Computational Origami

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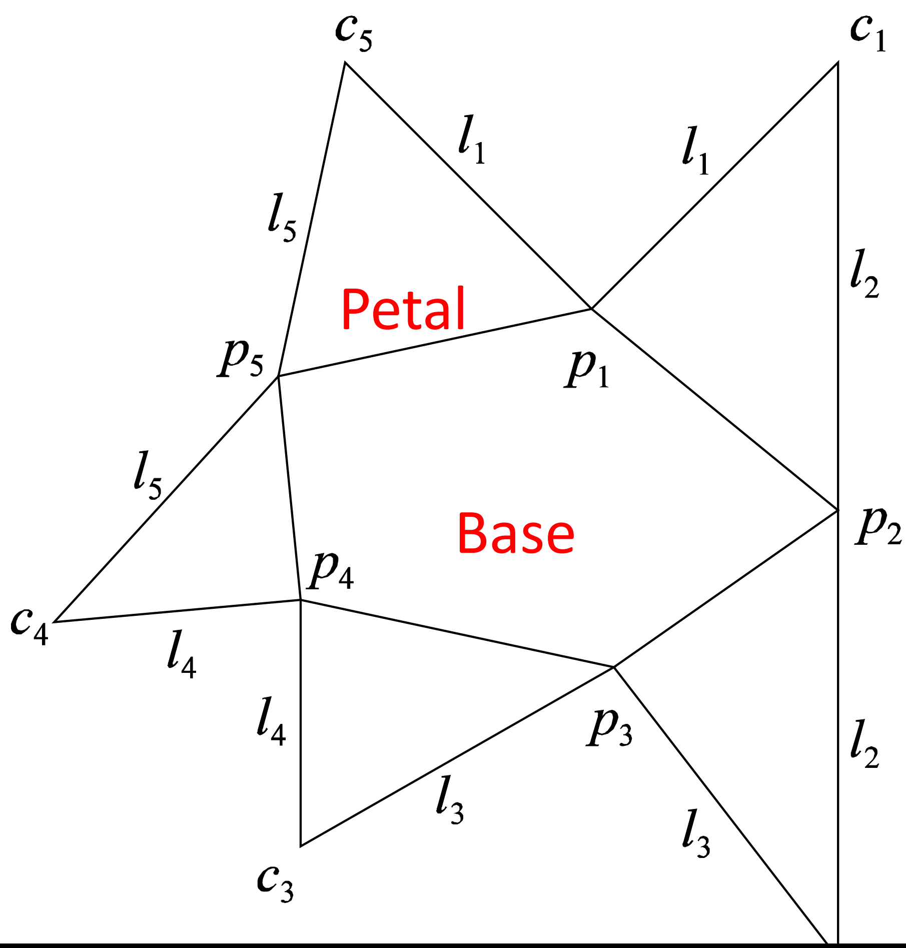
- Schedule
  - January 27 (13:30-15:10)
    - Introduction to Computational Origami
    - Polygons and Polyhedra folded from them
  - January 29 (10:50-12:30)
    - Computational Complexity of Origami algorithms
  - February 3 (9:00-10:40)
    - Advanced topics
      1. **(Bumpy) Pyramid Folding**
      2. Zipper Unfoldability
    - 13:30-15:10 (Office Hour at I67-b)

Bumpy Pyramid Folding

Reference:

Zachary R. Abel, Erik D. Demaine, Martin L. Demaine, Hiro Ito, Jack Snoeyink, and Ryuhei Uehara. Bumpy Pyramid Folding, COMPUTATIONAL GEOMETRY: Theory and Applications, Vol 75, pp. 22-31, December 2018. DOI:10.1016/j.comgeo.2018.06.007

