Structures vs Prestructures

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joint work with

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1. A Generalized Yokoyama-Yoshikawa Theorem

Rice's Theorem and Its Analogies

- In this section, we investigate Rice Property of structures.
- The name "Rice property" came from a famous theorem so-called Rice's Theorem.

Rice Property

- Let $\mathcal{N} = (N; \{f_i; i \in I\})$: a recursive structure, $\equiv$: its (possibly non-rec.) congruence relation.
- Def. $\mathcal{N}$ has the Rice property w.r.t. $\equiv$ iff $\neg\exists k \geq 1 \exists$ non-trivial rec $R \subset N^k$ compatible with $\equiv$.
  - Here, R: nontrivial iff $R \neq \emptyset, N^k$.
- Prop. $\mathcal{N}$ has the Rice property w.r.t. $\equiv$ $\iff$ $\neg\exists$ non-trivial rec $R \subset N$ compatible with $\equiv$.
  - Prf($\Leftarrow$). We prove by induction on k.
  - Let $R \subset N^{k+1}$: rec. & compatible with $\equiv$.
  - $\forall \vec{x} \in N^k$, rec. unary relation $R(\vec{x}, y)$ on y must be trivial.
  - $\therefore [\exists y R(\vec{x}, y)] \subset N^k$ is rec & k-ary relation.
  - By IH, it is trivial and so is R. □

Rice's Theorem

- Thm(Rice 1953). $\mathbb{N}$ has the Rice property w.r.t. $\equiv_{\mathrm{func}}$, i.e., $\neg\exists$ non-trivial rec $R \subset \mathbb{N}$ compatible with $\equiv_{\mathrm{func}}$.
  - Here, $e \equiv_{\mathrm{func}} k :\iff \varphi_e = \varphi_k$,
  - φ_e: the rec. partial func. defined by e-th Turing machine.

Theories and the Rice Property

- We consider structures associated with formal theories.
- Let T: a rec. axiomatizable theory of the classical logic.
- Def. $\varphi \equiv_T \psi$ iff $T \vdash \varphi \leftrightarrow \psi$
- Def. $\mathcal{M}(T) := (\mathrm{Sent}(T); \vee, \wedge, \neg, \dots)$.
  $\mathcal{M}(T)/\equiv_T$ is called the Lindenbaum algebra of T.
- Fact. $\mathcal{M}(T)/\equiv_T$ is a boolean algebra.

Yokoyama-Yoshikawa Theorem

- Thm(essentially, Bernardi 1981).
  $\mathcal{M}(\mathrm{PA})$ has the Rice property w.r.t. $\equiv_{\mathrm{PA}}$.
- Let T: a rec. axiomatizable theory of the classical logic.
  Thm(Yokoyama-Yoshikawa 2011). TFAE:
  - $\mathcal{M}(T)$ has the Rice property w.r.t. $\equiv_T$,
  - $\neg\exists$ rec. completion of T.
    (completion means consistent and complete ext.)
  - $\neg\exists$ rec. $F \subset \mathrm{Sent}(T)$ compatible with $\equiv_T$ s.t.
    $F/\equiv_T$ is a prime filter of $\mathcal{M}(T)/\equiv_T$.

Generalizing YY Theorem

- If we weaken the logic, we obtain a finer equivalence relation $\equiv'_T$.
- In general, the finer a congruence is, the harder a structure has the Rice property w.r.t. the congruence.
- But we show that YY theorem still holds as long as the Lindenbaum algebras of the logic are distributive lattices.

Generalized YY Theorem

- Let $\mathcal{D} = (D; \vee, \wedge, \top, \bot)$: a rec. structure,
  $\equiv$: congruence s.t. $\mathcal{D}/\equiv$ is a distr. lat. with top and bottom.
- Thm. $\exists$ non-trivial rec. $Q \subset D$ compatible with $\equiv$ $\iff$
  $\exists$ rec. P of D compat. with $\equiv$ s.t. $P/\equiv$ is a prime filter.
  - Prf, Idea($\Rightarrow$). Let $\{x_n\}_{n\in\mathbb{N}} = D$.
  - We may assume $\bot \notin Q$ and $x_0 \in Q$.
  - Define "descending" intervals $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ by
    $(a_0, b_0) := (\bot, x_0)$, and letting $y_n := (a_n \vee x_n) \wedge b_n$,
    $$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, y_n) & \text{if } y_n \in Q, \\ (y_n, b_n) & \text{o.w.} \end{cases}$$
  - One can check $P := \{x_n : y_n \in Q\}$ is compatible with $\equiv$ and
    $P/\equiv$ is a prime filter. □

