Categorical approach to first-order modal logic

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Background: Neighborhood-Sheaves?

an objective: to understand
neighborhood-sheaf semantics [Kishida '11]

- a semantics of first-order modal logic (FOML)
- works for modal logic MC (and stronger ones)
- doesn't seem to work for logics weaker than MC
- But why?

N.B. Neighborhood-sheaf semantics is NOT a topos-theoretic semantics for FOML.

This talk

Categorical/coalgebraic model of FOML?

- Intention: by considering general settings we could clarify what is really necessary
- The talk consists of two parts:
  1. What do we need to define a semantics of FOML?
  2. How do we construct such a structure?

Outline

What is needed to interpret FOML?

FOML = prop. logic + modality + terms (+ quantifiers)

Modality is interpreted using

- a T-coalgebra for a functor T : **Sets** $\to$ **Sets**, and
- a natural transformation $\sigma : \mathcal{P} \to \mathcal{P} \circ T$ (so-called predicate lifting)
  - $\mathcal{P}$: contravariant powerset functor

Terms are interpreted using

- a cartesian category $\mathscr{S}$

To combine these structures we need

- a functor $U : \mathscr{S} \to \mathbf{CoAlg}(T)$

Kripke frame as coalgebra

Definition (coalgebra)

For a functor T, a T-coalgebra is a morphism $c : X \to TX$.

Ordinary Kripke frame

Kripke frame is (W, R), where $R \subseteq W \times W$.

Coalgebraic form

Kripke frame is a $\mathcal{P}$-coalgebra $R : W \to \mathcal{P}W$.

Ordinary and coalgebraic form are equivalent:

$$\mathcal{P}(W \times W) \simeq 2^{W \times W} \simeq (2^W)^W \simeq (\mathcal{P}W)^W$$

Kripke semantics in coalgebraic form

(W, R): a Kripke frame
formulas are interpreted as a subset $[\![\varphi]\!] \subseteq W$

Ordinary Kripke semantics

when R is seen as a relation,

- $w \in [\![\Box\varphi]\!]$ is defined as $\forall v.(w\,R\,v \implies v \in [\![\varphi]\!])$

Coalgebraic form

when R is seen as a map,

- $w \in [\![\Box\varphi]\!]$ is defined as $R(w) \subseteq [\![\varphi]\!]$

Modality as Predicate Lifting

- $[\![\Box\varphi]\!] = \{w \mid R(w) \subseteq [\![\varphi]\!]\} = R^{-1}\{Q \mid Q \subseteq [\![\varphi]\!]\}$
- define $\sigma : \mathcal{P}X \to \mathcal{P}(\mathcal{P}X)$ by $\sigma(P) = \{Q \mid Q \subseteq P\}$
- then $[\![\Box]\!] = R^{-1} \circ \sigma$
- similarly $\tau(P) = \{Q \mid Q \cap P \neq \emptyset\}$ defines $[\![\Diamond]\!] = R^{-1} \circ \tau$
- topological interior operator is induced from
  - W: a topological space
  - $N(w)$: the set of neighborhoods of $w \in W$
  - $\rho(P) = \{\mathcal{U} \subseteq \mathcal{P}\mathcal{P}W \mid P \in \mathcal{U}\}$

  as $\mathbf{int}(P) = \{w \mid P \in N(w)\} = N^{-1} \circ \rho(P)$
  (topological interpretation of S4 modal logic)

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Modality for a general coalgebra

In general,

- a functor T : **Sets** $\to$ **Sets**
- a coalgebra $c : X \to TX$
- a natural transformation (predicate lifting) $\sigma : \mathcal{P} \to \mathcal{P} \circ T$

induces a modality $c^{-1} \circ \sigma : \mathcal{P}X \to \mathcal{P}(TX) \to \mathcal{P}X$.

Interpreting terms in a cartesian category

Terms are interpreted as functions (or morphisms).

- $[\![t]\!] : D^n \to D$, if t is a term with n variables
  - e.g. $[\![x + y + 2]\!] : \mathbb{N}^2 \to \mathbb{N}$

$[\![t]\!]$ is defined as:

- constants $[\![c]\!] : 1 \to D$ (given)
- m-ary functions $[\![f]\!] : D^m \to D$ (given)
- variables $[\![x_i]\!] = \pi_i : D^n \to D$
- $[\![f(t_1, \ldots, t_m)]\!]$ is a composition $[\![f]\!] \circ \langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle : D^n \to D^m \to D$

This makes sense in any cartesian category $\mathscr{S}$ and object $D \in \mathscr{S}$.