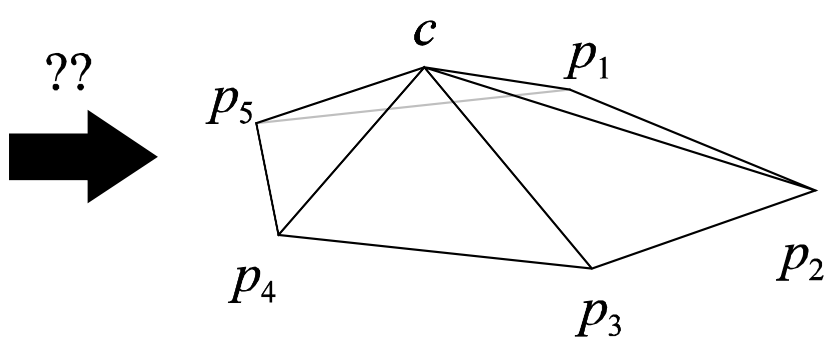
(Bumpy) Pyramid folding problem



Computational Origami

Input:

- Base of convex polygon
- Petals with matching lengths



Output:

it folds into (bumpy) pyramid?

Background story

My old puzzle friend sent me a postal mail…

1. Origami Problem 1:

   Input: triangle with three petals (edges are matching)

   Output: Does it fold to tetrahedron?

   Ans : "YES" if edges are sufficiently long.

2. Origami Problem 2:

   Input: quadrilateral with four petals

   Output : Does it fold to pyramid?

   Obs: In general, we have two ways of folding, and one gives convex, and the other gives concave?

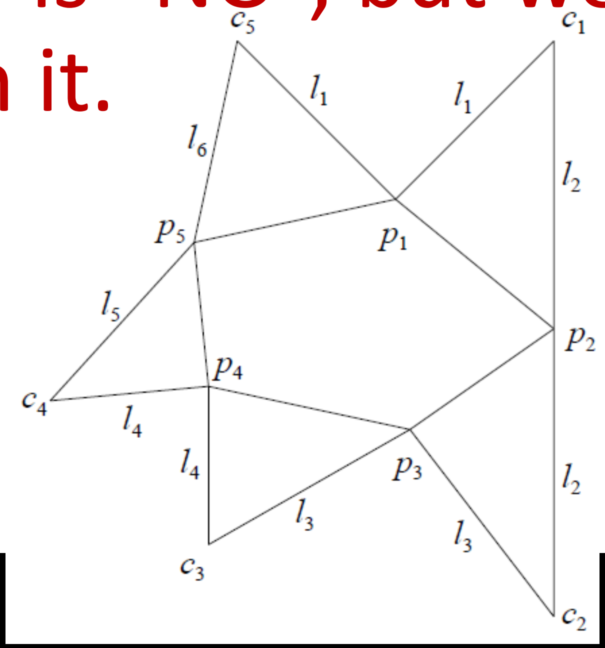
(Bumpy) pyramid folding problem

Input: A convex n-gon with petals

Problem 1: Does it fold to n-gonal pyramid?

In general, the answer is “NO”, but we have some solids from it.

Each triangulation of the base gives distinct convex/concave solid…



(Bumpy) pyramid folding problem

Input: A convex n-gon with petals

Problem 2: In the case of "NO,"

Problem 2-1: Can we have a convex polyhedron?

Problem 2-2: Can we find a polyhedron with maximum volume?

Meta-Problem 2: Can the solutions of 2-1 and 2-2 be different?

Meta^2-Problem 2 : When can we have the same solution for 2-1 and 2-2?

Related Results

1. Alexandrov's Theorem (1941)

For every convex polyhedral metric, there exists a unique polyhedron (up to a translation or a translation with a symmetry) realizing this metric.

- Polynomials for its volume: Sabitov 1998.
- Constructive proof: Bobenko, Izmestiev 2008.
- Poly-time algorithm: Kane, et al. 2009,

…which runs in $O(n^{456.5})$ time.

⇒Our problem is a *special case* in the framework

Observation 1

Origami Problem 1:

Input: A triangle with petals

Question: Does it fold to triangular pyramid?

