Complexity Classes Characterized by Many Computation Paths

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1 Introduction

Counting classes, which are widely investigated (e.g. [Sim77, Gil77, Val79b, Val79a, Ang90, Tod94, Wag86, FFK94]), consist of languages defined in terms of the number of accepting (or rejecting) computations of nondeterministic polynomial-time Turing machines. As well-known counting classes, NP, coNP, $\oplus$P, PP, and #P, we can define BPP and RP as the counting classes, which are usually defined as the probabilistic classes. For these counting classes, the number of accepting paths or rejecting paths are characterized by $c \cdot 2^p$ or $c \cdot 1 (= c \cdot 1^p)$ for some constant c with $0 < c < 1$.

In this paper, we consider the counting classes characterized by the number of paths equal to γ^p, for some γ with $1 < \gamma < 2$. More precisely, we introduce new classes, ManyRP and ManyBPP, which have γ^p incorrect paths, where p is the length of a computation. Here, ManyRP (or ManyBPP) corresponds to the class which has more accepting paths than RP (or BPP, respectively.)

Recently, the complexity classes characterized by semi-random sources are investigated[Ueh95]. Intuitively, the complexity classes can be extended by using a *kind* semi-random source. More precisely, it hold that:

$$^{\exists}\delta\text{-RP} = \text{NP}, \text{ and } ^{\exists}\delta\text{-BPP} = {}^{\exists}\delta\text{-PP} = \text{PSPACE},$$

where $^{\exists}\delta$-C is defined by the existence of a δ-random source which satisfies the condition for the random class C.

In this paper we define the class $^{\exists}\delta$-ManyBPP, which is extended ManyBPP by using a semi-random source, and will show that

$$^{\exists}\delta\text{-BPP} = {}^{\exists}\delta\text{-ManyBPP} \text{ implies } \text{P}^{\#\text{P}_{[1]}} = \text{PSPACE}.$$

It is unlikely that the latter equation holds. In this sense, it is easier to show the negative result for the former equation. This result can be seen as a passive evidence of the difference of BPP and ManyBPP.

2 Preliminaries

2.1 Complexity classes characterized by many computation paths

We assume a standard Turing machine model. For formal definitions of a *nondeterministic Turing machine* (NTM) and a *probabilistic Turing machine* (PTM), see [HU79]. In this paper, without loss of generality, we assume that an NTM (or PTM) is standardized as follows: Let M be a precise, polynomially bounded NTM with exactly two choices per step. All length of computation paths are equal each other.

For an NTM M, we denote by $\text{Acc}_M[x]$ (or $\text{Rej}_M[x]$) the number of accepting (or rejecting, respectively) paths of M on input x. Here we introduce new classes, ManyRP(γ) and ManyBPP(γ) as follows:

Definition 1 *For a given γ with $1 < \gamma < 2$, a language L is in the class* ManyRP(γ) *if there is an NTM M and a polynomial p such that for every x of length n,*

(i) the length of the computation paths of M are equal to $p(n)$,

(ii) for $x \in L$, $\text{Rej}_M[x] < \gamma^{p(n)}$, and

(iii) for $x \notin L$, $\text{Acc}_M[x] = 0$.

Definition 2 *For a given γ with $1 < \gamma < 2$, a language L is in the class* ManyBPP(γ) *if there is an NTM M, and a polynomial p such that for every x of length n,*

