Computational Complexity of a Pop-up Book*

Ryuhei Uehara† Sachio Teramoto†

Abstract

Origami is the centuries-old art of folding paper, and recently, it is investigated as computer science: Given an origami with creases, the problem to determine if it can be flat after folding all creases is NP-hard. Another hundreds-old art of folding paper is a pop-up book. A model for the pop-up book design problem is given, and its computational complexity is investigated. We show that both of the opening book problem and the closing book problem are NP-hard.

Keywords: Computational Complexity, Origami, Paper folding, Pop-up book.

1 Introduction

Origami is the centuries-old art of folding paper. Recently, some mathematicians and computer scientists have started to study origami. For example, a geometric approach to origami design has taken, and one of useful techniques is known as *TreeMaker* program by Lang [8]. On the other hand, "global flat foldability" of an origami is considered. The problem to find appropriate overlap order to fold a given origami flat is NP-hard [1]. The paper folding problem can be generalized. For example, folding a map seems to be similar to the problem of origami. The reader can find a comprehensive survey of the complexity of folding an origami and related results due to Demaine & Demaine [3] and Demaine & O'Rourke [4].

Another hundreds-old art of folding paper is a pop-up book. Contemporary pop-up book artists invent many sculpture of great beauty and intricacy (see, e.g., [9]). A pop-up book has two major differences comparing to origami. First, it has two surface covers with a hinge, and the essential movement depends on them. Hence the movement is strongly restricted (see, e.g., [7, 2] for possible movements). Second, a book is not only closed (or folded) but also opened (or unfolded). For a pop-up book designer, the problem is to design sculptures by a paper between two covers, and make the book be able to be opened and closed. Moreover, to see a page of the book, we usually open or close the page once. That is, we do not repeat the movements open and close to see a page in the book. From the viewpoint of the "computation" of the movement, this point also strongly restricts ourselves.

In this paper, we first give a model for the pop-up book design problem. Next, we show that both of the opening book problem and the closing book problem are NP-hard. We note that our results do not use the overlap order technique used in [1] to show the NP-hardness of the foldability problem of an origami.

2 Definitions

An input of the problems is a paper sculpture between a book structure. That is, a book consists of two *(surface) covers* which are joined by a *hinge*, and some *paper objects* are fixed between the covers. A paper object between the covers has some *faces* and *creases*. In our model, creases are given as a part of input, and we are not allowed to make a new crease. A crease can be folded in both ways, and it is allowed to not be folded (unless making a new crease). Given input is the (possible) design of a pop-up book. That consists of two surface covers with a fixed degree, say θ_0, and our objective is "opening" or "closing" the book. More precisely, for given degree θ_1, we aim to make the degree of the book from θ_0 to θ_1 without making a new crease. Now, we denote by $\textsc{Pop}(\theta_0, \theta_1)$ the problem to decide if a given pop-up book with two covers of degree θ_0 can be opened or closed to degree θ_1 without making a new crease. The *size* of an input (or a pop-up book) is defined by the summation of the number of lines (or edges of papers), the number of (predefined) creases, and the number of corners. In this paper, all

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†School of Information Science, JAIST, Ishikawa 923-1292, Japan. {uehara,s-teramo}@jaist.ac.jp

