Computational Origami

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Today…

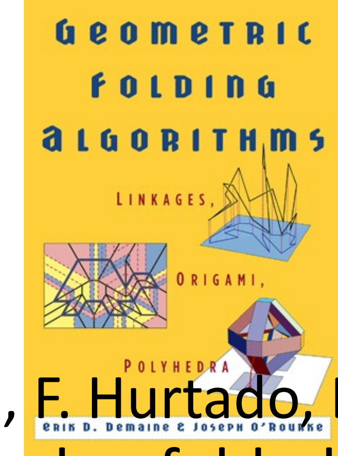
1. Basic facts for unfolding
2. Polygons foldable two or more boxes
3. Common unfolding of regular polyhedra (or Platonic solids)
   1. Common unfolding of a regular tetrahedron and a cube
   2. Common unfolding of a regular tetrahedron and Johnson-Zalgaller solids

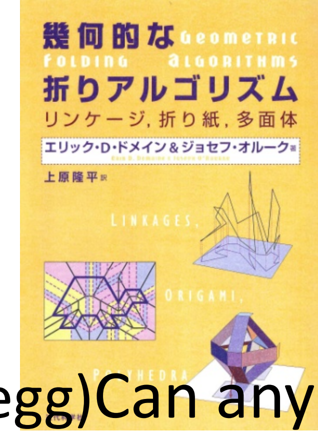
Construction of Common Unfolding of a Regular Tetrahedron and a Cube

Reference:

- Toshihiro Shirakawa, Takashi Horiyama,and Ryuhei Uehara: On Common Unfolding of a Regular Tetrahedron and a Cube, *Japan Conference on Discrete and Computational Geometry (JCDCG 2011)*, 2011/11/28-29, Tokyo, Japan.
- 正4面体と立方体の共通の展開図に関する研究(On Common Unfolding of a Regular Tetrahedron and a Cube)，白川俊博・堀山貴史・上原隆平，『折り紙の科学』，Vol.4, No.1, pp.45-54, 2015.

Introduction

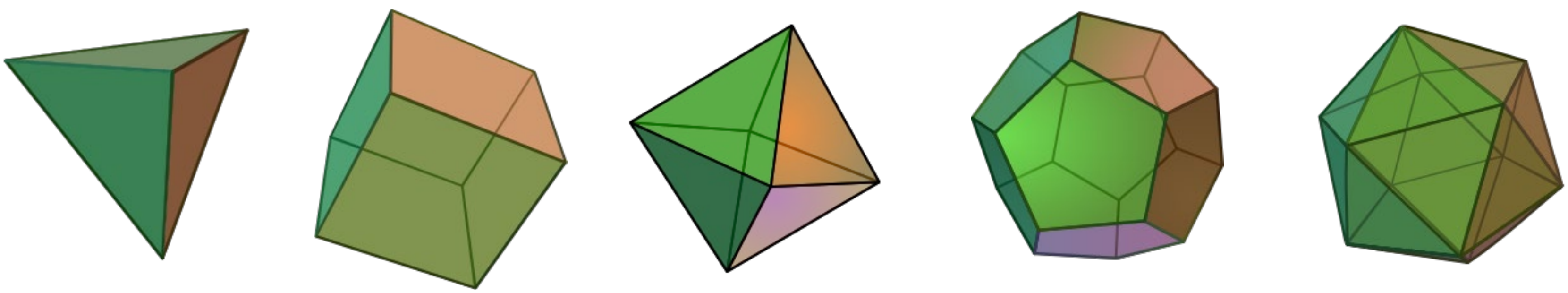




- Open problem 25.6

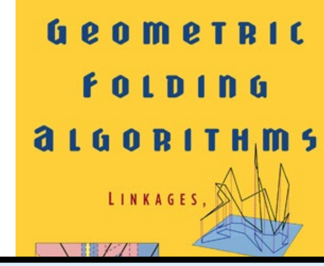
  (by M. Demaine, F. Hurtado, E. Pegg)Can any **Platonic solid** be cut open and unfolded to a polygon that may be refolded to a **different Platonic solid** ?

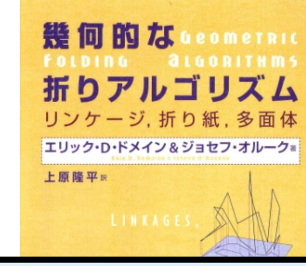
For ex., may a cube be so dissected to a tetrahedron ?



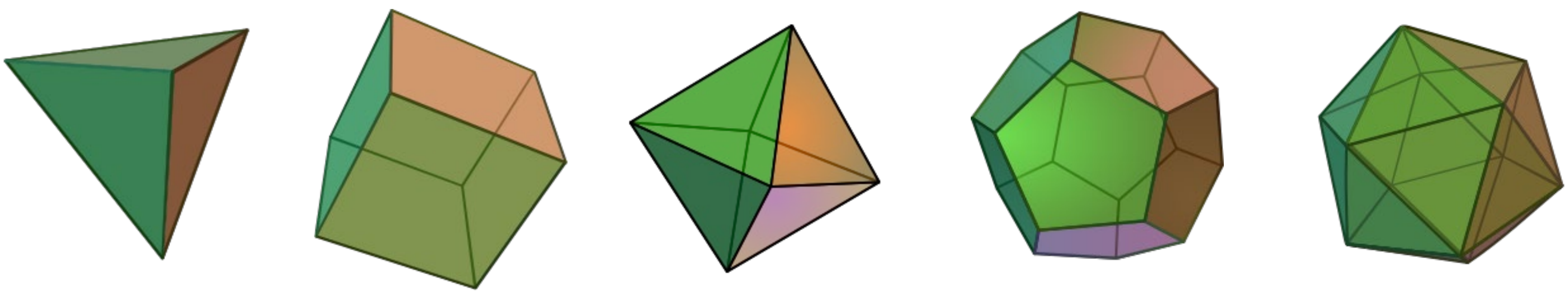
Tetrahedron　Cube (Hexahedron)　Octahedron　Dodecahedron　Icosahedron

Introduction





Is there any polygon that can be folded to two (or more) different Platonic solids ?



