The complexity of a Pop-up book

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Abstract

Origami is the centuries-old art of folding paper, and recently, it is investigated as science. In this paper, another hundreds-old art of folding paper, a pop-up book, is studied. A model for the pop-up book design problem is given, and its complexity is investigated. We show that both of the opening book problem and the closing book problem are NP-hard.
Keywords: Computational Complexity, Origami, Paper folding, Pop-up books.

1 Introduction

Origami is the centuries-old art of folding paper. Recently, some mathematicians and computer scientists have started to study origami. For example, a geometric approach to origami design has taken, and one of useful techniques is known as *TreeMaker* program by Lang [7]. On the other hand, "global flat foldability" of an origami is considered. The problem to find appropriate overlap order to fold a given origami flat is NP-hard [1]. The paper folding problem can be generalized. For example, folding a map seems to be similar to the problem of origami. The reader can find a comprehensive survey of the complexity of folding an origami and related results due to Demaine & Demaine [2] and Demaine & O'Rourke [3].

Another hundreds-old art of folding paper is a pop-up book. Contemporary pop-up book artists invent many sculpture of great beauty and intricacy (see, e.g., [8]). A pop-up book has two major differences comparing to origami. First, it has two surface covers with a hinge, and the essential movement depends on them. Hence the movement is strongly restricted (see, e.g., [6] for possible movements). Second, a book is not only closed (or folded) but also opened (or unfolded). For a pop-up book designer, the important thing is to design sculptures by a paper between two covers, and make the book be able to be opened and closed. Moreover, to see a page of the book, we usually open or close the page once. That is, we do not repeat the movements open and close to see a page in the book. From the viewpoint of the "computation" of the movement, this point also strongly restricts ourselves.

In this paper, we first give a model for the pop-up book design problem. Next, we show that both of the opening book problem and the closing book problem are NP-hard. We note that our results do not use the overlap order technique used in [1] to show the NP-hardness of the foldability problem of an origami.

2 Definitions

An input of the problems is a paper sculpture between a book structure. That is, a book consists of two *(surface) covers* which are joined by a *hinge*, and some *paper objects* are fixed between the covers. A paper object between the covers has some *faces* and *creases*. In our model, creases are given as a part of input, and we are not allowed to make a new crease. Creases can be folded in both ways, and it is allowed to not be folded (unless making a new crease). Given input is the (possible) design of a pop-up book. That consists of two surface covers with a fixed degree, say θ_0, and our objective is "open" or/and "close" the book. More precisely, for given degree θ_1, we aim to make the degree of the book from θ_0 to θ_1 without making a new crease. Now, we denote by $\textsf{Pop}(\theta_0, \theta_1)$ the problem to decide if a given pop-up book with two covers of degree θ_0 can be open or close to degree θ_1 without making a new crease. The *size* of an input (or a pop-up book) is defined by the summation of the number of lines (or edges of papers), the number of (predefined) creases, and the number of corners. In this paper, all borders of a paper consist of straight lines. That is, we do not deal with the case that the border of a paper makes a curve.

3 Closing a pop-up book

In this section, we show NP-hardness of the closing a pop-up book. More precisely, main theorem in this section is the following:

Theorem 1 *The problem* $\textsf{Pop}(\theta_0, \theta_1)$ *is* NP*-hard for any* $\theta_0 > \theta_1 \geq 0$.

We reduce from a well known NP-complete problem, NAE3SAT defined as follows [4, LO3].

Input: A formula F consists of m clauses $c_1, c_2, \ldots, c_m$ of 3 literals with n variables $x_1, x_2, \ldots, x_n$.

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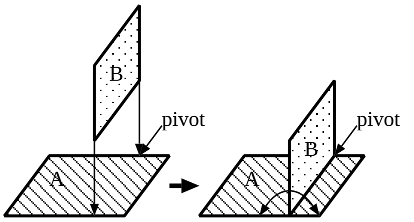