Answer: “Yes” if edges match, and “long enough”

…When it has a positive volume by Sabitov’s polynomial!!

$$V^2 = \frac{1}{144}[l_1^2 l_5^2 (l_2^2 + l_3^2 + l_4^2 + l_6^2 - l_1^2 - l_5^2) + l_2^2 l_6^2 (l_1^2 + l_3^2 + l_4^2 + l_5^2 - l_2^2 - l_6^2) + l_3^2 l_4^2 (l_1^2 + l_2^2 + l_5^2 + l_6^2 - l_3^2 - l_4^2) - l_1^2 l_2^2 l_4^2 - l_2^2 l_3^2 l_5^2 - l_1^2 l_3^2 l_6^2 - l_4^2 l_5^2 l_6^2].$$

Observation 2

Origami Problem 2:

Input: An quadrilateral with petals

Question: Does it fold to quadrangular pyramid?

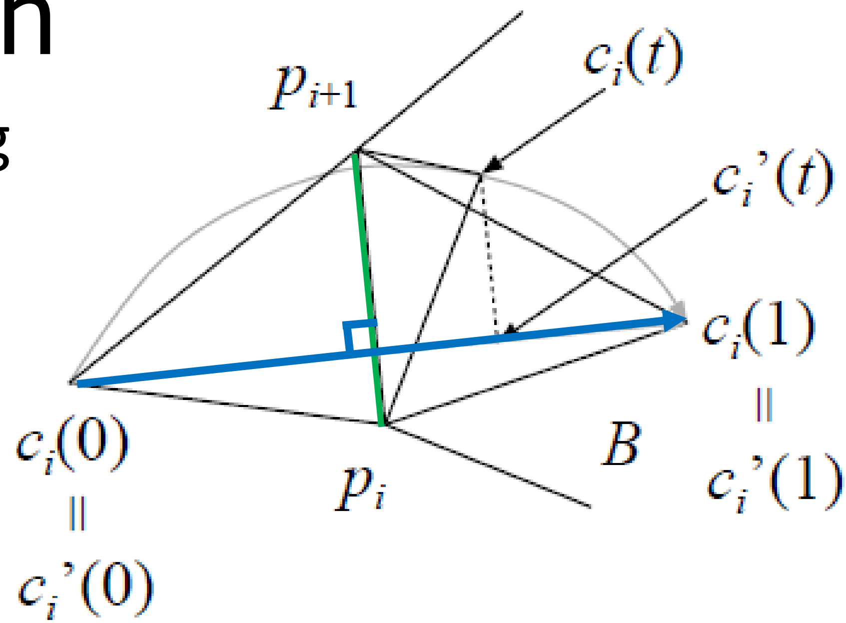
Obs: We can obtain two triangular pyramids if we split along one of two diagonals

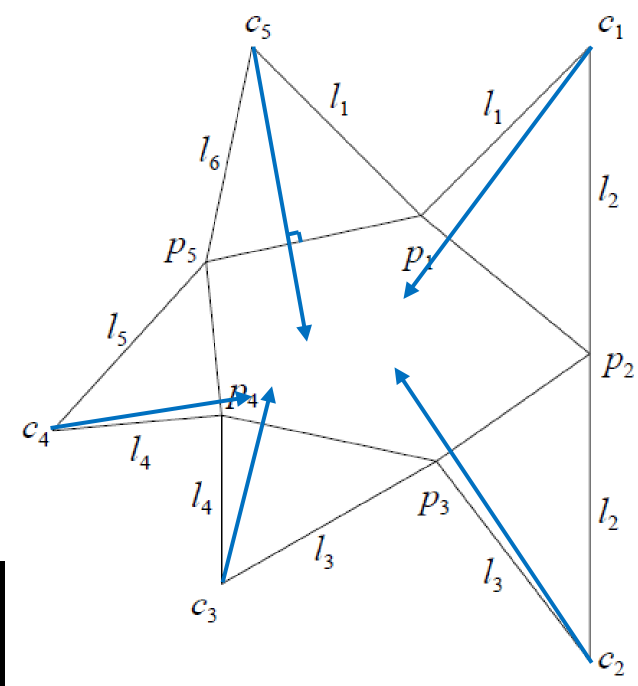
1. They have the same volume in 2 ways: Pyramid!!
2. They have different volumes: One is convex and the other is concave.
3. (Sometimes we have only convex one)
4.