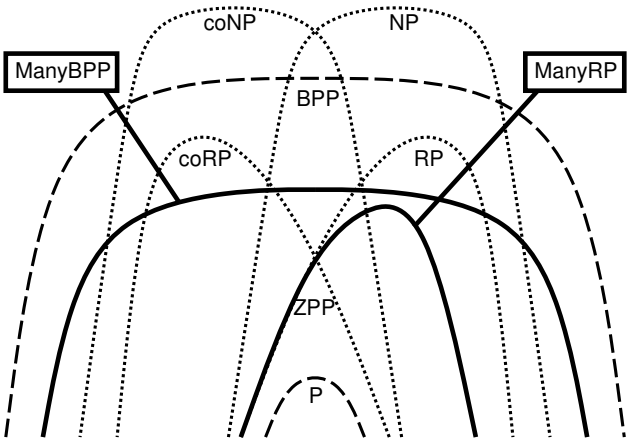

Figure 1: Relations

(*i*) *the length of the computation paths of M are equal to $p(n)$,*
(*ii*) *for $x \in L$, $\mathrm{Rej}_M[x] < \gamma^{p(n)}$, and*
(*iii*) *for $x \notin L$, $\mathrm{Acc}_M[x] < \gamma^{p(n)}$.*

Notice the following;

(*i*) if the probabilities do not depend on the length of the computation paths, the classes coincident with RP and BPP, respectively[Zac82, Zac86];

(*ii*) $\mathsf{ManyRP}(2) = \mathsf{NP}$ holds, and $\mathsf{ManyBPP}(2)$ makes
no sense, and $\mathsf{ManyRP}(\gamma) = \mathsf{ManyBPP}(\gamma) = \mathsf{ZPP}$ if $0 \leq \gamma \leq 1$; and

(*iii*) $\mathsf{ManyRP}(\gamma) \subseteq \mathsf{ManyRP}(\gamma')$, and $\mathsf{ManyBPP}(\gamma) \subseteq \mathsf{ManyBPP}(\gamma')$ if $\gamma \leq \gamma'$ since $\gamma^p \leq \gamma'^p$.

We define the classes ManyRP and ManyBPP by $\mathsf{ManyRP} = \cup_{1<\gamma<2}\mathsf{ManyRP}(\gamma)$ and $\mathsf{ManyBPP} = \cup_{1<\gamma<2}\mathsf{ManyBPP}(\gamma)$. By definitions, the following proposition holds:

Proposition 1

$$\mathsf{ZPP} \subseteq \mathsf{ManyRP} \subseteq \mathsf{RP}, \text{ and}$$

$$\mathsf{ManyRP} \subseteq \mathsf{ManyBPP} \subseteq \mathsf{BPP}.$$

The inclusions are summarized as Figure 1. For the class RP and BPP, we can improve the probability by repeating the algorithm enough times. On the other hand, for ManyBPP and ManyRP, we can not improve the probability such a sense, since it's probability depends on the length of the computation path. However, the following proposition holds:

Proposition 2 *Suppose that there is a number $\gamma(1 < \gamma < 2)$ and $k(0 < k < 1)$, an NTM M, and polynomials p and q such that for every x of length n,*

(*i*) *the length of the computation paths of M are equal to $p(n)$,*
(*ii*) *for $x \in L$, $\mathrm{Rej}_M[x] < q(n)\gamma^{kp(n)}$, and*
(*iii*) *for $x \notin L$, $\mathrm{Acc}_M[x] < q(n)\gamma^{kp(n)}$.*
Then, $L(M) \in \mathsf{ManyBPP}$.

Proof. Let γ be a number with $1 < \gamma < 2$ and k be a number with $0 < k < 1$. Here, $q(n)\gamma^{kp(n)} = q(n)\left(\gamma^k\right)^{p(n)} < \left(\frac{2+\gamma^k}{2}\right)^{p(n)}$ holds for large $p(n)$ since $\frac{2+\gamma^k}{2} > \gamma^k > 1$. Thus $L(M) \in \mathsf{ManyBPP}\left(\frac{2+\gamma^k}{2}\right)$, since $1 < \frac{2+\gamma^k}{2} < 2$. ∎

A similar proposition for ManyRP also holds, which are omitted here.