Tetrahedron | Cube (Hexahedron) | Octahedron | Dodecahedron | Icosahedron

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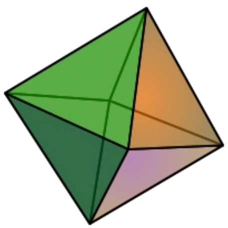
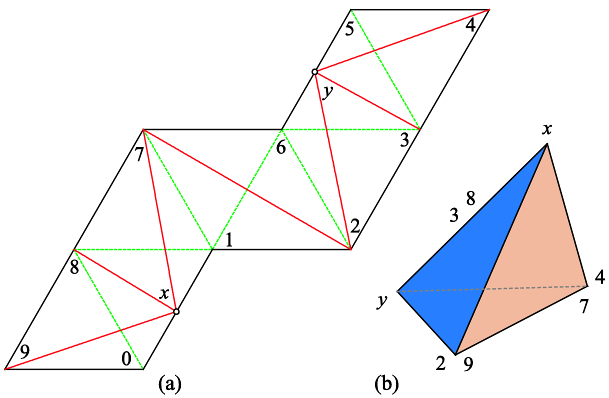
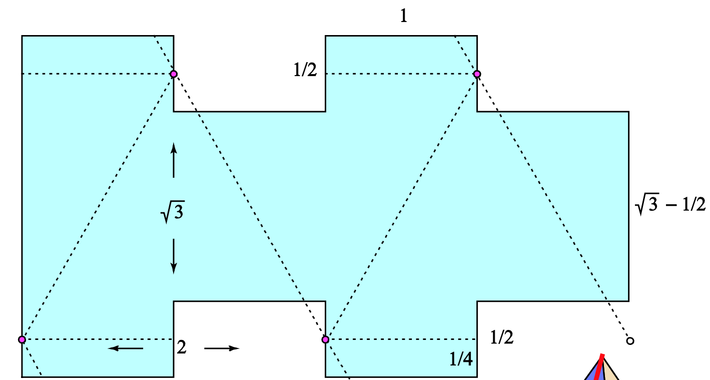
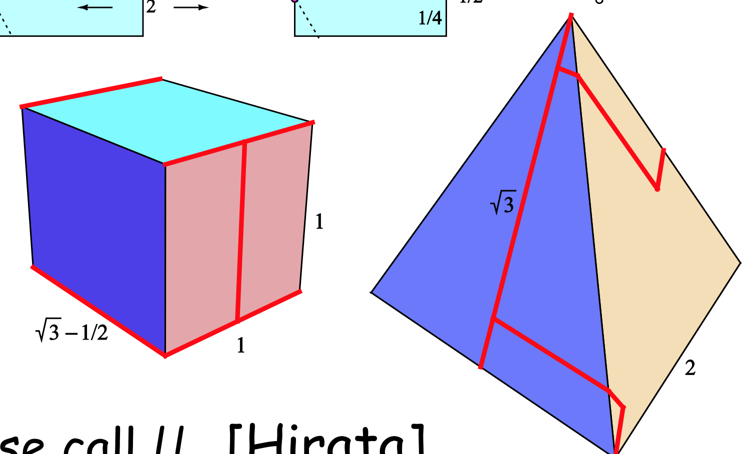




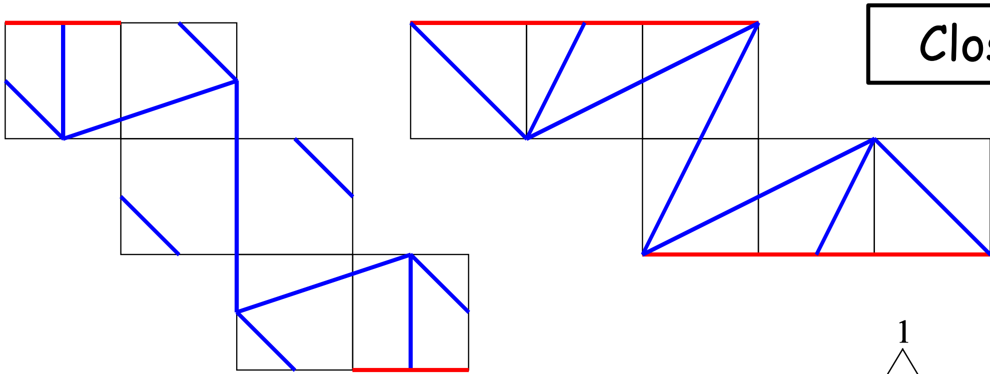
Close call ! [O'Rourke]
Regular Octahedron
⇔ Tetramonohedron
(tetrahedron that consists of congruent triangles)





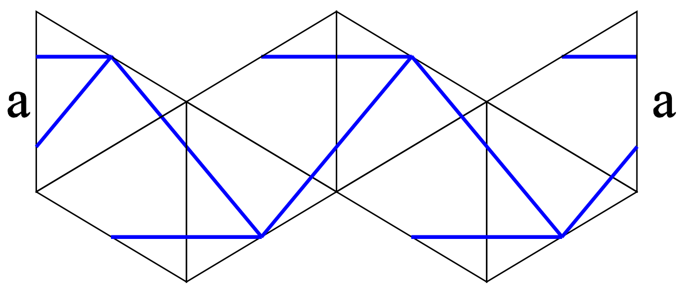
Close call !! [Hirata]
Regular Tetrahedron
⇔ Box 1 x 1 x 1.232

Is there any polygon that can be folded to two (or more) different Platonic solids ?