It seems to be difficult to go further by only Sabitov…

Crucial Observation

- When you fold a petal along an edge of the base, the locus of the vertex follows the perpendicular of the edge of the base, and reach to “the other side”





⇒We can compute the loci of the vertices when you fold a pyramid (with the flat base)

(Bumpy) pyramid folding problem

Problem 1:

Input: A convex n-gon with petals

Question: Does it fold to n-gonal pyramid?

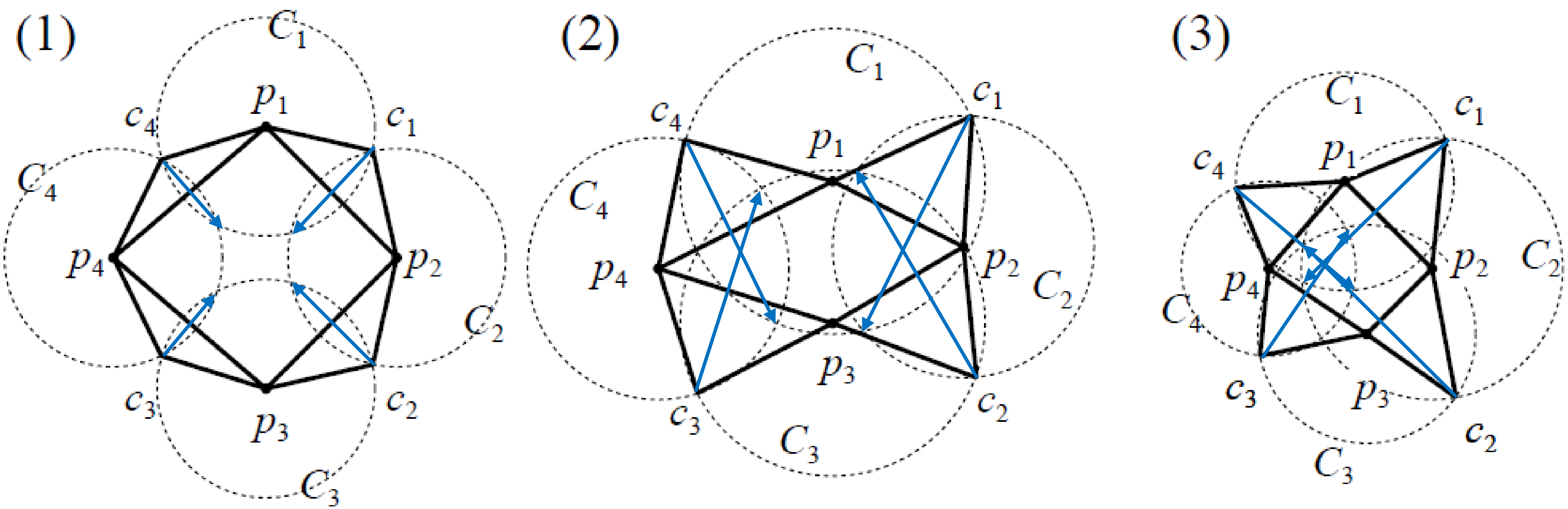
[Ans] YES iff all perpendiculars meet at one point, which corresponds to the apex.

(+each vertex has enough height)

It can be computed in linear time.

Hereafter, we consider the general case, i.e., the base should be “bumpy”

Case study: n=4



(1) Petals are too short, so no pyramid cannot be folded

(2) Petals are bit short, so only one convex bumpy pyramid can be folded

(3) Petals are enough long, so one convex bumpy pyramid and another concave one can be folded

(Bumpy) pyramid folding problem

Problem 2:

Input: A convex n-gon with petals

Assumption: It cannot fold to n-gonal pyramid

Q2-1: Does it fold to convex polyhedron?