2.2 Complexity classes characterized by semi-random sources

M. Santha and U.V. Vazirani introduced, as an extremely general model of an imperfect source of randomness, a "slightly random source" in [SV84], and they renamed it "semi-random source" in [SV86]. A semi-random source is assumed that the previous bits output by the source can condition the next bit in an arbitrarily bad way. Accordingly, the next bit is output by the flip of a coin whose bias is fixed by an adversary who has complete knowledge of the history of the process. The adversary is limited to choosing a bias in $[\delta, 1-\delta]$ with some positive number $0 \leq \delta \leq \frac{1}{2}$. More precisely:

Definition 3 ([SV84]) *Let δ be a number such that $0 \leq \delta \leq \frac{1}{2}$. A* semi-random source *with parameter δ outputs bits $X_1X_2\cdots$, such that for all i and for all $x_1, x_2, \cdots$,*

$\delta \leq \Pr[X_i = x_i \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1}] \leq 1-\delta$.

A semi-random source with parameter δ will be termed δ-random source.

The influence by using a semi-random source, instead of a fair random source, over randomized classes has been investigated. (The terminology of the classes below are unified as in [Ueh95].) The classes $^{\forall}\delta$-RP, $^{\forall}\delta$-BPP, and $^{\forall}\delta$-PP, where, for the

random class C, the class ${}^{\forall}\delta$-C is a set of all languages which satisfy the condition for C by using *any* δ-random source, are investigated in [VV85], [Vaz86], and [Ueh95]. In the papers, U.V. Vazirani and V.V Vazirani showed that ${}^{\forall}\delta$-RP = RP[VV85], U.V. Vazirani showed that ${}^{\forall}\delta$-BPP = BPP[Vaz86], and R. Uehara showed that ${}^{\forall}\delta$-PP = BPP[Ueh95] for any δ with $0 < \delta < \frac{1}{2}$.

The classes ${}^{\forall}\delta$-RP, ${}^{\forall}\delta$-BPP, and ${}^{\forall}\delta$-PP request to satisfy the conditions for *all* δ-random sources. The symbol "$\forall$" means it. In this sense, the *dual* classes, which are characterized by the symbol "$\exists$, " were investigated.

Definition 4 ([Ueh95]) *A language L is in ${}^{\exists}\delta$-BPP if there exists a PTM M such that; for $x \in L$, M accepts with the probability greater than $\frac{3}{4}$ for* at least one *δ-random source, and for $x \notin L$, M accepts with the probability less than $\frac{1}{4}$ for* all *δ-random sources.*

The classes ${}^{\exists}\delta$-RP and ${}^{\exists}\delta$-PP are defined by the same manner as above, and omitted here. In the paper, he showed that ${}^{\exists}\delta$-RP = NP and ${}^{\exists}\delta$-BPP = ${}^{\exists}\delta$-PP = PSPACE for any δ with $0 < \delta < \frac{1}{2}$. We define ${}^{\exists}\delta$-ManyBPP(γ) as follows:

Definition 5 *A language L is in ${}^{\exists}\delta$-ManyBPP(γ) if there exists a PTM M such that; for $x \in L$, M rejects with the probability less than $(\frac{\gamma}{2})^p$ for* at least one *δ-random source, and for $x \notin L$, M accepts with the probability less than $(\frac{\gamma}{2})^p$ for* all *δ-random sources.*

We define the class ${}^{\exists}\delta$-ManyBPP by ${}^{\exists}\delta$-ManyBPP = $\cup_{1<\gamma<2}$ ${}^{\exists}\delta$-ManyBPP(γ).

2.3 Best tree and Worst tree

For a PTM with δ-random source, it is not clear how to assign the probability to the computation paths to maximize the probability that a given PTM accepts. It depends on the distribution of the accepting paths in the computation tree of the PTM. We define some notation to deal with the computation paths which are regarded as a simple full binary tree whose edges are labeled the value between δ and $1-\delta$.

Definition 6 *A* computation tree *is a full binary tree whose leaves are labeled by "accept" or "reject ".*

We call the path to a leaf labeled by "accept" (or "reject") is an accepting (or a rejecting, respectively) path. For a computation tree T, we denote by $|T|$ the number of the accepting paths of T.