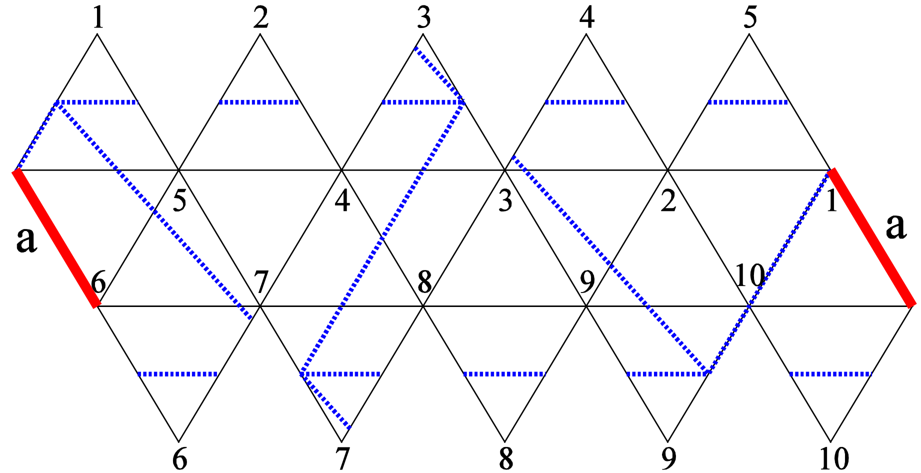


Close calls

Cube (Regular hexahedron)
⇔ Tetramonohedron
1 : 0.972 : 0.972



Regular Octahedron
⇔ Tetramonohedron
1.0072 : 0.9965 : 0.9965



Regular Icosahedron
⇔ Tetramonohedron 1 : 1.145 : 1.25

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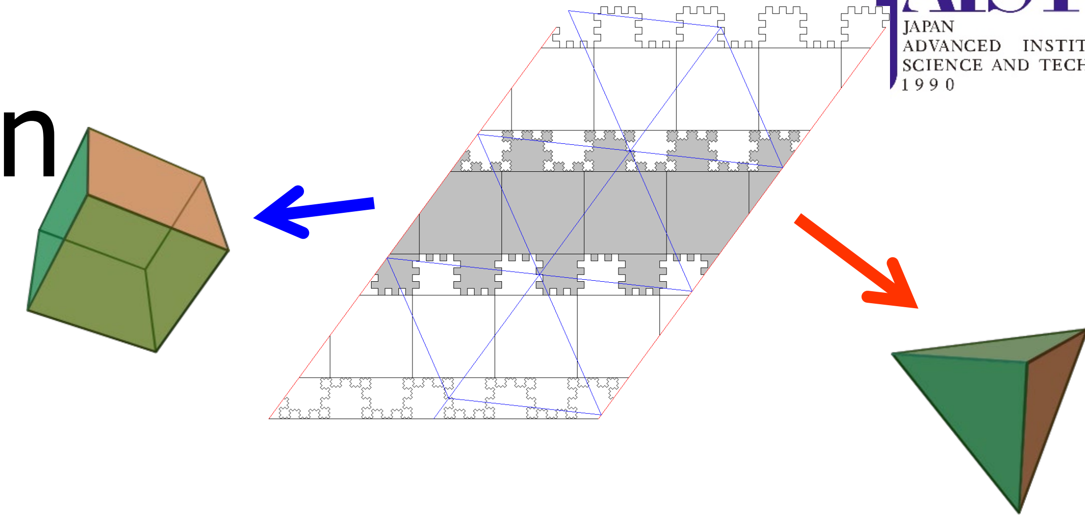
Introduction



Our Results

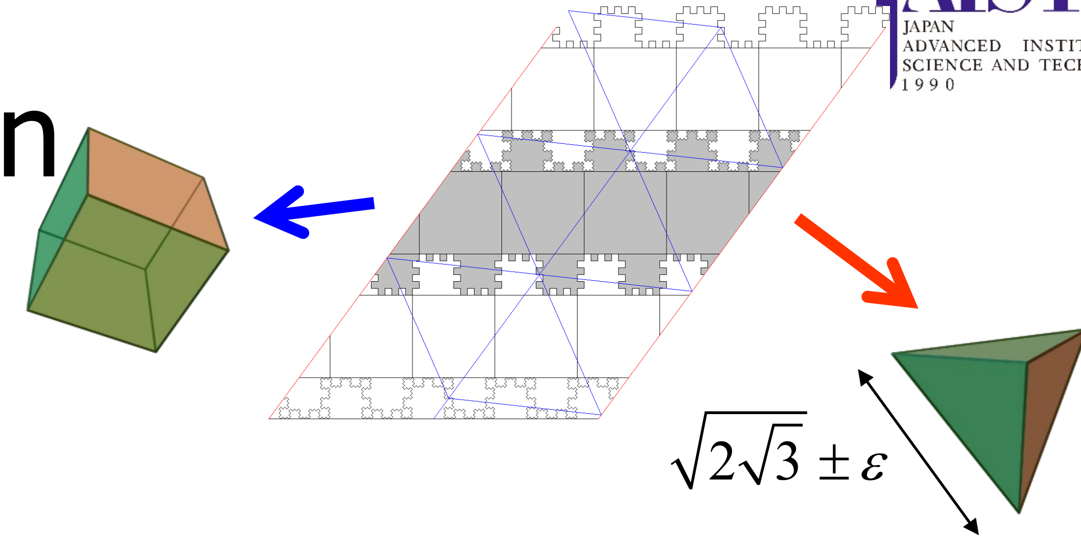
A procedure that produces points s.t.

(1) they seem to converge to a polygon that can fold to a cube and a regular tetrahedron (with infinitely many points)

- we solve the open problem in a sense!
- … but we left some conjectures

(2)

Introduction



Our Results

A procedure that produces points s.t.

(1) they seem to converge to a polygon that can fold to a cube and a regular tetrahedron (with infinitely many points)

(2) they certainly form a polygon that folds to a cube and an almost regular tetramonohedron with error $\varepsilon < 2.89 \times 10^{-1796}$

The Key Theorem

- Theorem [Akiyama 2007, Akiyama Nara 2007]

Let **P** be a development of a **regular tetrahedron**.

Then, **P** is a **tiling**. That is, **P** **fills a plane**.

The Key Theorem

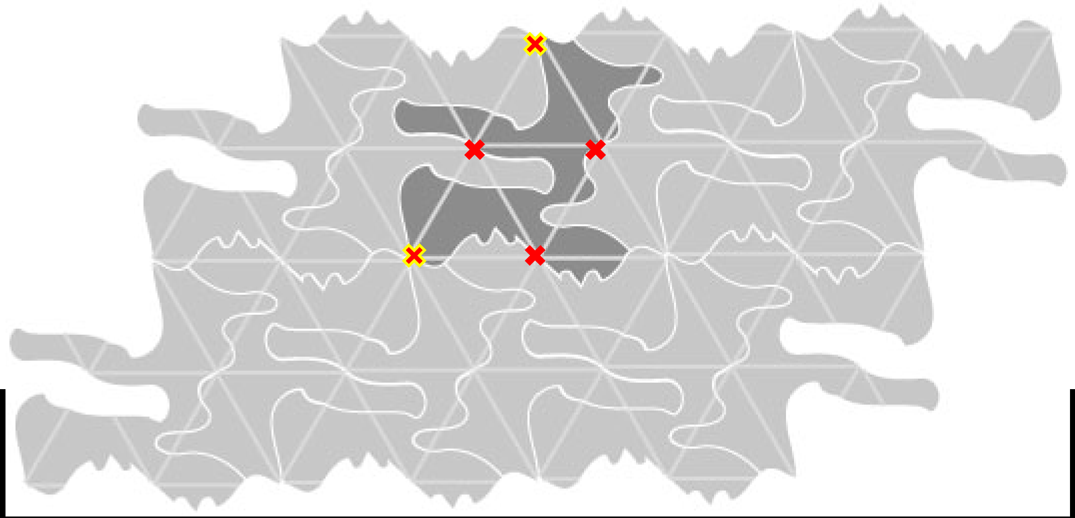
- Theorem [Akiyama 2007, Akiyama Nara 2007]

  P is a development of a **regular tetrahedron** iff

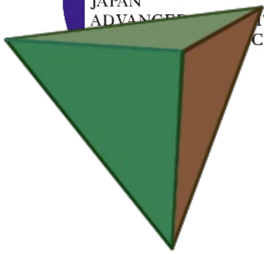
  (1) **P** has a **p2 tiling**, i.e., tiling by 180° rotations

  (2) **4** of the **rotation centers** define the triangular lattice

  (3) no two of the 4 rotation centers belong to the same equivalent class on the tiling