Q2-2: find polyhedron with max. volume.

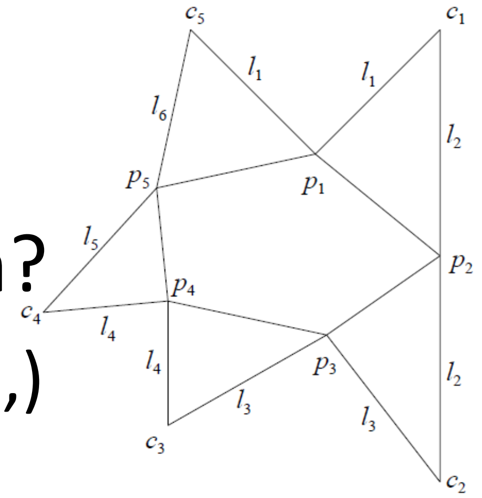
[Ans] Poly-time solvable. — Simple DP: $O(n^3)$

MetaQ2-1: Two answers of Q2-1 and Q2-2 are different?

[Ans] Yes. Concave can bigger than convex!

MetaQ2-2: When? …Don’t know…

(Bumpy) pyramid folding problem



Input: A convex n-gon with petals

Q2-1: it folds to convex polyhedron?

[Theorem] (If petals are sufficiently long,)

1. Always "YES"
2. Folding way (order of gluing petals)

can be computed in linear time

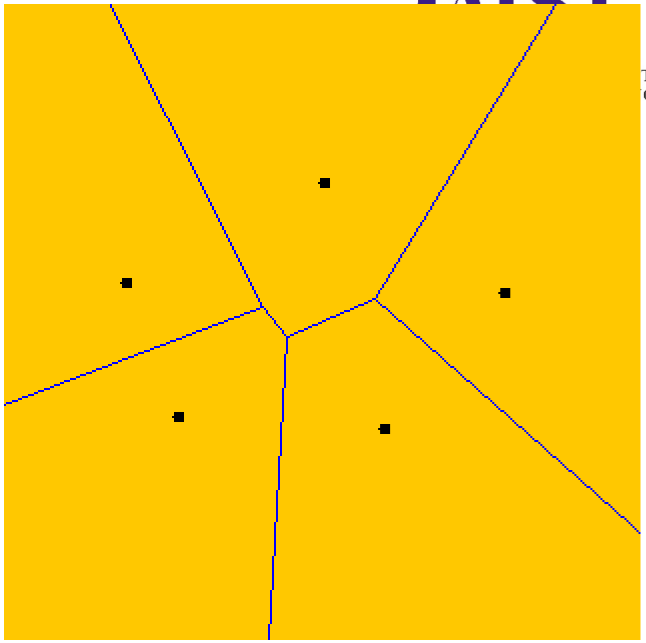
[Core idea of proof]

The folding way corresponds to the

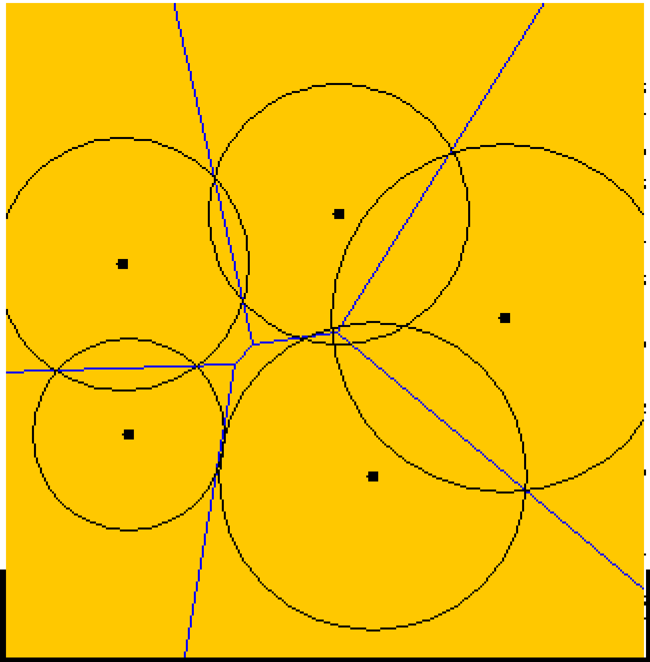
Power Diagram (generalized Voronoi diagram)!