Definition 7 *For each δ with $0 \leq \delta \leq \frac{1}{2}$, a* δ-assignment F *to a computation tree is a mapping from the set of edges of the tree to the interval $[\delta, 1-\delta]$ such that the two edges leaving each internal node are assigned numbers adding up to 1.*

Definition 8 *Let T be a computation tree, and F be a δ-assignment to T. The* probability of a node *of T for F is defined by the product of each value which is mapped from the edge, on the path from root to the node, by F. The* probability of T *for F, denoted by* $\Pr[T \mid F]$*, is defined by the sum of every probability of the leaf labeled "accept".*

For a given computation tree, we consider an assignment which maximizes the probability of the tree:

Definition 9 *For a given computation tree T,* a maximal assignment $F_{\max}(T)$ *is defined by the following rules:*
(i) For the node whose sons are leaves; assign $(1-\delta)$ to an edge incidenting a leaf labeled "accept" and assign δ to another edge, if there exists at least one leaf labeled "accept"; or assign $(1-\delta)$ and δ to edges if both are labeled "reject".
(ii) For the internal node whose sons are the subtrees whose assignments are already defined; let T_0 and T_1 are the subtrees; assign $(1-\delta)$ to the edge incidenting the root of T_0 (or T_1) and assign δ to the edge incidenting the root of T_1 (or T_0), if $\Pr[T_0 \mid F_{\max}(T_0)] > \Pr[T_1 \mid F_{\max}(T_1)]$ (or otherwise, respectively).

By using the induction for the depth of the tree, it is easily shown that $\Pr[T \mid F_{\max}(T)] \geq \Pr[T \mid F']$ for any δ-assignment F'. Notice that to maximize the probability, it is sufficient to consider δ-assignments which only assign the values δ and $1-\delta$.

Definition 10 *Let t be an integer. The computation tree T with t accepting paths is the* worst *if $\Pr[T \mid F_{\max}(T)] \leq \Pr[T' \mid F_{\max}(T')]$ holds for any computation tree T' with t accepting paths.*

The computation tree T with t accepting paths is the best *if* $\Pr[T \mid F_{\max}(T)] \geq \Pr[T' \mid F_{\max}(T')]$ *holds for any computation tree T' with t accepting paths.*

We note that a worst tree gives the maximal number of the accepting paths for a given probability, and a best tree gives the minimal number of the accepting paths for a given fixed probability. Here, we can easily show the following lemma for the probability of a best tree.

Lemma 3 *For a best tree T of depth p with t accepting paths, it holds that*

$\Pr[T \mid F_{\max}(T)] = \sum_{i=0}^{t_0} \binom{p}{i} \delta^i (1-\delta)^{p-i} + t_1 \delta^{t_0+1} (1-\delta)^{p-t_0-1}$,

where $t = \sum_{i=0}^{t_0} \binom{p}{i} + t_1$ *with* $t_1 < \binom{p}{t_0+1}$.

To construct a worst tree, we consider to draw the computation tree as a planar tree, whose root is drawn on the top.

Definition 11 *A computation tree T with t accepting paths is* unbalanced *if it can be drawn such that the first tth leaves in order from right side are labeled "accept."*

Notice that for a given unbalanced tree T, $F_{\max}(T)$ assigns $(1-\delta)$ to each edge to a right son, and δ to each edge to a left son. For an unbalanced tree, the following lemma was shown.

Lemma 4 ([Ueh95]) *Any unbalanced tree is the worst.*

By using this lemma, the following lemma for the probability of a worst tree was shown.