The Key Theorem



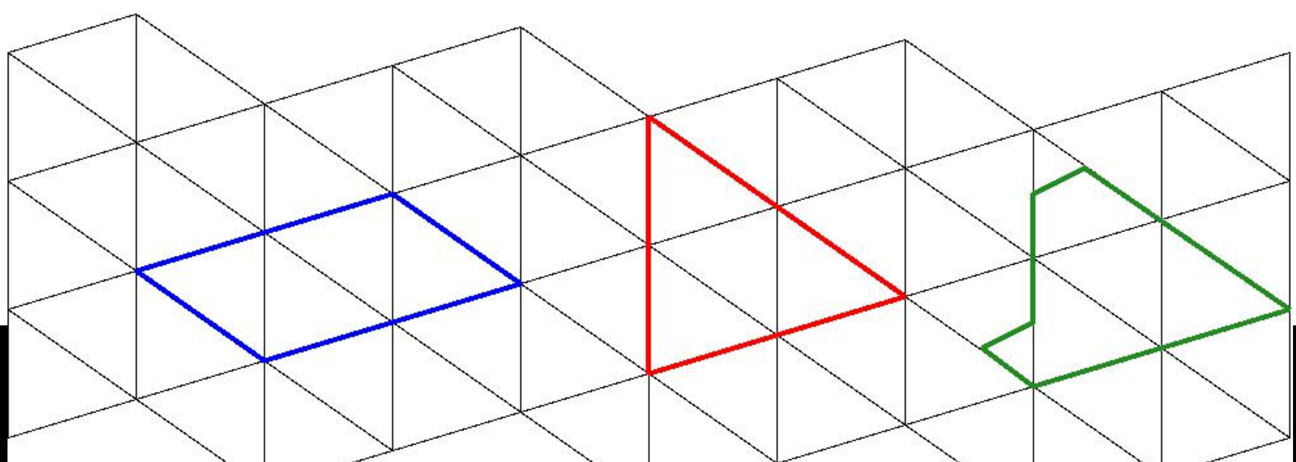
- Theorem [Akiyama 2007, Akiyama Nara 2007]

  P is a development of a **regular tetramonohedron** if

  (1) **P** has a **p2 tiling**, i.e., tiling by 180° rotations

  (2) **4** of the **rotation centers** define a nonregular triangular lattice

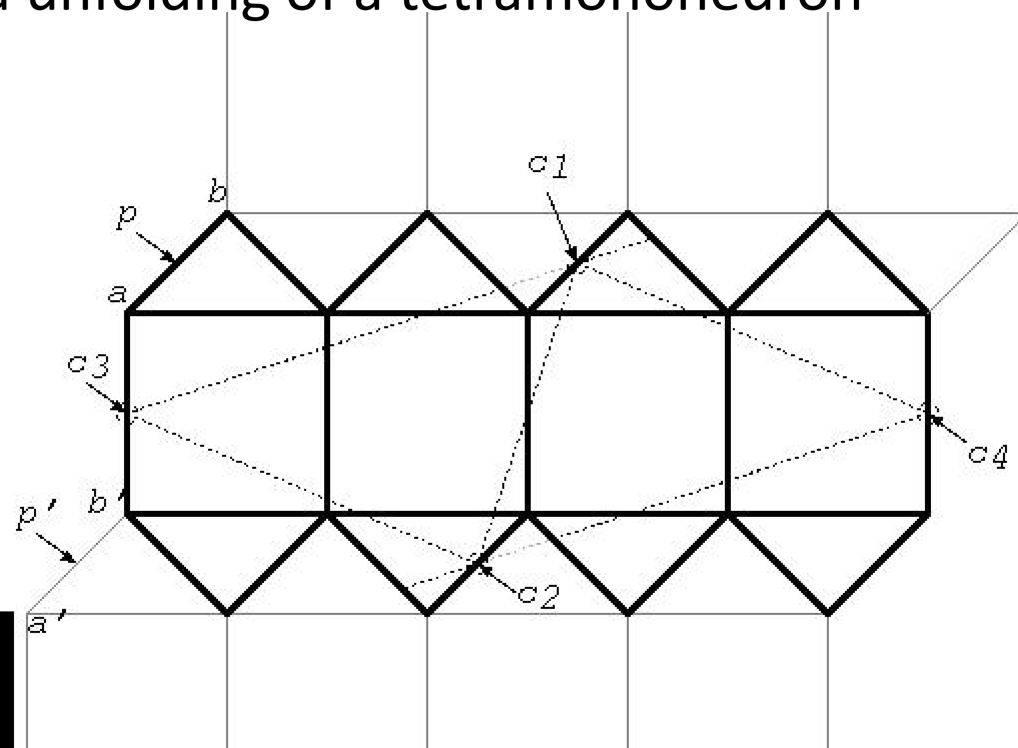
  (3) no two of the 4 rotation centers belong to the same equivalent class on the tiling



Construction of a development

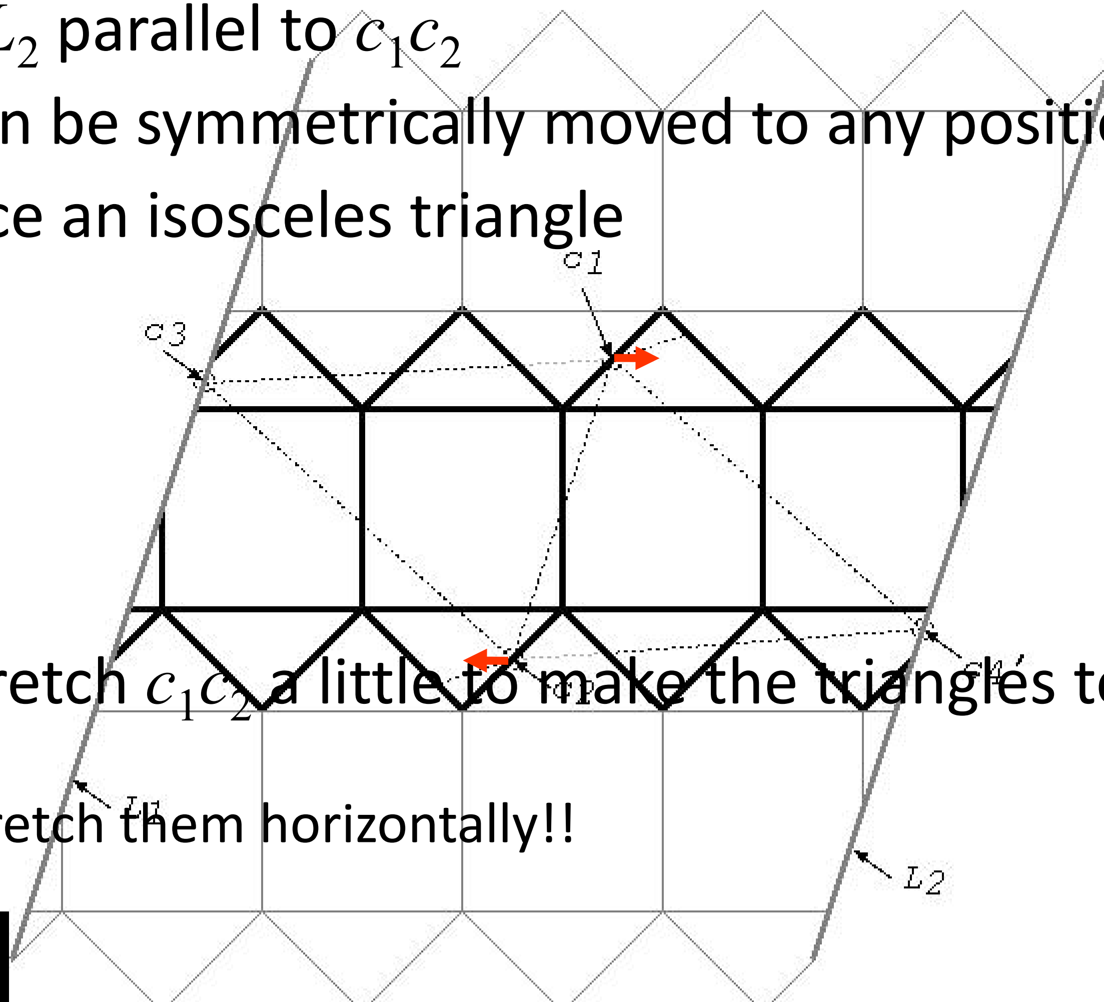
- We modify the unfolding of a cube with the following invariants:
  - it is a development of a cube
  - it is a p2 tiling = a unfolding of a tetramonohedron

