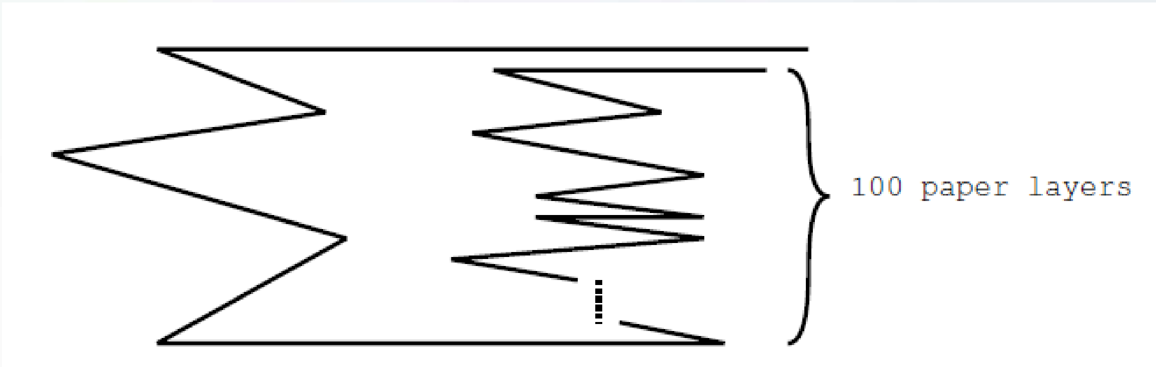
- Computational Origami
  - Its “complexity” is not well investigated from the viewpoint of theoretical computer science...
- Focus on quite simple case!
  - 1D: paper strip
  - ~~Creases at unit intervals~~
  - General M/V pattern

Non-unit intervals!!

Not only M/V, but also lengths between creases are given

For non-unit interval creases…

Goodness = the number of paper layers at a crease?



How can we count the paper layers?

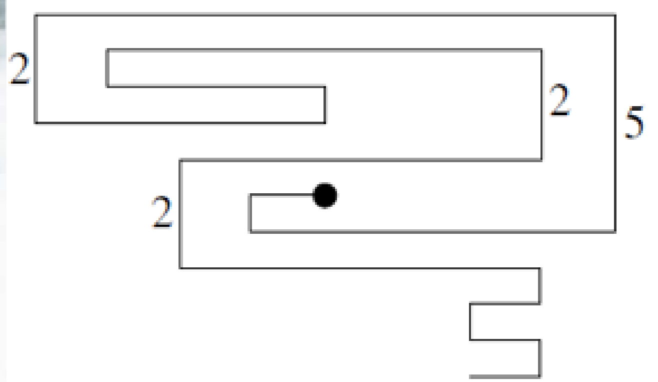
For non-unit interval creases…

Goodness = the number of paper layers at a crease?

We introduce three new “widths” of a folded state:

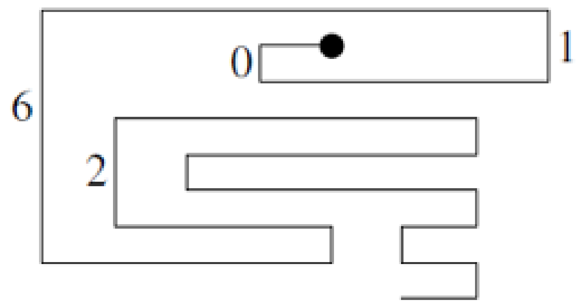
For VMVMVVMMMM, e.g., we have;

Minimum *max crease width*



$h = 11$, $m = 5$, $t = 11$

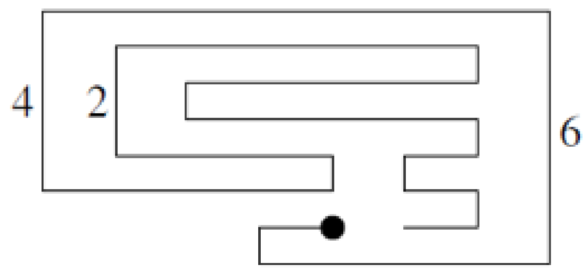
Minimum *total crease width*



$h = 9$, $m = 6$, $t = 9$

Minimum **height**

New



$h = 8$, $m = 6$, $t = 12$

Main results

Computational Complexities of new problems

	Unit interval model in [Umesato, et.al TCS, 2012]	General model in this talk
max crease width	NP-complete ⇒	NP-complete
total crease width	open	NP-complete [this talk]
height	trivial	NP-complete [this talk]

Proof Idea

FPT algorithm: "If a folded state with height ≦k?" can be checked in $O(2^{O(k \log k)} n)$ time.

Minimize height is NP-complete

Proof: Polynomial time reduction from 3-Partition.