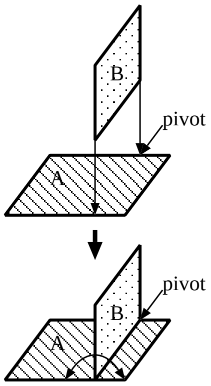

Figure 1: REVSTOP gadget

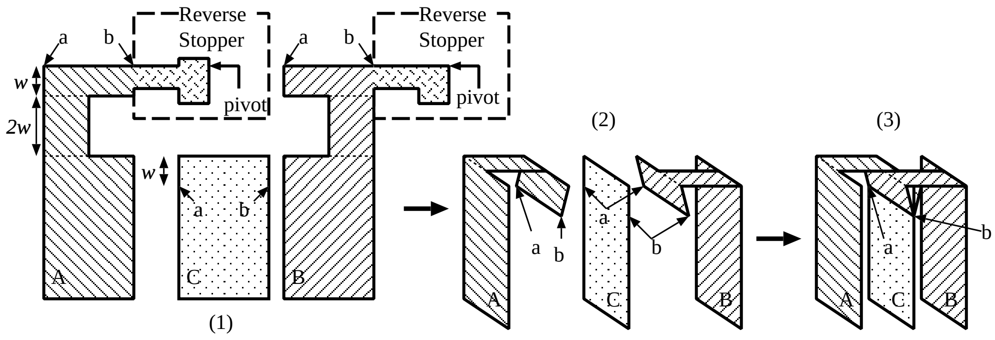

Figure 2: CLAUSE$_c$ gadget

borders (and creases) of a paper consist of straight lines. That is, we do not deal with the case that the border of a paper makes a curve.

3 Closing a pop-up book

In this section, we show NP-hardness of the closing a pop-up book. More precisely, main theorem in this section is the following:

Theorem 1 *The problem* Pop(θ_0, θ_1) *is* NP*-hard for any* $\theta_0 > \theta_1 \geq 0$.

We reduce from a well known NP-complete problem, NAE3SAT defined as follows [5, LO3].

Input: A formula F consists of m clauses $c_1, c_2, \ldots, c_m$ of 3 literals with n variables $x_1, x_2, \ldots, x_n$.

Output: "Yes" if there is a truth assignment such that each clause has at least one true literal and at least one false literal.

To reduce the problem, we make three kinds of gadgets called REVSTOP, CLAUSE$_c$, and VARIABLE$_c$ by paper. The REVSTOP is described in Figure 1; for the face A, the face B can be flipped from degree 0 to degree 180 centered at the line *pivot*. The CLAUSE$_c$ is described in Figure 2. A CLAUSE$_c$ consists of three parts (Figure 2(1)). On the papers A and B, the right upper parts form REVSTOP. To see easily, they are omitted in Figure 2(2) and (3). Figure 2 (3) is the final form of the CLAUSE$_c$ (with REVSTOP). The VARIABLE$_c$ is described in Figure 3; two bottom lines will be glued to two surface covers, respectively. The neutral position is depicted in Figure 3(0). Since the bottom lines have the same height, we have four possible cases to fold the VARIABLE$_c$ flat shown in Figure 3(1)-(4). Among them, the cases (3) and (4) will be inhibited by other gadgets. Hence we will represent the true and false assignments by the forms (1) and (2), respectively. We call two lines labeled by "a" and "c" in the gadget *ridges*. When two foldings (1) and (2) are exchanged, the heights of two ridges (ex)change $2w$.

Now, we construct a paper sculpture, or a design of a pop-up book, from a formula F (Figure 4). For each $i = 1, 2, \ldots, n$, the VARIABLE$_c$ X_i for x_i are glued to two covers at the bottom lines. Initially, each VARIABLE$_c$ is in a neutral position; two ridges are at the same height. For a clause $c_j = (\ell_{i_1}, \ell_{i_2}, \ell_{i_3})$ with $\ell_i = x_i$ or $\ell_i = \bar{x}_i$, the CLAUSE$_c$ C_j is connected to VARIABLE$_c$ X_{i_1}, X_{i_2}, and X_{i_3} as follows: If $\ell_{i_1} = x_{i_1}$, the bottom line of A in Figure 2 is connected to the right ridge of the VARIABLE$_c$ X_{i_1}. If $\ell_{i_2} = \bar{x_{i_2}}$, the bottom line of C in Figure 2 is connected to the left ridge of the VARIABLE$_c$ X_{i_2}. The bottom line of B in Figure 2 is connected to the ridge of the VARIABLE$_c$ X_{i_3} similarly. The connections are done in a natural way; see Figure 4 for the clause $c_j = (x_1, x_2, \bar{x}_n)$. In Figure 4, the ridges imply x_1 is true, x_2 is false, and x_n is true. We note that each VARIABLE$_c$ is in a neutral position, and all ridges have the same height. Thus, each CLAUSE$_c$ is also in a neutral position as Figure 2(3). We do not glue the gadgets to the covers except the bottom lines of VARIABLE$_c$s. After connecting CLAUSE$_c$s and VARIABLE$_c$s, each VARIABLE$_c$ cannot be folded in the form in Figure 3(3) and (4) without making a new crease. The reduction can be done in a polynomial time of the size of F.