Theories of a weaker logical system

- Cor(Generalized YY Theorem). For a rec. axiomatizable theory (T, F) of a logical system whose Lindenbaum algebra is a distr. lat. with top and bot., TFAE:
  - $\mathcal{M}(T, F)$ has the Rice property w.r.t. $\equiv_{(T,F)}$,
  - $\neg\exists$ rec. completion of (T, F).
    (Completion means a pair $T' \supset T$ and $F' \supset F$ compatible with $\equiv_{(T,F)}$ s.t. $T'/\equiv_{(T,F)}$ is a prime filter and $F'/\equiv_{(T,F)}$ is its complement.)
- This ends the first section.

2. A Computable Nonstandard Pre-Model of Arithmetic

Tennenbaum's Theorem

- Thm(Tennenbaum 1959). $\neg\exists$ rec. nonstandard $\mathcal{M} \models$ PA.
- Thm(McAloon 1982). $\neg\exists$ rec. nonstandard $\mathcal{M} \models \mathrm{I}\Delta_0$.
- We shall show:
  Thm. $\exists$ "simple" rec. $\mathcal{M}$ $\exists$ (non-rec.) congruence $\equiv$ s.t. $\mathcal{M}/\equiv$ is a nonstandard model of $\mathrm{I}\Delta_0$.

- Def. $\mathrm{PrimRec}$: the set of symbols of the prim. rec. func. $\mathrm{PrimRec}_1$: the set of symbols of the unary prim. rec. func.
- We can think $\mathrm{PrimRec}$,$\mathrm{PrimRec}_1$ are rec. sets.
- Def. Term: the set of all terms starting from elements in $\mathrm{PrimRec}_1$ and built up by means of elements in $\mathrm{PrimRec}$. $\mathcal{PR} := (\mathrm{Term}; \{(\vec{t} \mapsto f(\vec{t})) : f \in \mathrm{PrimRec}\})$.
- Prop. $\mathcal{PR}$ is a rec. structure. □

Filters

- Def. For a filter $\mathcal{F} \subset \mathrm{Pow}(\mathbb{N})$,
  $\forall^{\mathcal{F}} n, P(n)$ iff $\{n \in \mathbb{N} : P(n)\} \in \mathcal{F}$.
- Note that
  - $\mathbb{N} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$ means $\forall^{\mathcal{F}} n, \top$ and $\exists^{\mathcal{F}} n, \top$ hold;
  - Closed under super set and intersection means
    $[\forall^{\mathcal{F}} n, P(n)] \wedge [\forall^{\mathcal{F}} n, Q(n)] \iff [\forall^{\mathcal{F}} n, [P(n) \wedge Q(n)]]$.
- Def. Let $\mathcal{F}, \mathcal{S} \subset \mathrm{Pow}(\mathbb{N})$.
  $\mathcal{F}$: $\mathcal{S}$-ultra filter iff $\mathcal{F}$ is a filter and
  - $A \in \mathcal{S} \Longrightarrow A \in \mathcal{F} \vee \mathbb{N} \setminus A \in \mathcal{F}$.
- Def. ultra filter $=$ $\mathrm{Pow}(\mathbb{N})$-ultra filter.
  This means $\forall^{\mathcal{F}} n, P(n) \iff \exists^{\mathcal{F}} n, P(n)$.
- Def. filter $\mathcal{F}$: principal iff $\exists A, \mathcal{F} = \{B \subset \mathbb{N} : B \supset A\}$.
  This means $\forall^{\mathcal{F}} n, P(n) \iff \forall n \in A, P(n)$.
  Otherwise, $\mathcal{F}$ is non-principal.

