# Reduction System for Extensional Λμ-Calculus

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# Result

- A new reduction system
  for (a variant of) λμ-calculus
  - confluence
  - subject reduction
  - strong normalization

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$\Lambda\mu_{cons}$

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  - proof terms for CND

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  - proof terms for CND
- η destroys confluence [David&Py01]
  - λμ does not enjoy separation thm
- Λμ-calculus [Saurin05,10]
  - separation, confluence, and SN

# η and confluence

# βη for λ

$$t,u ::= x \mid \lambda x.t \mid tu$$

$$(\lambda x.t)u \rightarrow_\beta t[x:=u]$$

$$\lambda x.tx \rightarrow_\eta t \qquad (x \notin FV(t))$$

**Theorem** (Confluence)
βη is confluent

# βημ for λμ

$t,u ::= x \mid \lambda x.t \mid tu \mid \mu a.t \mid ta$

$$(\lambda x.t)u \to_\beta t[x:=u]$$
$$(\mu a.t)b \to_\beta t[a:=b]$$
$$\lambda x.tx \to_\eta t \qquad (x \notin FV(t))$$
$$\mu a.ta \to_\eta t \qquad (a \notin FV(t))$$
$$(\mu a.t)u \to_\mu \mu b.t[va:=(vu)b]$$

# Example

car = $\lambda x.\mu a.x$

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car $t_1$ $t_2$ $t_3$ b

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car $t_1$ $t_2$ $t_3$ b

= $(\lambda x.\mu a.x)$ $t_1$ $t_2$ $t_3$ b

# Example

$car = \lambda x.\mu a.x$

$car\ t_1\ t_2\ t_3\ b$

$= \quad (\lambda x.\mu a.x)\ t_1\ t_2\ t_3\ b$

$\to_\beta \quad (\mu a.\ t_1)\ t_2\ t_3\ b$

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$\rightarrow^*_\mu \quad t_1[va{:=}v\ t_2\ t_3\ b] \qquad (a \notin t_1)$

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$= \quad t_1$

# Example

$E = [\,]\ t_1\ t_2\ \ldots\ t_n\ a$

# Example

$E = [\,]\ t_1\ t_2\ \ldots\ t_n\ a$

$E[\mu b.t]$

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$= \quad (\mu b.t)\ t_1\ t_2\ \ldots\ t_n\ a$

$\to^{*}_{\mu}\ (\mu c.t[vb{:=}v t_1 t_2 \ldots t_n c])a$

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$\to_\beta\ \ t\,[vb{:=}v t_1 t_2 \ldots t_n a]$

# Example

$E = [\,]\ t_1\ t_2\ \ldots\ t_n\ a$

$E[\mu b.t]$

$$
\begin{aligned}
&= (\mu b.t)\ t_1\ t_2\ \ldots\ t_n\ a \\
&\to^*_\mu (\mu c.t[vb{:=}v t_1 t_2 \ldots t_n c])a \\
&\to_\beta\ t\,[vb{:=}v t_1 t_2 \ldots t_n a] \\
&= t[vb := E[v]]
\end{aligned}
$$

# Example

$E = [\,]\ t_1\ t_2\ \ldots\ t_n\ a$

a stream with
initial segment $t_1 \ldots t_n$
and tail part a

$E[\mu b.t]$

$$= \quad (\mu b.t)\ t_1\ t_2\ \ldots\ t_n\ a$$

$$\to^*_\mu\ (\mu c.t[vb{:=}vt_1t_2\ldots t_nc])a$$

$$\to_\beta\ \ t\,[vb{:=}vt_1t_2\ldots t_na]$$

$$= \quad t[vb := E[v]]$$

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$E = [\,]\ t_1\ t_2\ \ldots\ t_n\ a$

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$E[\mu b.t]$

$$
\begin{aligned}
&= \quad (\mu b.t)\ t_1\ t_2\ \ldots\ t_n\ a \\
&\to^{*}_{\mu}\ (\mu c.t[vb{:=}vt_1t_2\ldots t_nc])a \\
&\to_{\beta}\ t\,[vb{:=}vt_1t_2\ldots t_na] \\
&= \quad t[vb := E[v]]
\end{aligned}
$$