Initial unfolding of a cube



Construction of a development

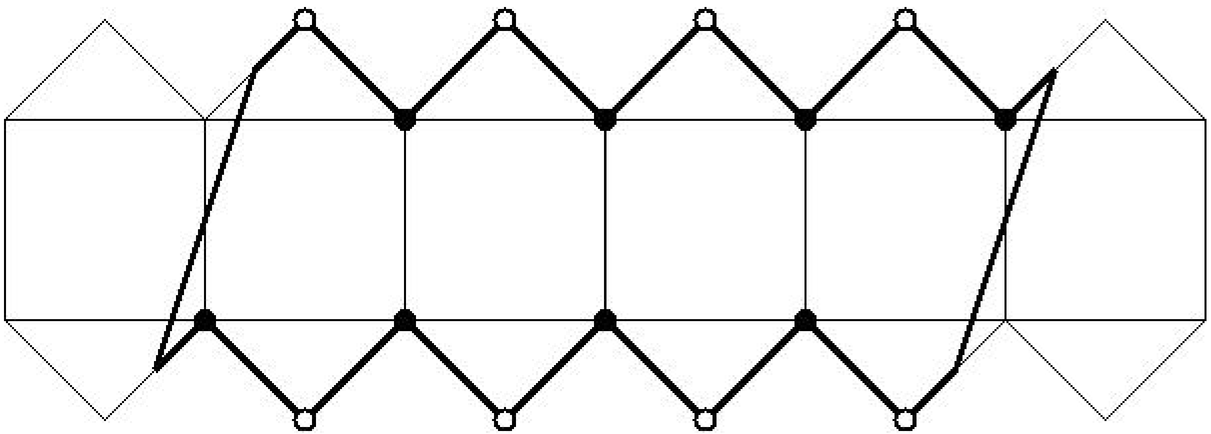
- make L_1 and L_2 parallel to c_1c_2
  … c_3 and c_4 can be symmetrically moved to any position
- make each face an isosceles triangle



- we have to stretch c_1c_2 a little to make the triangles to regular ones.
  - …so we'll stretch them horizontally!!