Congruences via Filters

- Def. Let $\mathcal{F} \subset \mathrm{Pow}(\mathbb{N})$ be a filter and $t, s \in \mathrm{Term}$.
  $t \equiv_{\mathcal{F}} s$: $\iff \forall^{\mathcal{F}} n, \mathbb{N} \models t(n) = s(n)$.
- Prop. $\forall$ filter $\mathcal{F}$, $\equiv_{\mathcal{F}}$ is a congruence relation of $\mathcal{PR}$.
- Def. $\mathrm{PrimRecSet}$: the set of all prim. rec. sets.
- The main theorem of this section is:
  Thm. Let $\mathcal{F}$ be a non-principal $\mathrm{PrimRecSet}$-ultra filter.
  $\mathcal{PR}/\equiv_{\mathcal{F}}$ is a nonstandard model of $\mathrm{I}\Delta_0(\mathrm{PrimRec})$.

Łoś' Theorem

- Let $\mathcal{U}$: a ultra filter.
  Thm(Łoś). For all formula $\varphi(\vec{x})$ and $\vec{f} \in \mathbb{N}^{\mathbb{N}}$, TFAE:
  - $\mathbb{N}^{\mathbb{N}} / \equiv_{\mathcal{U}} \models \varphi(\vec{f})$,
  - $\forall^{\mathcal{U}} n, \mathbb{N} \models \varphi(\overrightarrow{f(n)})$.

  (We may write $[\varphi(\overrightarrow{f(x)}) \iff \forall^{\mathcal{U}} x, \varphi(\overrightarrow{f(x)})]$.)
- In $\mathcal{PR}/\equiv_{\mathcal{F}}$, we interpret $t \leq s$ as $\exists x, t + x = s$.
- Let $\mathcal{F}$: a $\mathrm{PrimRecSet}$-ultra filter.
  Lem(restricted Łoś' theorem). For all $\varphi(\vec{x}) \in \Delta_0$ and $\vec{t} \in \mathrm{Term}$, TFAE:
  - $\mathcal{PR}/\equiv_{\mathcal{F}} \models \varphi(\vec{t})$,
  - $\forall^{\mathcal{F}} n, \mathbb{N} \models \varphi(\overrightarrow{t(n)})$.

A Proof of Ristricted Łoś' Theorem

- Let $\mathcal{F}$ be a $\mathrm{PrimRecSet}$-ultra filter.
  Lem(restricted Łoś' theorem). For all $\varphi(\vec{x}) \in \Delta_0$ and $\vec{t} \in \mathrm{Term}$, TFAE:
  - $\mathcal{PR}/\equiv_{\mathcal{F}} \models \varphi(\vec{t})$,
  - $\forall^{\mathcal{F}} n, \mathbb{N} \models \varphi(\overrightarrow{t(n)})$.
- Prf. We show by induction on the complexity of formulas φ.
  - Atomic $t = s$: just by the definition of $\equiv_{\mathcal{F}}$.
  - $\varphi \wedge \psi$: since $[\forall^{\mathcal{F}} n, \dots] \wedge [\forall^{\mathcal{F}} n, \dots] \iff [\forall^{\mathcal{F}} n, [\dots \wedge \dots]]$
  - $\neg\varphi$: note $\{n \in \mathbb{N} : \mathbb{N} \models \varphi(\overrightarrow{t(n)})\} \in \mathrm{PrimRecSet}$ so that $[\forall^{\mathcal{F}} n, \dots] \iff [\exists^{\mathcal{F}} n, \dots]$. Thus $[\neg\forall^{\mathcal{F}} n, \dots] \iff [\forall^{\mathcal{F}} n, \neg\dots]$
  - $\exists y \leq s(\vec{x}), \varphi(y, \vec{x})$: to prove the latter condition implies the former, we need a choice function, but in $\mathcal{PR}/\equiv$, we have a choice $n \mapsto (\mu k \leq s(\overrightarrow{t(n)}))[\mathbb{N} \models \varphi(k, \overrightarrow{t(n)})]$!!! □