Figure 1: REVERSESTOPPER gadget

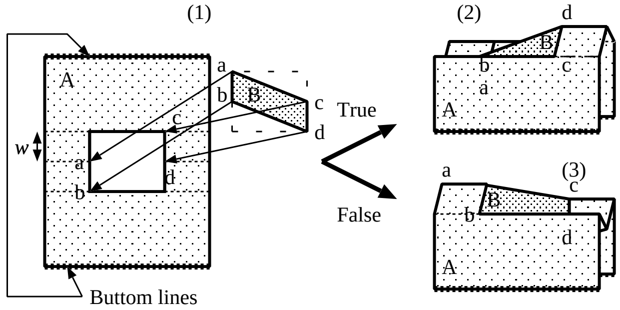

Figure 2: VARIABLE gadget

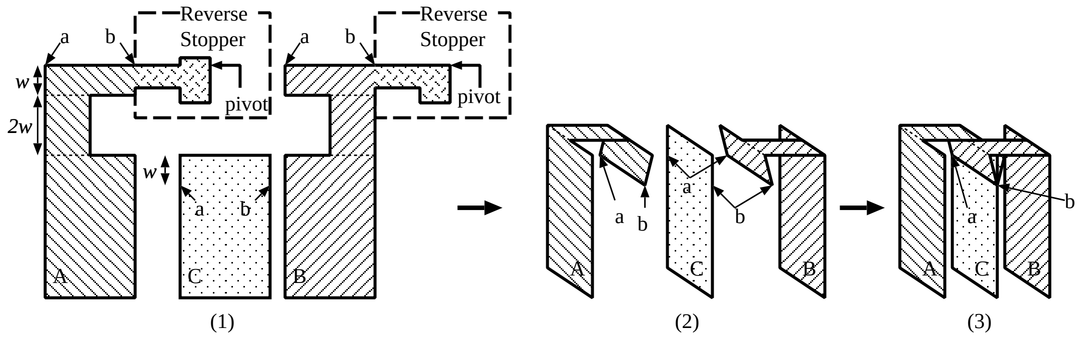

Figure 3: CLAUSE gadget

Output: "Yes" if there is a truth assignment such that each clause has at least one true literal and at least one false literal.

To reduce the problem, we make three kinds of gadgets called REVERSESTOPPER, VARIABLE, and CLAUSE by paper. The REVERSESTOPPER is described in Figure 1; for the face A, the face B can be flipped from degree 0 to degree 180 centered at the line *pivot*. The VARIABLE is described in Figure 2; for the face A, we attach the parallelogram B (Figure 2(1)). In VARIABLE, the parallelogram B is not stable when A is flat. However, there are two stable folding for the gadget shown in Figure 2(2) and (3). Since B is made by paper and hence it can be bended, two stable foldings can be exchanged by a flipping of mountain and valley of A. In those stable foldings, VARIABLE can be folded flat, and any other form cannot be folded flat. We assign the forms (2) and (3) to the true and false assignments, respectively. We call the top of mountain and the bottom of valley in the gadget *ridges*. When two stable foldings are exchanged, the heights of two ridges (ex)change $2w$. The CLAUSE is described in Figure 3. A CLAUSE consists of three parts (Figure 3(1)). On the papers A and B, the right upper parts form REVERSESTOPPER. To see easily, they are omitted in Figure 3(2) and (3). Figure 3 (3) is the final form of the CLAUSE (with REVERSESTOPPER).

Now, we construct a paper sculpture, or a design of a pop-up book, from a formula F (Figure 4). For each $i = 1, 2, \dots, n$, the VARIABLE X_i for x_i are glued to two covers at the bottom lines. Initially, each VARIABLE is in a neutral position; two ridges are at the same height, and the parallelogram B is bent. For a clause $c_j = (\ell_{i_1}, \ell_{i_2}, \ell_{i_3})$ with $\ell_i = x_i$ or $\ell_i = \bar{x_i}$, the CLAUSE C_j is connected to VARIABLE X_{i_1}, X_{i_2}, and X_{i_3} as follows: If $\ell_{i_1} = x_{i_1}$, the bottom line of A in Figure 3 is connected to the right ridge of the VARIABLE X_{i_1}. If $\ell_{i_2} = \bar{x_{i_2}}$, the bottom line of C in Figure 3 is connected to the left ridge of the VARIABLE X_{i_2}. The bottom line of B in Figure 3 is connected to the ridge of the VARIABLE X_{i_3} similarly. The connections are done in a natural way; see Figure 4 for the clause $c_j = (x_1, x_2, \bar{x_n})$. In Figure 4, the ridges imply $x_1 = false$, $x_2 = true$, and $x_n = false$. We note that each VARIABLE is in a neutral position, and all ridges are on the same line. Thus, each CLAUSE is also in a newtral position as Figure 3(3). We do not glue the gadgets to the covers except the bottom lines of VARIABLEs. The reduction can be done in a polynomial time of the size of F.