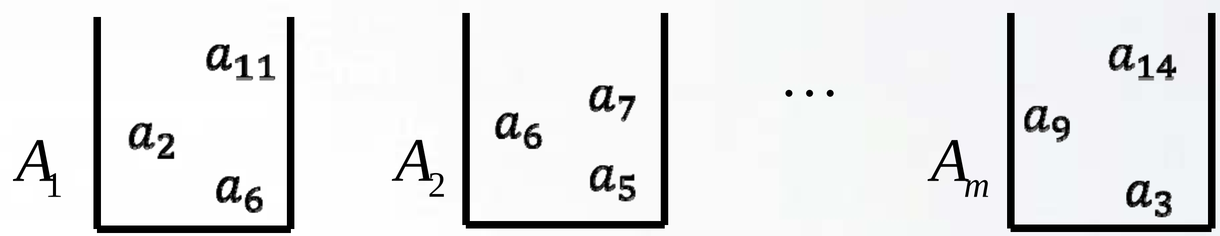
3-Partition:

$(B/4 < a_j < B/2)$

Input: Set of integers $A = \{a_1, a_2, \ldots, a_{3m}\}$ and integer B

Question: Is there a partition of A to $A_1, \ldots, A_m$ such that $|A_i|=3$ and $\sum_{a_j \in A_i} a_j = B$

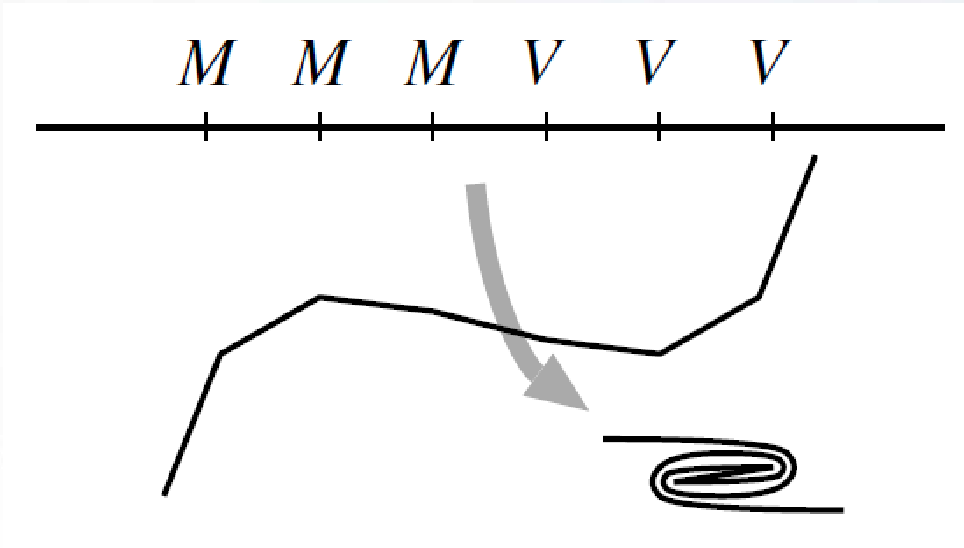
$A = \{a_1, a_2, \ldots, a_{3m}\}$



Minimize height is NP-complete

Proof: Polynomial time reduction from 3-Partition.