$\mathcal{PR}/\equiv_{\mathcal{F}} \models$ Basic Axioms

- Let $\mathcal{F}$: a $\mathrm{PrimRecSet}$-ultra filter. Let $\mathcal{U} \supset \mathcal{F}$: a ultra filter.
- Lem(restricted Łoś' theorem). For all $\varphi(\vec{x}) \in \Delta_0$ and $\vec{t} \in \mathrm{Term}$, TFAE:
  - $\mathbb{N}^{\mathbb{N}}/\equiv_{\mathcal{U}} \models \varphi(\vec{t})$,
  - $\mathcal{PR}/\equiv_{\mathcal{F}} \models \varphi(\vec{t})$,
  - $\forall^{\mathcal{F}} n, \mathbb{N} \models \varphi(\overrightarrow{t(n)})$,
  - $\forall^{\mathcal{U}} n, \mathbb{N} \models \varphi(\overrightarrow{t(n)})$.
- Cor. $\mathbb{N} \prec_{\Delta_0} \mathcal{PR}/\equiv_{\mathcal{F}} \prec_{\Delta_0} \mathbb{N}^{\mathbb{N}}/\equiv_{\mathcal{U}}$ and $\mathbb{N} \prec \mathbb{N}^{\mathbb{N}}/\equiv_{\mathcal{U}}$.
- Cor. $\mathbb{N} \prec_{\Pi_1} \mathcal{PR}/\equiv_{\mathcal{F}}$. Thus $\mathcal{PR}/\equiv_{\mathcal{F}} \models \mathrm{I}\Delta_0(\mathrm{PrimRec})$.
- Prop. If $\mathcal{F}$ is non-principal, then $\mathcal{PR}/\equiv_{\mathcal{F}}$ is nonstandard.
  $\because$ $n \mapsto (x \mapsto n)$ is the unique embedding from $\mathbb{N}$ to $\mathcal{PR}/\equiv_{\mathcal{F}}$.
  Thus $x \mapsto x$ is nonstandard element in $\mathcal{PR}/\equiv_{\mathcal{F}}$. □

An Etra Property

- Let $\mathcal{F}$: non-principal.
- Thm. $\mathcal{PR}$ has the Rice property w.r.t. $\equiv_{\mathcal{F}}$.

Future Works

- Question. $\mathcal{PR}/\equiv_{\mathcal{F}} \not\models \mathrm{I}\Sigma_1$?
- We would like to develop separating technik for systems of arithmetic $\mathrm{I}\Sigma_n$ and $\mathrm{I}\Delta_n$ and to investigate weaker systems as our future works.

That's all, thank you!

$\mathcal{PR}/\equiv_{\mathcal{F}} \models \mathrm{L}\Delta_0(\mathrm{PrimRec})$

- We can prove $\mathcal{PR}/\equiv_{\mathcal{F}}$ satisfies $\mathrm{L}\Delta_0(\mathrm{PrimRec})$ directly.
- Lem. For all $\mathrm{PrimRecSet}$-ultra filter $\mathcal{F}$,
  $\mathcal{PR}/\equiv_{\mathcal{F}}$ satisfies Δ_0 Least Number Principle.
  - Prf(Sketch). Let $\varphi \in \Delta_0(\mathrm{PrimRec})$ and $\vec{t} \in \mathrm{Term}$.
  - Suppose that $\mathcal{PR}/\equiv_{\mathcal{F}} \models \exists x, \varphi(x, \vec{t})$.
  - Choose f s.t. $\mathcal{PR}/\equiv_{\mathcal{F}} \models \varphi(f, \vec{t})$.
  - Define a prim. rec. func. g by
    $g(n) := (\mu k \leq f(n))[\mathbb{N} \models \varphi(k, \overrightarrow{t(n)})]$.
  - Using Restricted Łoś's Theorem, it is shown that g is the least element in $\mathcal{PR}/\equiv_{\mathcal{F}}$ s.t. $\varphi(g, \vec{t})$ is true. □
- Thm. Let $\mathcal{F}$ be a non-principal $\mathrm{PrimRecSet}$-ultra filter.
  $\mathcal{PR}/\equiv_{\mathcal{F}}$ is a nonstandard model of $\mathrm{I}\Delta_0(\mathrm{PrimRec})$.