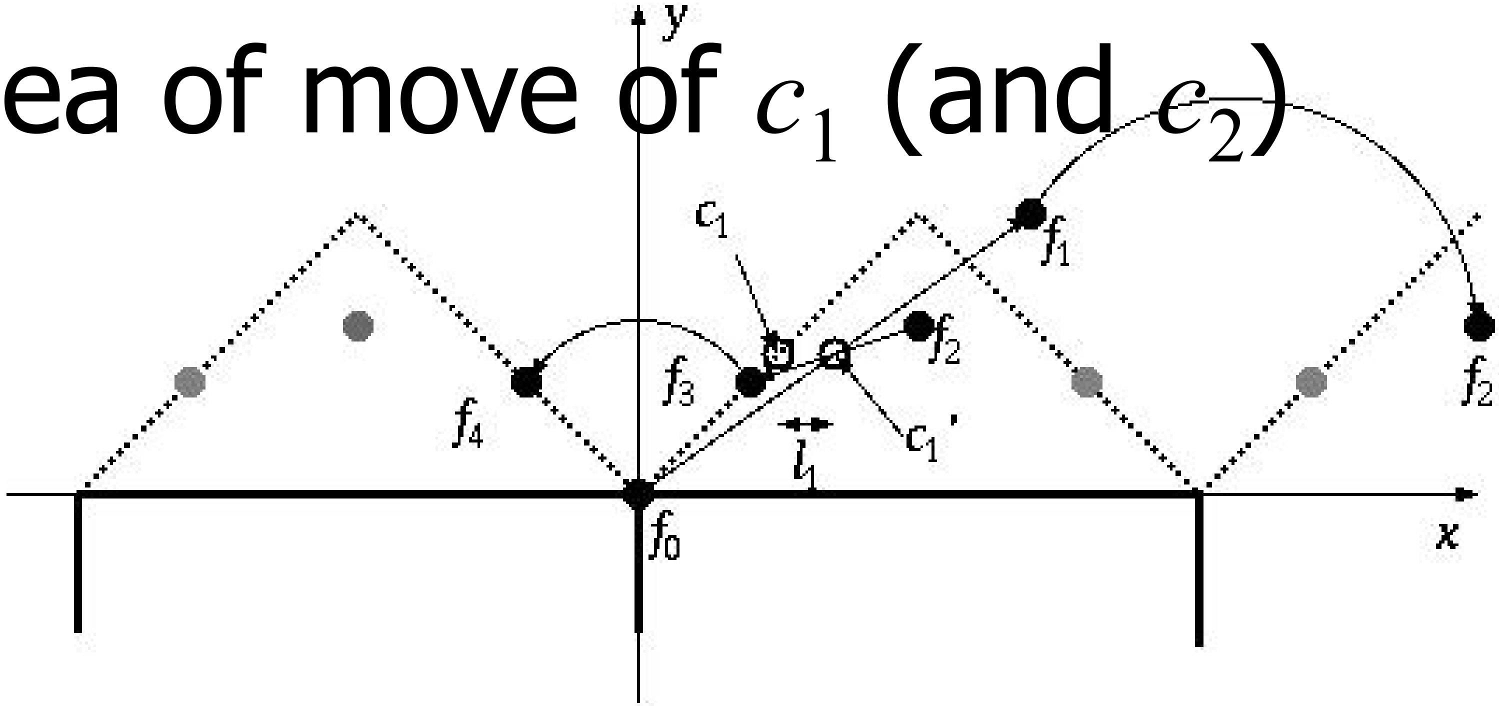
Construction of a development

- We have some “fixed points” that have to be on an edge of the unfolding



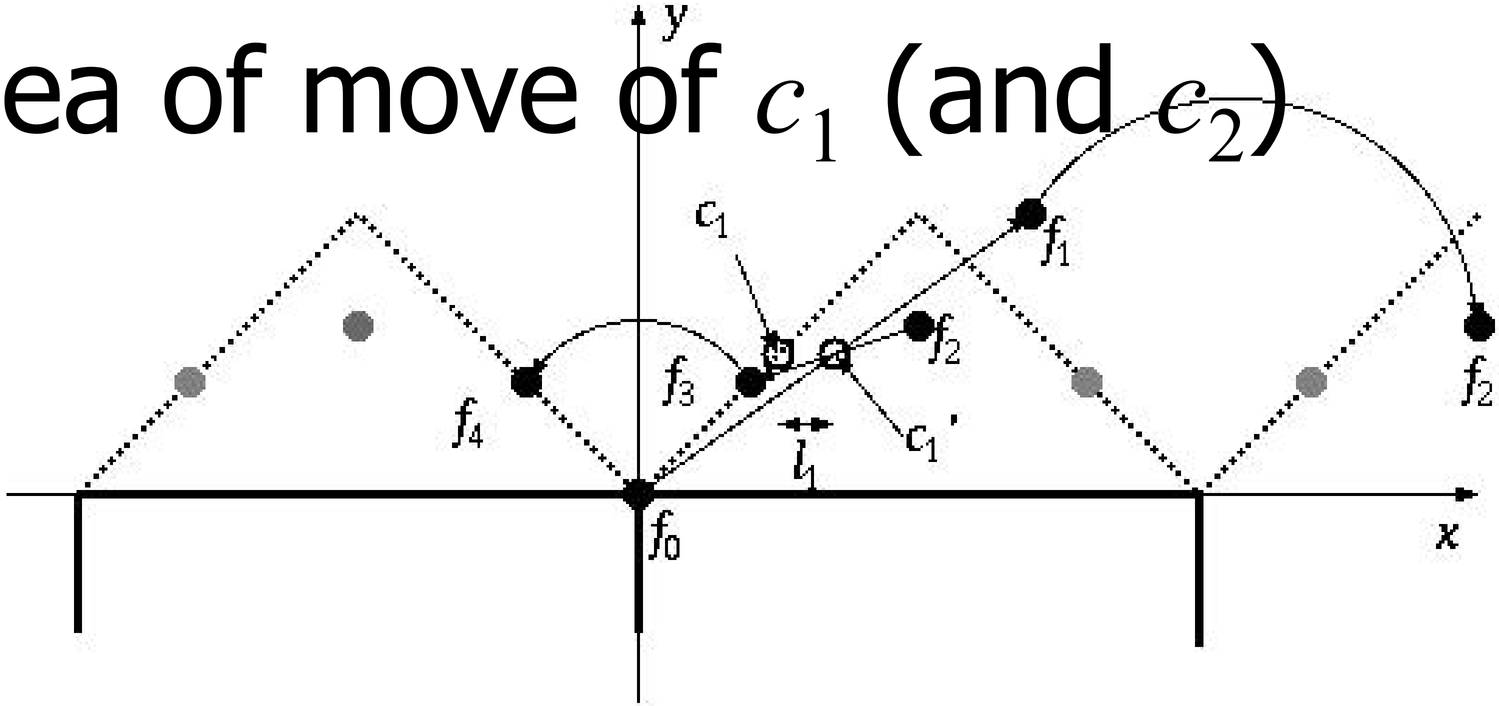
○ : they have to make a center of a top/bottom square
● : they form a vertex of the unit cube

Idea of move of c_1 (and c_2)



- "shift" the center c_1 to c_1' with distance l_1
  - fixed points have their images since they should form a p2 tiling.
  - these images of fixed points generate the unfolding of a cube and a tetramonohedron with center c_1'

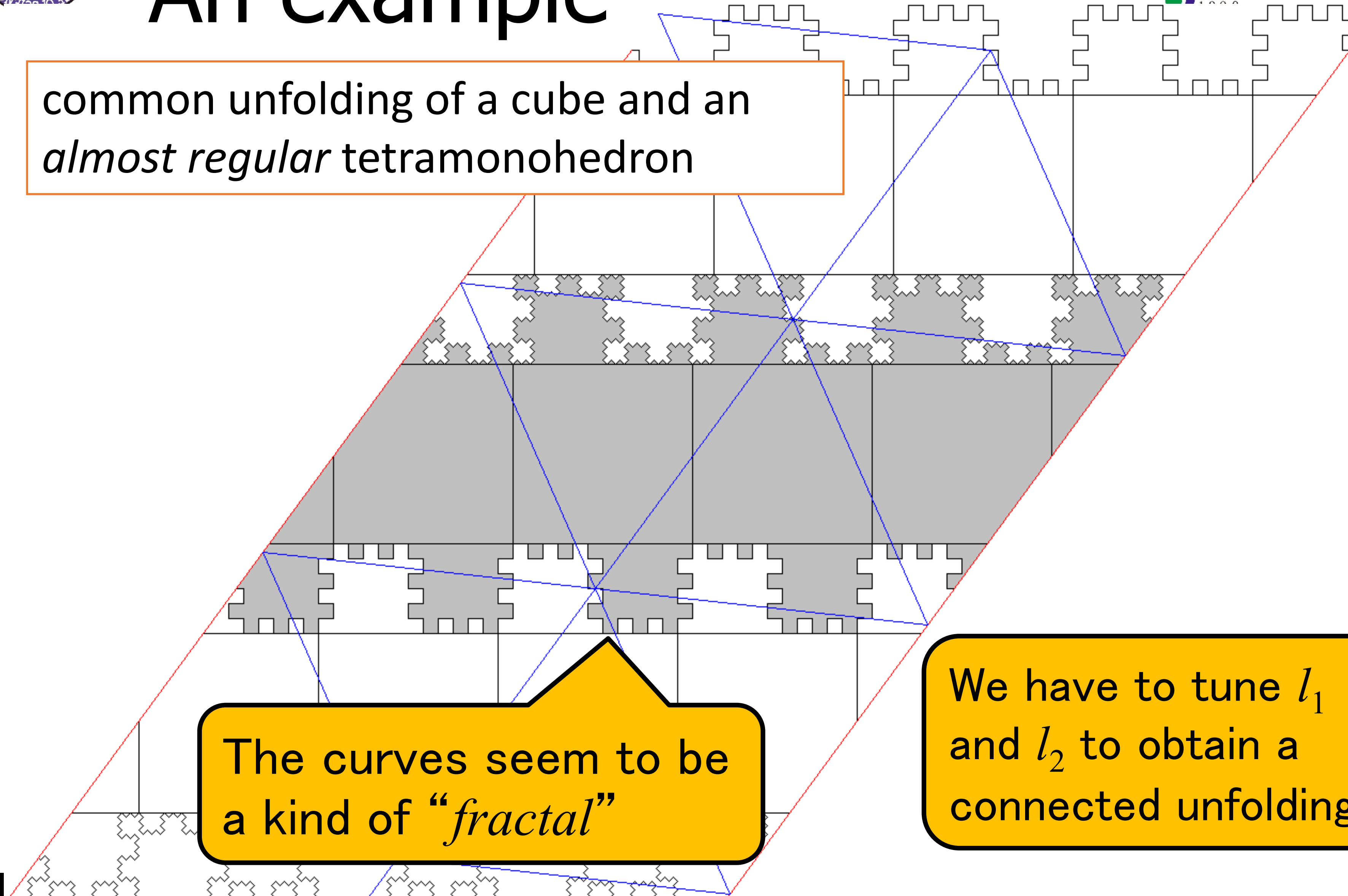
Idea of move of c_1 (and c_2)