Lemma 5 ([Ueh95]) *Let T be a worst tree of depth p with t accepting paths. Let $t_0, t_1, \cdots, t_k$ be the integers such that $t = 2^{t_k} + \cdots + 2^{t_1} + 2^{t_0}$, with $t_k > \cdots > t_1 > t_0 \geq 0$. These t_i's are uniquely defined by using binary representation of t. Here, it holds that*

$$\Pr[T \mid F_{\max}(T)] = \sum_{i=0}^{k} \delta^i (1-\delta)^{p-t_{k-i}-i}.$$

3 Main Result

Here, we show crucial theorem in this paper:

Theorem 6

$$^{\exists}\delta\text{-ManyBPP} \subseteq \mathsf{P}^{\#\mathsf{P}[1]} \text{ for } 0.35 < \delta < \frac{1}{2}.$$

Proof. Let M be a PTM with $L(M) \in {}^{\exists}\delta$-ManyBPP. By definition, there exists γ with $1 < \gamma < 2$ such that for $x \in L$, $\Pr[M \text{ rejects } x] < (\frac{\gamma}{2})^{p(n)}$ for at least one δ-random source, and for $x \notin L$, $\Pr[M \text{ accepts } x] < (\frac{\gamma}{2})^{p(n)}$ for any δ-random sources. It is sufficient to show that

$$\max_{x \notin L} \mathrm{Acc}_M[x] < \min_{x \in L} \mathrm{Acc}_M[x]$$

holds for large n.

Let t be a number such that $\max_{x \notin L} \mathrm{Acc}_M[x]$, and s a number such that $\min_{x \in L} \mathrm{Acc}_M[x]$. It is sufficient to show that $t < s$ holds for any δ-random sources and for sufficiently large n.

By definition, t is given by a worst tree. Thus, by using Lemma 5, $(\frac{\gamma}{2})^p \geq \sum_{i=0}^{k} \delta^i (1-\delta)^{p-t_{k-i}-i} \geq (1-\delta)^{p-\log t}$ holds (to simplify, we assume that $\log t = \lfloor \log t \rfloor$, which is not essential.) Hence,

$$t \leq \left(\left(\frac{2}{\gamma} (1-\delta) \right)^{\frac{1}{\log(1-\delta)}} \right)^p.$$

We note that for input x with $x \in L$ and $\mathrm{Acc}_M[x] \geq 2^{p-1}$, M accepts with the probability greater than $(1-\delta)$, which is greater than $(\frac{\gamma}{2})^p$ for large p. Thus we can assume that $t < 2^{p-1}$.

Let s be a number such that $\min_{x \in L} \mathrm{Acc}_M[x]$. By definition, s is given by a best tree. Thus, by using Lemma 3, by letting $s = \sum_{i=0}^{s_0} \binom{p}{i}$ (to simplify, we assume that $s = \sum_{i=0}^{s_0} \binom{p}{i} + s_1$ with $s_1 = 0$, which is not essential,) $1 - (\frac{\gamma}{2})^p \leq \sum_{i=0}^{s_0} \binom{p}{i} \delta^i (1-\delta)^{p-i}$ holds.

Let $s_0 = bp$. Here, since $t < 2^{p-1}$ and $\sum_{i=0}^{p/2} \binom{p}{i} = 2^{p-1}$ hold, we can assume that $b < \frac{1}{2}$. Moreover, Chernoff bounds[Che52] shows that $\sum_{i=0}^{b'p} \binom{p}{i} \delta^i (1-\delta)^{p-i} \leq c^p$ for some constant c with $0 < c < 1$ if $b' < \delta$. Thus, $\delta \leq b < \frac{1}{2}$.

Here, by using the technique for answer to exercise 9.42 in [GKP89] and since $\delta \leq b < \frac{1}{2}$, $1 - (\frac{\gamma}{2})^p \leq \sum_{i=0}^{s_0} \binom{p}{i} \delta^i (1-\delta)^{p-i} = (1 -$

$\delta)^p \sum_{i=0}^{s_0} \binom{p}{i} \left(\frac{\delta}{1-\delta}\right)^i \leq (1 - \delta)^p \binom{p}{s_0} \sum_{i=0}^{s0} \left(\frac{b}{1-b}\right)^i \left(\frac{\delta}{1-\delta}\right)^{s_0-i} \leq (1-\delta)^p \binom{p}{s_0}(s_0 + 1)$ holds. Thus,

$$\begin{aligned} s &= \sum_{i=0}^{s_0} \binom{p}{i} > \binom{p}{s_0} \geq \frac{1-\left(\frac{\gamma}{2}\right)^p}{(s_0+1)(1-\delta)^p} \\ &> \left(1-\left(\frac{\gamma}{2}\right)^p\right) \frac{1}{\delta p+1} \left(\frac{1}{1-\delta}\right)^p \end{aligned}$$

holds.