μ ≒ function on
streams

# η versus μ

$\lambda x.(\mu a.y)x$

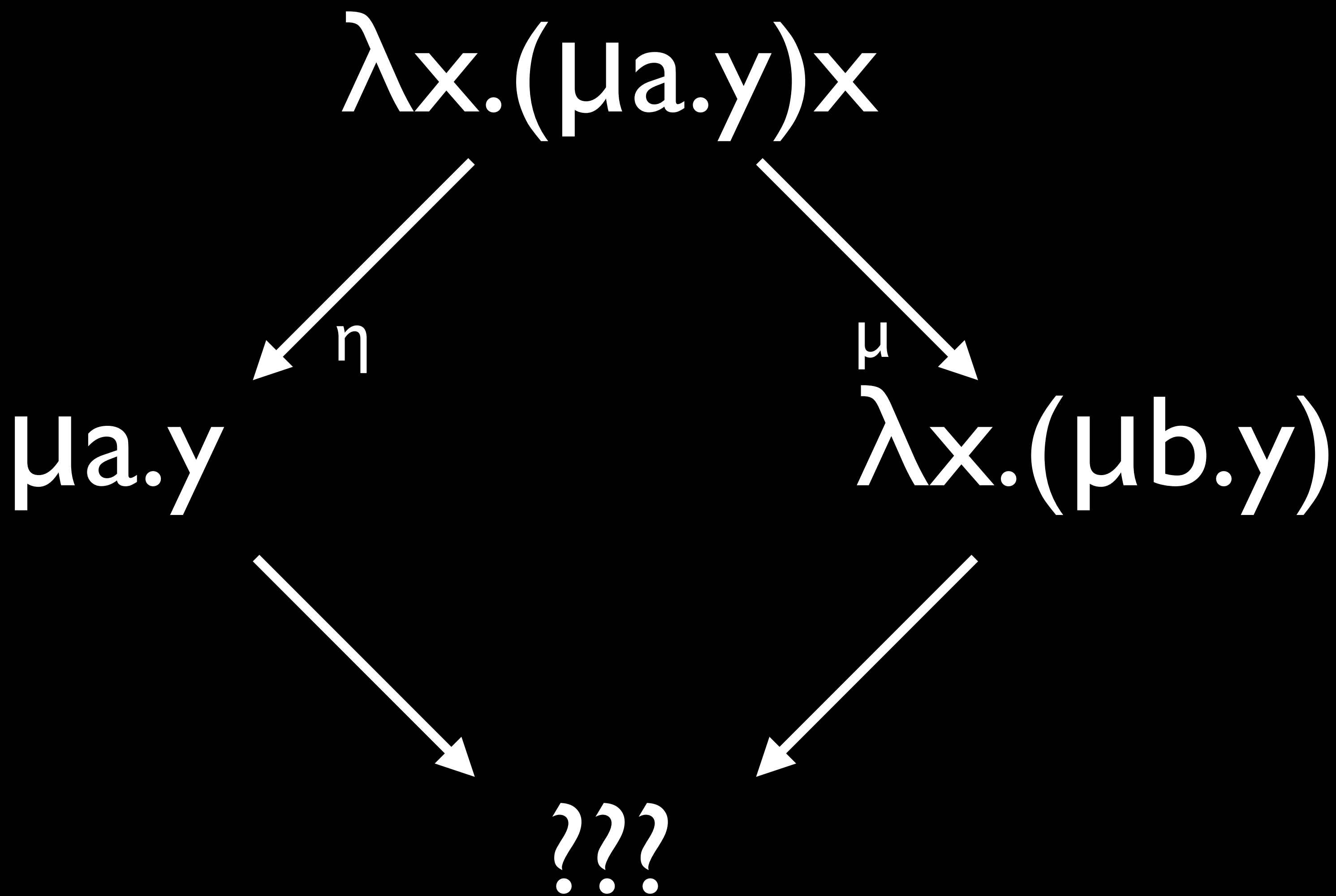
# η versus μ

λx.(μa.y)x