Now we are ready to show the key lemma:

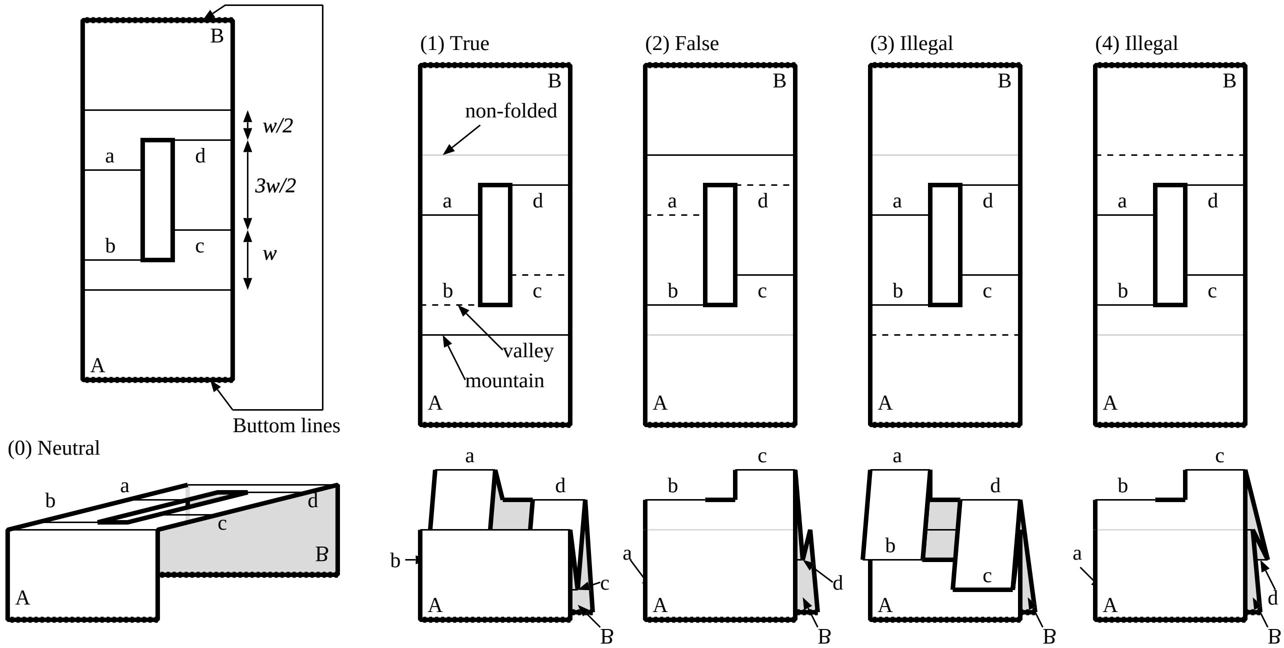

Figure 3: VARIABLE$_c$ gadget

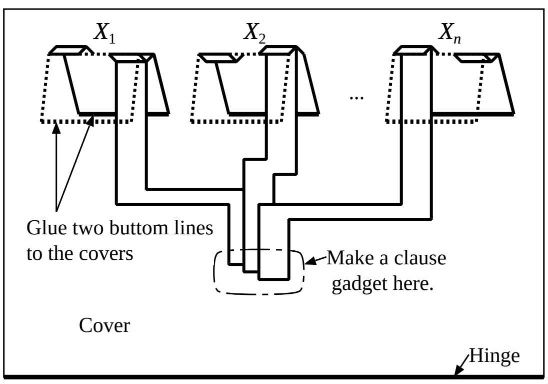

Figure 4: Construction from F

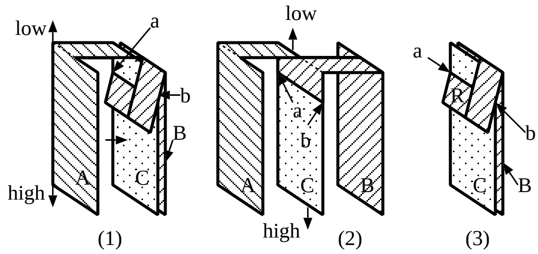

Figure 5: Foldable and unfoldable cases

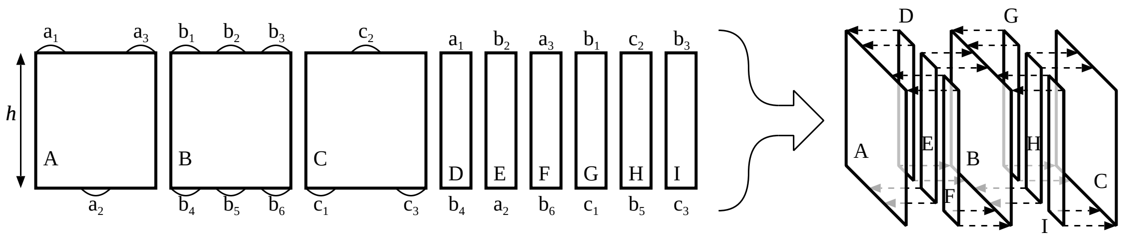

Figure 6: VARIABLE$_o$ gadget

Lemma 2 *The pop-up book constructed above can be closed completely if and only if there is a truth assignment of F such that each clause has at least one true literal and at least one false literal.*

Proof. Each ridge of a VARIABLE$_c$ can be *high* when it is on the top of the mountain, and *low* when it is on the bottom of the valley. To fold each VARIABLE$_c$ flat, one of two ridges is high and the other ridge is low. Hence the parts A, C, B of a CLAUSE$_c$ can take only two states, say, *high* and *low*.

We first show feasible cases for a CLAUSE$_c$. When B and C correspond to the same height, and A corresponds to the different height, C is let come near to B, and then A can be moved up or down $2w$ height to fold them flat (Figure 5(1)). On the other hand, when A and B correspond to the same height and C takes the different height, A and B are let go farther to both sides, and then C can be moved up or down $2w$ height to fold them flat (Figure 5(2)). Using the symmetric way, a CLAUSE$_c$ can be fold flat when one of A, B, and C is high and one of them is low.

The other ways to fold them flat can be classified in two cases. The first case is three different heights; from the form in Figure 5(2), we can fold A, C, and B flat with three different heights in this order or vice versa. However, this case is impossible since three parts can take either high or low from the restriction by the VARIABLE$_c$s. The last case is the case that A, B, and C have the same height. This folding can be done if A and B are folded symmetrically as shown in Figure 5(3) where the face A, which forms a symmetric shape of B, is omitted to see the case easier. However, this case is also impossible. In the case, two symmetric faces, marked by R in Figure 5(3), of A and B have to make 360 degree. However, the "reverse" movement is inhibited by the REVSTOP in Figure 2(1).

Therefore, the CLAUSE$_c$ C_j can be folded flat if and only if one variable takes the different value from the other two variables. Hence the pop-up book can be closed if and only if F is a yes instance of NAE3SAT.