Here, when $0.35 < \delta$, it is easily seen that $\left(\frac{2}{\gamma}(1-\delta)\right)^{\frac{1}{\log(1-\delta)}} < \frac{1}{1-\delta}$, and $1 < \frac{1}{1-\delta}$, consequently, $t < s$ for large n. ∎

Corollary 1

$$\textit{If } {}^{\exists}\delta\text{-}\mathsf{ManyBPP} = {}^{\exists}\delta\text{-}\mathsf{BPP} \textit{ then}$$

$$\mathsf{P}^{\#\mathsf{P}[1]} = \mathsf{P}^{\mathsf{PP}} = \mathsf{P}^{\#\mathsf{P}} = \mathsf{PSPACE}.$$

Proof. It is clear that $\mathsf{P}^{\#\mathsf{P}[1]} \subseteq \mathsf{P}^{\#\mathsf{P}} = \mathsf{P}^{\mathsf{PP}} \subseteq \mathsf{PSPACE}$. Suppose that ${}^{\exists}\delta\text{-}\mathsf{ManyBPP} = {}^{\exists}\delta\text{-}\mathsf{BPP}$. Let L be a language in PSPACE. Since ${}^{\exists}\delta\text{-}\mathsf{BPP} = \mathsf{PSPACE}$ for all δ with $0 < \delta < \frac{1}{2}$([Ueh95]), there exists a PTM M with $L(M) \in {}^{\exists}\delta\text{-}\mathsf{BPP}$ with $\delta > 0.35$, and by assumption, there also exists a PTM M' with $L(M') \in {}^{\exists}\delta\text{-}\mathsf{ManyBPP}$ with $\delta > 0.35$. Thus $L(M') \in \mathsf{P}^{\#\mathsf{P}[1]}$. This implies the corollary. ∎

4 Concluding Remarks

We introduce new counting classes, ManyBPP and ManyRP, characterized by the number of paths equal to γ^p, where $1 < \gamma < 2$. They are not likely coincident with the other classes, such as RP, BPP, or ZPP. There exist problems for these classes:

- As shown in Figure 1, the relation between NP and BPP is similar to the relation between RP and ManyBPP.

  Does $\mathsf{RP} \subseteq \mathsf{ManyBPP}$ imply $\mathsf{RP} = \mathsf{ManyRP}$, as $\mathsf{NP} \subseteq \mathsf{BPP}$ imply $\mathsf{NP} = \mathsf{RP}$ shown in [Ko82]?

- For $1 < \gamma < \gamma' < 2$, does $\mathsf{ManyRP}(\gamma) = \mathsf{ManyRP}(\gamma')$ (or $\mathsf{ManyBPP}(\gamma) = \mathsf{ManyBPP}(\gamma')$) hold?

- For $0 < \delta < \delta' < \frac{1}{2}$, does ${}^{\exists}\delta\text{-}\mathsf{ManyBPP} = {}^{\exists}\delta'\text{-}\mathsf{ManyBPP}$ (or ${}^{\exists}\delta\text{-}\mathsf{ManyBPP} = {}^{\exists}\delta'\text{-}\mathsf{ManyBPP}$) hold?

- Does ${}^{\exists}\delta\text{-}\mathsf{ManyBPP}$ characterize the class $\mathsf{P}^{\#\mathsf{P}[1]}$? Notice that ${}^{\exists}\delta\text{-}\mathsf{BPP}$ characterizes the class PSPACE, and the simple characterization for $\mathsf{P}^{\#\mathsf{P}}$ is investigated by S. Toda in [Tod94].

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