Basic gadget

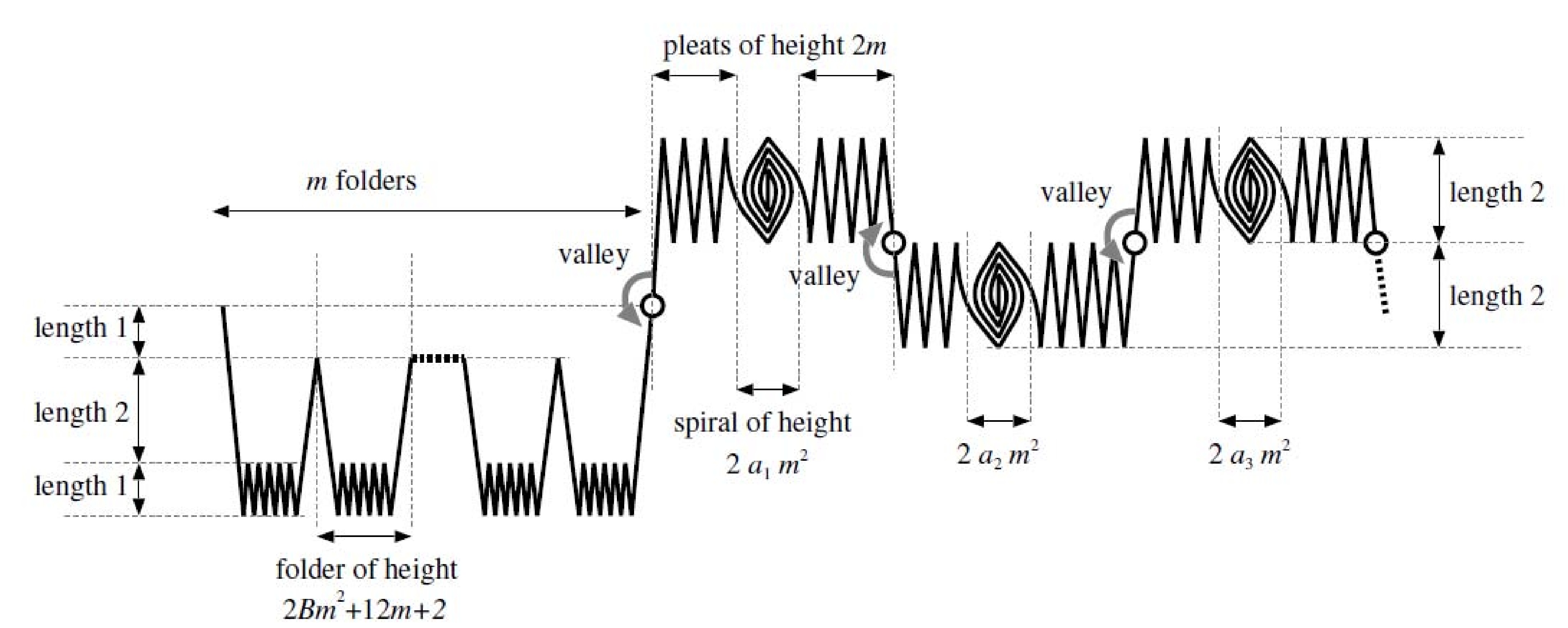


The way of folding is unique by bit longer end-edges

Minimize height is NP-complete

Proof: Polynomial time reduction from 3-Partition.

Overview



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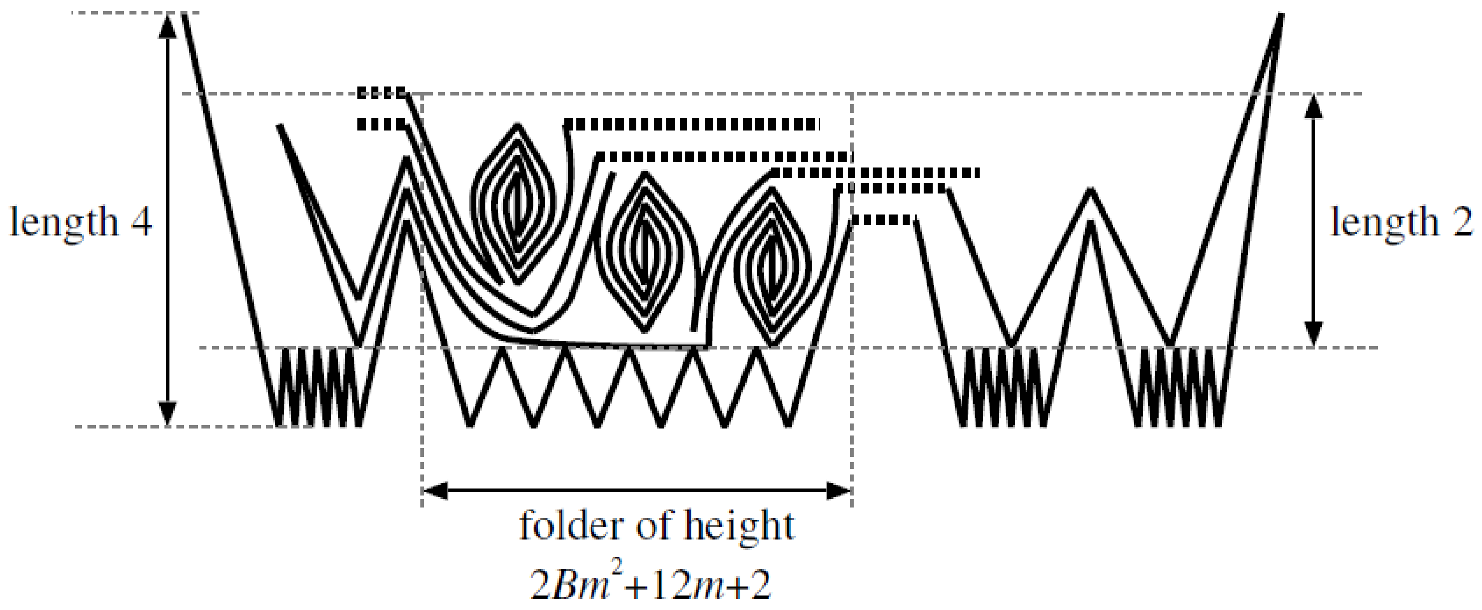
Proof: Polynomial time reduction from 3-Partition.

<u>Overview</u>



Summary

	Unit interval model in [Umesato, et.al TCS, 2012]	General model in this talk
max crease width	NP-complete ⇒	NP-complete
total crease width	open	NP-complete [this talk]
height	trivial	NP-complete [this talk]

FPT algorithm: height ≦k? can be checked in $O(2^{O(k \log k)} n)$ time.

Future work:
- Replace "open" into ???
- Extension to 2 dimension
  - Different measures of "thickness"?
- Estimation of the way of folding (~time complexity)
- "*Time-space trade off*" for computational origami