- "shift" the center c_1 to c_1' with distance l_1
  - l_1 is rational: finite points form a certain unfolding
  - l_1 is irrational: infinite points converge to an unfolding

An example

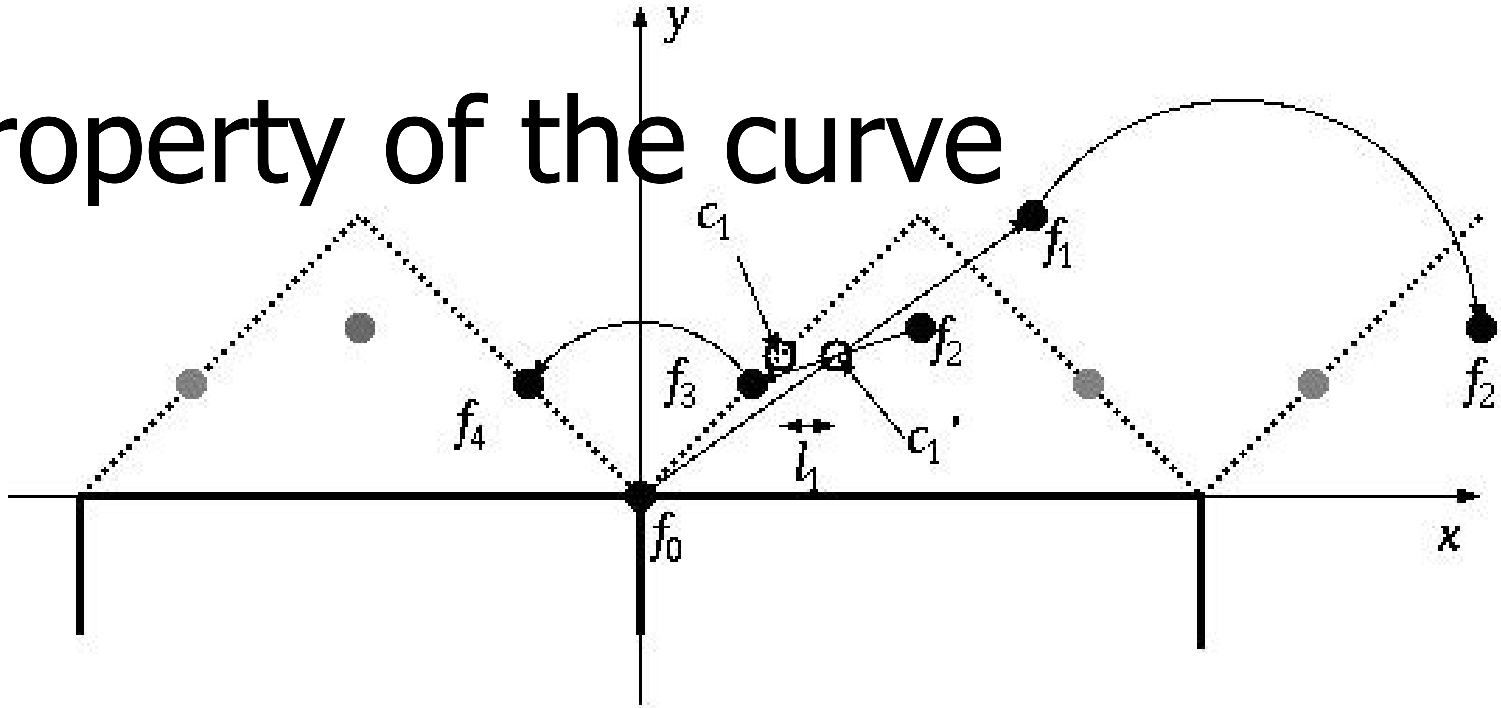
common unfolding of a cube and an *almost regular* tetramonohedron



The curves seem to be a kind of "*fractal*"

We have to tune l_1 and l_2 to obtain a connected unfolding

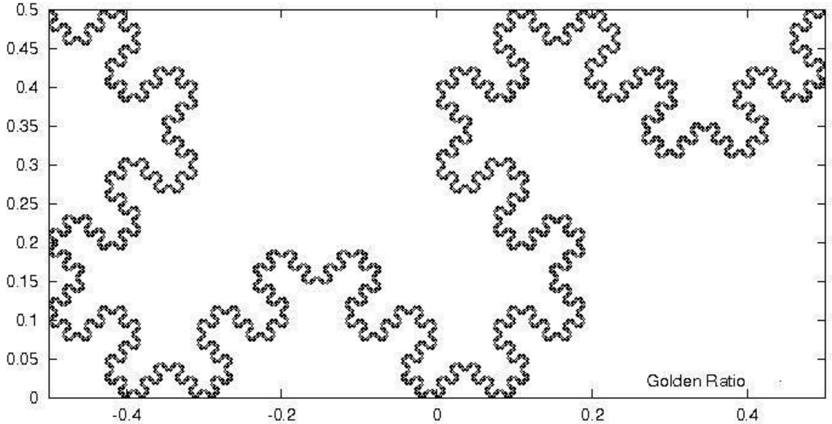
Property of the curve

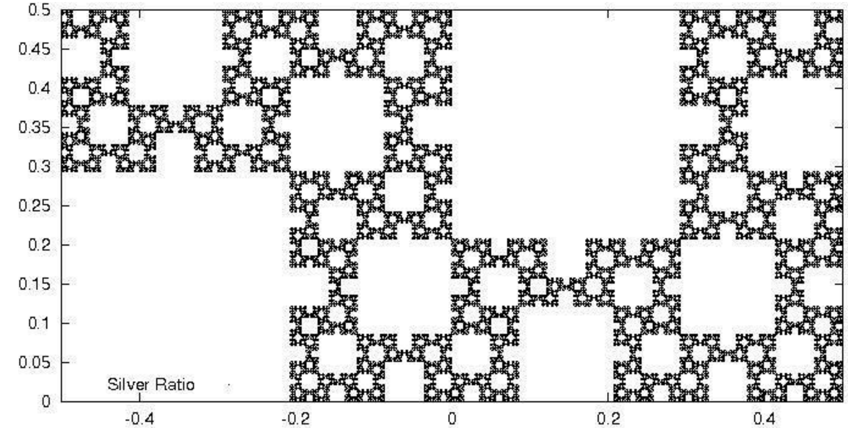


- [Observation/Conjecture] The "fractal curves" are defined by the value of l_1 in continued fraction form