Now we prove the main theorem in this section. In Lemma 2, making the gadgets small enough, we can prove the theorem if θ_0 is small enough and $\theta_1 = 0$. When $\theta_1 > 0$ and θ_0 is close enough to θ_1, we make the gadgets between two inner covers, and put some stable stands between the inner covers and surface covers. On the other hand, when θ_1 is large, we join the inner covers and surface covers by a long paper ribbon with one crease. It is easy to adjust the length of them to fit for given θ_1 and θ_0. This completes the proof of Theorem 1. ∎

4 Opening a pop-up book

In this section, we show NP-hardness of the opening a pop-up book. More precisely, main theorem in this section is the following:

Theorem 3 *The problem* Pop(θ_0, θ_1) *is* NP*-hard for any* $\theta_1 > \theta_0 \geq 0$.

We reduce from the 3SAT, well known NP-complete problem [5]. Let F be an instance of 3SAT, which consists of m clauses $c_1, c_2, \ldots, c_m$ of 3 literals with n variables $x_1, x_2, \ldots, x_n$. To reduce F, we make two kinds of gadgets called VARIABLE$_o$ and CLAUSE$_o$ by paper.

The VARIABLE$_o$ is described in Figure 6; that consists of three thick rectangles and six thin rectangles. Two edges of the same label are glued as in Figure 6. We note that the resultant gadget is completely flat. Let h be the common height of the rectangles. Next, two handles are glued to the VARIABLE$_o$ at height $h/2$ as in Figure 7(1). (Two handles can be fold flat at the center creases.) Then, there are only two ways to make two handles $2h$ apart shown in Figure 7(2) and (3). (It has the same structure to an old Asian wooden toy which consists of several boards banded like Figure 6, and they can be continuously flipped by twisting a handle.) We call the case