Power Diagram

- Voronoi Diagram:
  perpendicular bisector
  for each pair of points



- Power Diagram:
  Each vertex has its own
  "weight"="power"



Note : When points are convex, the diagram forms a tree, or acyclic.

http://pages.cpsc.ucalgary.ca/~laneb/Power/

(Bumpy) pyramid folding problem

Input: A convex n-gon with petals

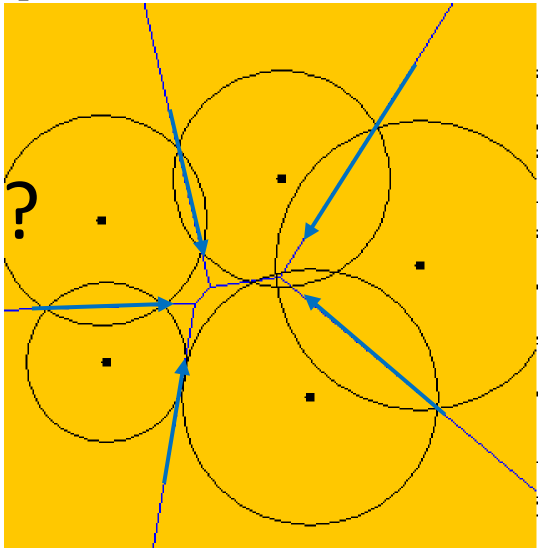
Q2-1: it folds to convex polyhedron?

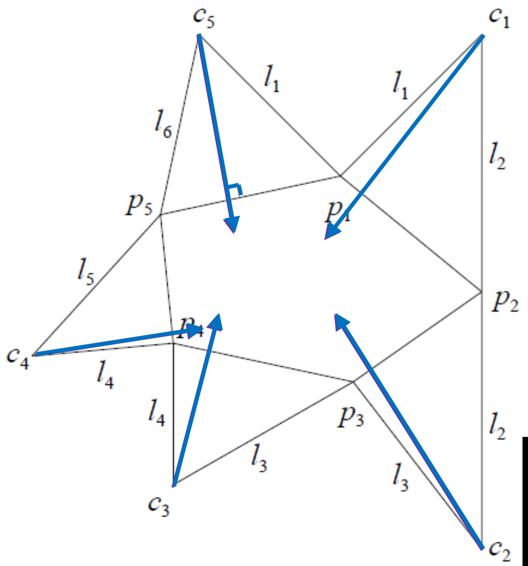
[Theorem] Always "YES"

[Core idea of proof]

The folding way can be computed by using Power Diagram

1. Each p_i of the base is "point"
2. Each l_i is "weight/power"
3. Power Diagram (or tree) gives the loci of c_i and new generated vertices!





Summary & Future work

Input: A convex n-gon with petals

Q1: Does it fold to n-gonal pyramid?

[Ans] Solvable in linear time.

In general, …

Q2-1: Does it fold to convex polyhedron?

[Ans] Always "YES", and folding way can be found in linear time.

Q2-2: find polyhedron with max. volume.

[Ans] $O(n^3)$ time DP algorithm.

Summary & Future work

Input: A convex n-gon with petals

MetaQ2-1: Two answers of Q2-1 and Q2-2 are different?

[Ans] Yes. Concave can bigger than convex!

MetaQ2-2: When?

...Don’t know...