$$l_1 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

Property of the curve





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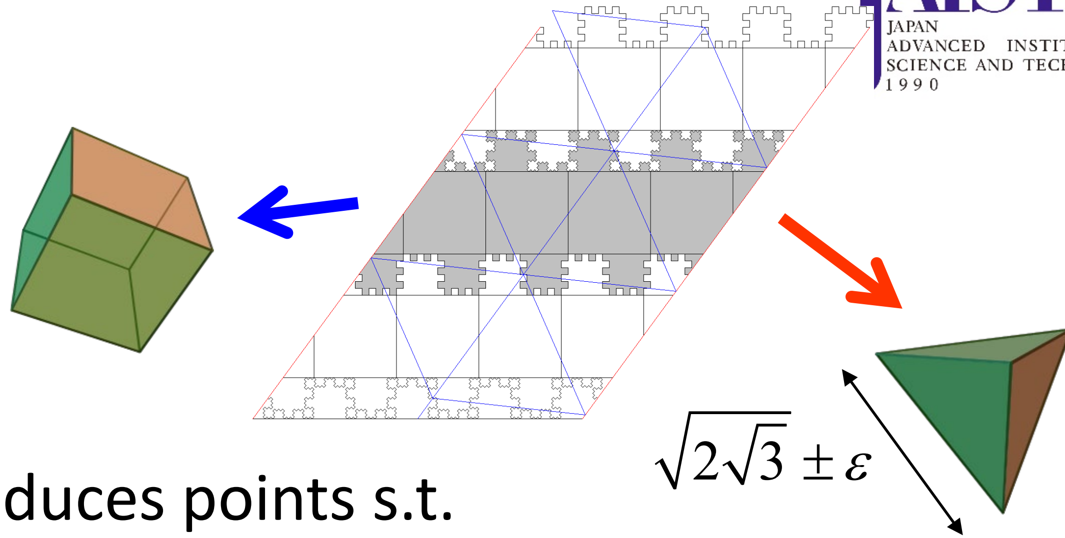
$$l_1 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

Summary

Our Results



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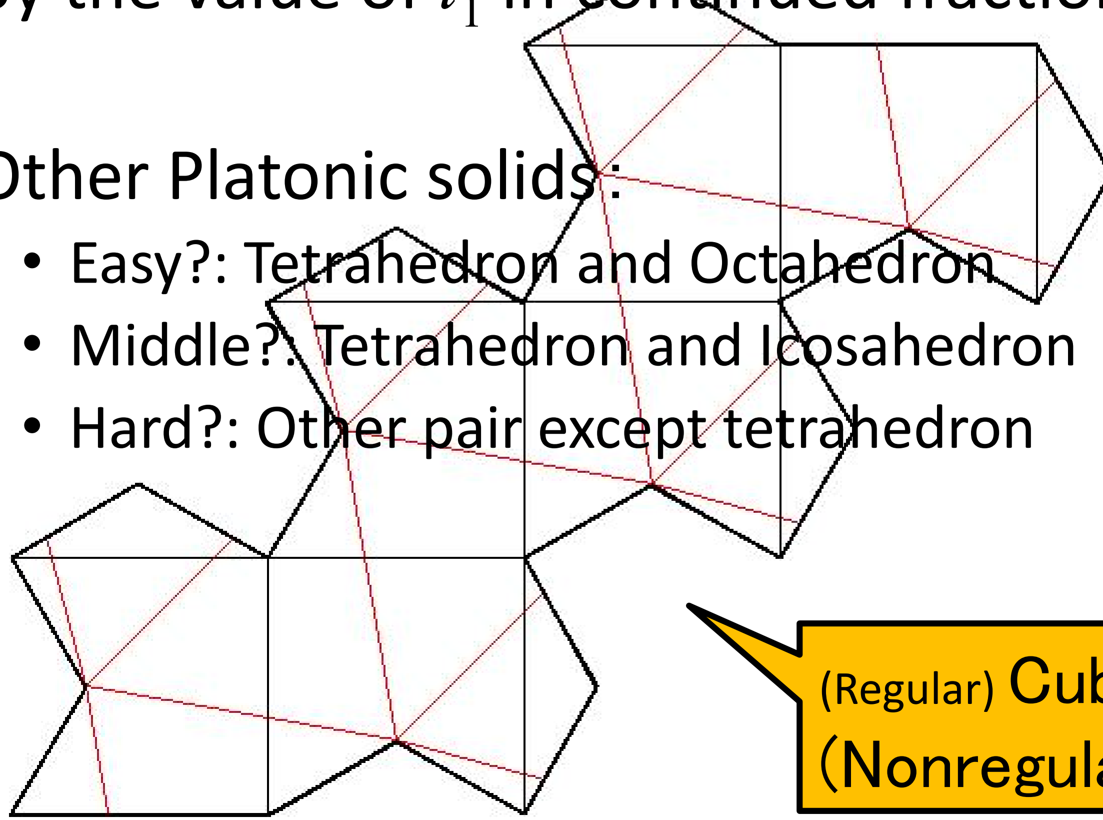
Future work

- [Obser Theorem jecture] The "curves" are defined by the value of l_1 in continued fraction form

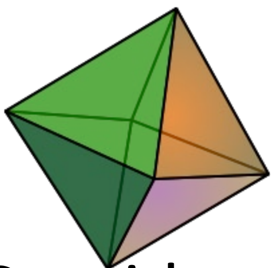
- Other Platonic solids:
  - Easy?: Tetrahedron and Octahedron
  - Middle?: Tetrahedron and Icosahedron
  - Hard?: Other pair except tetrahedron



(Regular) Cube and (Nonregular) Octahedron

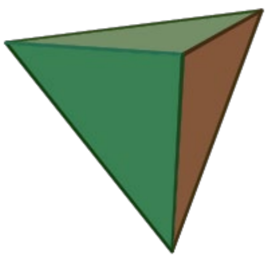
Simple Exercises

- Find "good" common unfolding of a regular octagon or a regular icosahedron AND a tetramonohedron. Estimate how "good" it is.



or

AND



- Consider why a dodecahedron has no common unfolding with a tetramonohedron or a non-regular icosahedron?