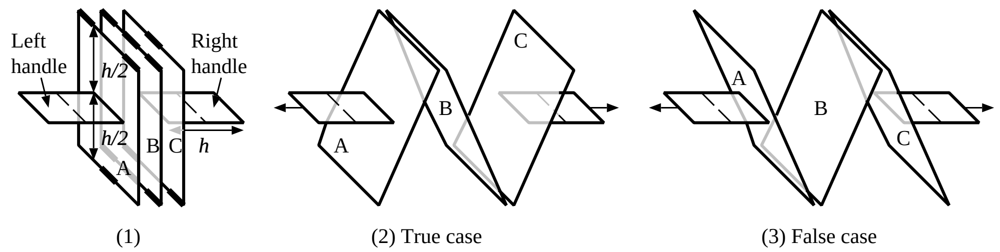

Figure 7: Handles with VARIABLE$_o$ gadget

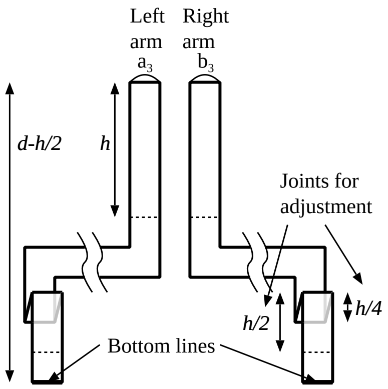

Figure 8: Arms for VARIABLE$_o$ gadget

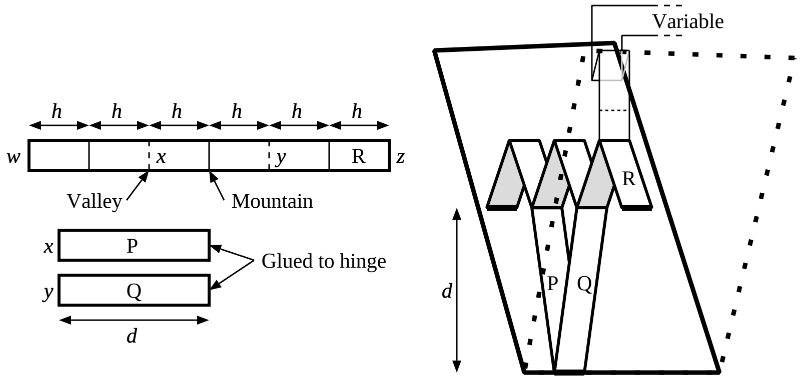

Figure 9: CLAUSE$_o$ gadget

(2) "True" and (2) "False." Now, we attach two kinds of arms in Figure 8 to the VARIABLE$_o$. (The number of arms will be described later.) The labeled edges are glued to the corresponding edges in Figure 6. (Precisely, the left arm is between A and F at a_3, and the right arm is between B and I at b_3.) The joints for adjustment are folded flat as in Figure 8.

Now, from the completely closed VARIABLE$_o$, when we make two handles $2h$ apart as in Figure 7(2), the left arm can go down at most $h/2$ since it is free except at the edge a_3, but the right arm has to go up $h/2$ since it is caught by B and C, and pulled up. Hence, the bottom line of the left arm can go down h with unfolding the joint, and the bottom line of the right arm cannot go down from the initial position. We note that, in the case, the left arm can choose to stay at the initial position with using the joint. Similarly, when we make two handles $2h$ apart as in Figure 7(3), the right arm can go down h, and the left arm cannot go down at all.

The CLAUSE$_o$ is described in Figure 9. A CLAUSE$_o$ consists of three ribbons P, Q, and R. The ribbon R has length $6h$, and both sides are glued to the covers at distance d from the hinge. The ribbon P joins the hinge and one of the valley on R, and the ribbon Q joins the hinge and another valley on R.

Now, we construct a paper sculpture, or a design of a pop-up book, from a formula F. For each $i = 1, 2, \ldots, n$, the VARIABLE$_o$ X_i for x_i are glued to two covers by two handles at distance $2d$ from the hinge. For a clause $c_j = (\ell_{i_1}, \ell_{i_2}, \ell_{i_3})$ with $\ell_i = x_i$ or $\ell_i = \bar{x}_i$, the CLAUSE$_o$ C_j is connected to VARIABLE$_o$ X_{i_1}, X_{i_2}, and X_{i_3} as follows: If $\ell_{i_1} = x_{i_1}$, one of three mountains on the ribbon R of C_j is connected to the bottom line of the left arm of X_{i_1}. If $\ell_{i_2} = \bar{x_{i_2}}$, another mountain on R is connected to the bottom line of the right arm of X_{i_2}. The last mountain of R is connected to X_{i_3} similarly. Hence, X_i has l_i left arms and r_i right arms, where l_i and r_i are the number of occurrences of x_i and $\bar{x}_i$ in F, respectively. We note that, with suitable choice of h and d, all gadgets can be folded flat, and the resultant pop-up book can be closed completely. The reduction can be done in a polynomial time of the size of F.

Now we are ready to show the key lemma:

Lemma 4 *The pop-up book constructed above can be opened if and only if there is a truth assignment of F such that each clause has at least one true literal.*

Proof. We try to open the book with the assignment for each VARIABLE$_o$. For each clause c_j, if at least one of three literals is true, the corresponding arm comes down to C_j, and hence it can be opened to θ with $d \sin\theta = h$. However, if none of them are true, no arms come close to C_j, and hence it cannot be opened. Hence F is satisfiable if and only if the pop-up book can be opened to θ. ∎

Now we prove the main theorem. In Lemma 4, letting $d \gg h$, we have the theorem for Pop$(0, \theta_1)$ for small $\theta_1 > 0$. We use the same trick in Section 3 for the other cases. This completes the proof of Theorem 3. ∎

5 Concluding remarks

For the problems for an origami and a pop-up book, we did not show that they are in NP. In fact, the problems might be PSPACE-hard in some model; they seem to be similar to the movement problems for 2-dimensional linkages, which is PSPACE-hard due to Hopcroft, Joseph, and Whitesides [6].

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