Coalgebras and cartesian category

We use

- coalgebra to interpret a modality
- cartesian category to interpret terms

related in some way. (N.B. $\mathbf{CoAlg}(T)$ is not cartesian.)
Consider

- a functor $T : \mathbf{Sets} \to \mathbf{Sets}$,
- a cartesian category $\mathscr{S}$, and
- a functor $U : \mathscr{S} \to \mathbf{CoAlg}(T)$
- a predicate lifting $\sigma : \mathcal{P} \to \mathcal{P} \circ T$

(such that $\mathcal{P}(|U(\cdot)|)$ is a hyperdoctrine)
Then we can define a semantics of FOML.

Interpreting FOML

- terms are interpreted in $\mathscr{S}$
- formula φ is interpreted as $[\![\varphi]\!] \subseteq |U(D^n)|$
  - $|U(D^n)|$ is the underlying set of $U(D^n)$
- $\wedge, \vee, \neg, \rightarrow$ are set-theoretic operations
- $\Box$ is interpreted as $\mathcal{P} \circ \sigma$
- $\forall, \exists$ are

$$\forall[\![\varphi]\!] = \{x \in |U(D^n)| \mid (U\pi)^{-1}(x) \subseteq [\![\varphi]\!]\}$$
$$\exists[\![\varphi]\!] = \{x \in |U(D^n)| \mid (U\pi)^{-1}(x) \cap [\![\varphi]\!] \neq \emptyset\}$$

where $\pi : D^{n+1} \rightarrow D^n$.
In other words, the right and left adjoints to $(U\pi)^{-1}$.

Outline

Constructing a concrete model of FOML?

A problem: How do we construct a cartesian category $\mathscr{S}$?

- Typically, choose a suitable subcategory of $\mathbf{CoAlg}(T)/X$ for a fixed $X \in \mathbf{CoAlg}(T)$.
- We will
  - describe a constant domain model
  - observe how Kripke/neighborhood sheaves are defined

Constant domain model

- Fix $c : X \to TX$.
- For a set S, $\pi_2 : S \times X \to X$ can be turned into an object of $\mathbf{CoAlg}(T)/X$:

$$S \times X \simeq \coprod_{a \in S} X \to \coprod_{a \in S} TX \to T\left(\coprod_{a \in S} X\right) \simeq T(S \times X)$$

  More concretely, $(a, x) \mapsto T(\iota_a)(c(x))$
  where $\iota_a : X \ni x \mapsto (a, x) \in S \times X$.
- $\mathscr{S} \subseteq \mathbf{CoAlg}(T)/X$ consisting of coalgebras of this form is cartesian, and this $\mathscr{S}$ gives a model of FOML.

Kripke/neighborhood sheaves

Occasionally "locally isomorphic" maps $D \to X$ form a cartesian category $\mathscr{S} \subseteq \mathbf{CoAlg}(T)/X$.

Example

- Kripke sheaf $p : D \to X$ is a p-morphism s.t. $\forall d \in D$, p is injective on $R_D(d)$
- neighborhood sheaf $p : D \to X$ is a p-morphism s.t. $\forall d \in D, \exists U \in N_D(d)$, p is injective on U

Fact

$p : D \to X$ is a Kripke/neighborhood sheaf iff $\Delta : D \to D \times_X D$ is a p-morphism (homomorphism)

Sheaves in $\mathbf{CoAlg}(T)/X$?

Similar formulation in a more general setting?

A Problem

Given T-hom $p : D \to X$, is $D \times_X D$ a T-coalgebra?

- we have $D \times_X D \to TD \times_{TX} TD$
- but not $D \times_X D \to T(D \times_X D)$

What do we need to define $D \times_X D \to T(D \times_X D)$?

An observation on sheaves

- in Kripke/neighborhood sheaf cases, there are weaker notion of maps between coalgebras
  - graph homs, continuous maps
- the categories of Kripke/neighborhood frames and such maps have pullbacks, hence their slices are cartesian
- so $D \times_X D$ is equipped with coalgebra structure
- define sheaf as $p : D \to X$ s.t. $\Delta : D \to D \times_X D$ is a homomorphism
- $\mathscr{S}$ is defined as the subcategory of sheaves

Open question: does this argument work in general?

Outline

Related work

Topos semantics for higher-order S4
[Awodey, Kishida, Kotzsch '14]

- uses a certain Heyting-algebra object instead of Ω
- geometric morphism induces an S4 modality

Summary and further questions

Summary:

- first-order modal logic is interpreted by
  - cartesian category $\mathscr{S}$ and
  - a functor $\mathscr{S} \to \mathbf{CoAlg}(T)$

Open question:

- recipe for constructing $\mathscr{S}$?
- a notion of "sheaf" for coalgebras?