Now we are ready to show the key lemma:

Lemma 2 *The pop-up book constructed above can be closed completely if and only if there is a truth assignment of F such that each clause has at least one true*

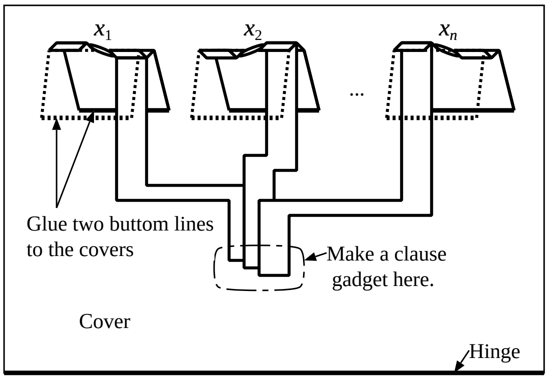

Figure 4: Construction from F

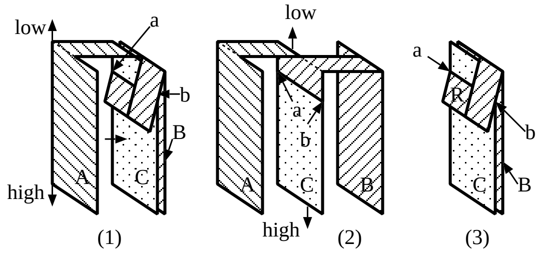

Figure 5: Foldable and unfoldable cases

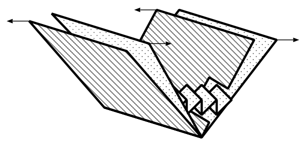

Figure 6: Cheat covers

literal and at least one false literal.

Proof. Each ridge of a VARIABLE can be *high* when it is on the top of the mountain, and *low* when it is on the bottom of the valley. To fold each VARIABLE flat, one of two ridges is high and the other ridge is low. Hence the parts A, C, B of a CLAUSE can take only two states, say *high* and *low*.

We first show feasible cases for a CLAUSE. When B and C correspond to the same height, and A corresponds to the different height, C is let come near to B, and then A can be moved up or down $2w$ height to fold them flat (Figure 5(1)). On the other hand, when A and B correspond to the same height and C takes the different height, A and B are let go farther to both side, and then C can be moved up or down $2w$ height to fold them flat (Figure 5(2)). Using the symmetric way, a CLAUSE can be fold flat when one of A, B, and C is high and one of them is low.

The other ways to fold them flat can be classified in two cases. The first case is three different heights; from the form in Figure 5(2), we can fold A, C, and B flat with three different heights in this order or vice versa. However, this case is impossible since three parts can take either high or low from the restriction by the Variables. The last case is the case that A, B, and C have the same height. This case can be done if A and B are folded symmetrically as shown in Figure 5(3) where the face A, which forms a symmetric shape of B, is omitted to see the case easier. However, this case is also impossible. In the case, two symmetric faces, marked by R in Figure 5(3), of A and B have to make 360 degree. However, the "reverse" movement is inhibited by the REVERSESTOPPER in Figure 3(1).

Therefore, the CLAUSE C_j can be folded flat if and only if one variable takes the different value from the other two variables. Hence the pop-up book can be closed if and only if F is a yes instance of NAE3SAT. □

Now we prove the main theorem in this section. In Lemma 2, making the gadgets small enough, we can prove the theorem if θ_0 is small enough and $\theta_1 = 0$. When $\theta_1 > 0$ and θ_0 is close enough to θ_1, we make the gadgets between two inner covers, and put some stable stands between the inner covers and surface covers. On the other hand, when θ_1 is large, we join the inner covers and surface covers by a long paper ribbon with one crease. It is easy to adjust the length of them to fit for given θ_1 and θ_0. This completes the proof of Theorem 1. □

4 Opening a pop-up book

In this section, we show NP-hardness of the opening a pop-up book. More precisely, main theorem in this section is the following:

Theorem 3 *The problem* Pop(θ_0, θ_1) *is* NP*-hard for any* $0 < \theta_0 < \theta_1$.

We note that θ_0 is greater than 0.

Proof. Between two surface covers, we add two inner covers and two paper springs shown in Figure 6 (two paper springs join one surface cover and one inner cover symmetrically). Then, opening the surface cover closes the inner covers. Hence we can use the gadgets in Theorem 1 again, and we have the theorem. □

5 Concluding remarks

The gadget in Figure 6 is a kind of cheat. Hence the problem $\textsf{Pop}(0, \theta_1)$ for $\theta_1 > 0$ is still open. The problem to open a completely closed book seems to be interesting since our gadgets do not work at all. We did not show that the problems in NP. In fact, our problem might be PSPACE-hard in some model; our problem (and some problems for origami) seems to be similar to the movement problems for 2-dimensional linkages, which is PSPACE-hard due to Hopcroft, Joseph, and Whitesides [5]. However, as noted in Introduction, we usually open or close a page of a pop-up book once, and hence the movement does not "repeat", which seems to be a key point. Therefore, realizing a computation with repeating by one close/open movement of a pop-up book is also interesting problem.

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