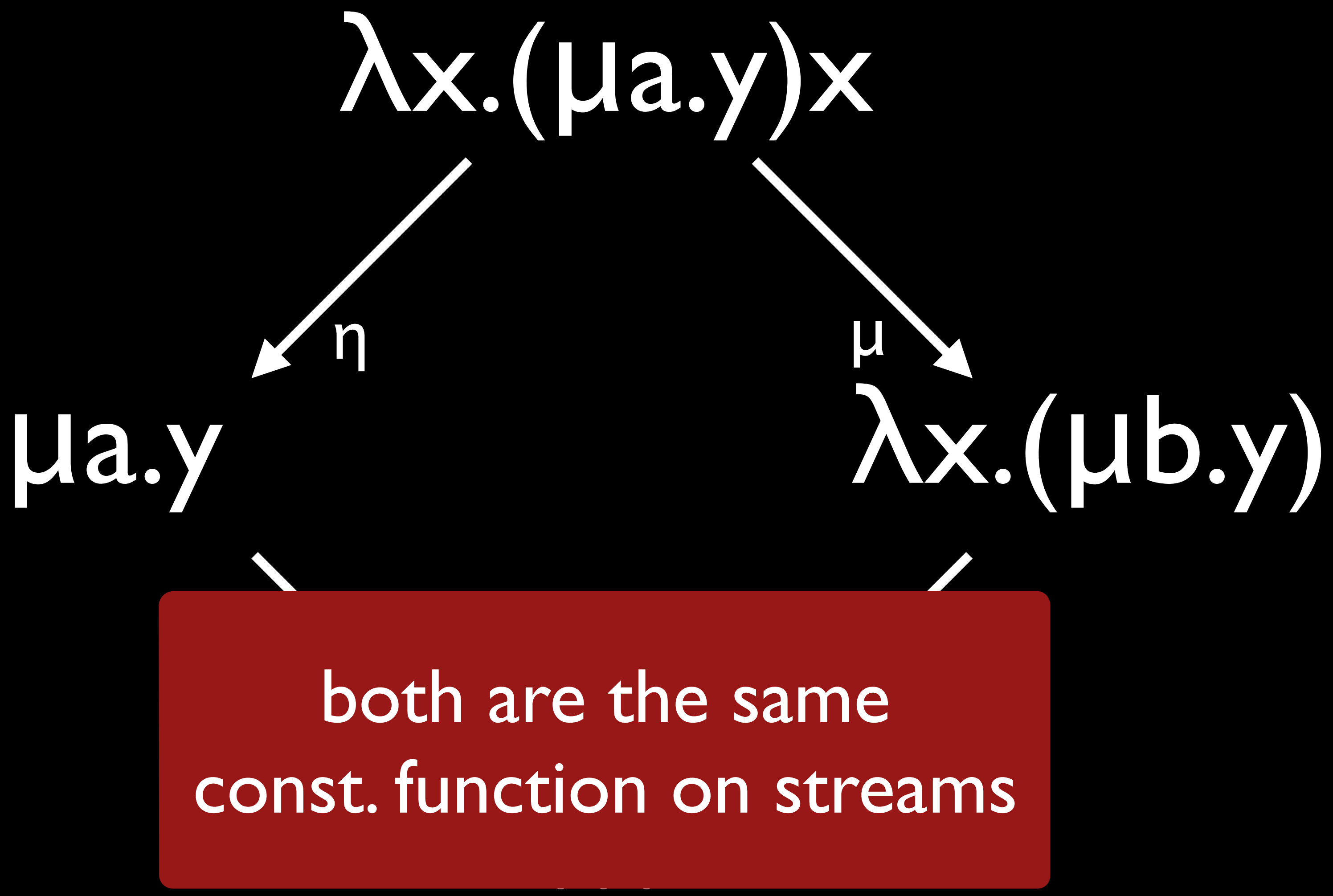
η ↙ ↘ μ

μa.y　　λx.(μb.y)

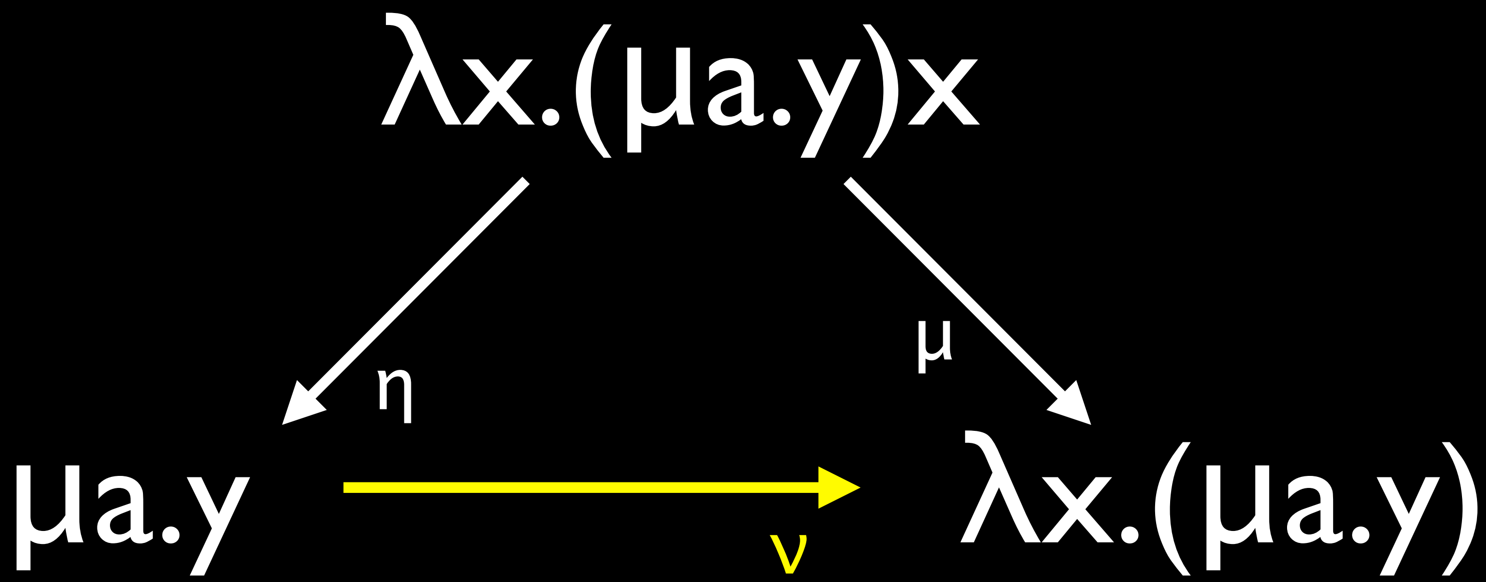
# η versus μ



# η versus μ

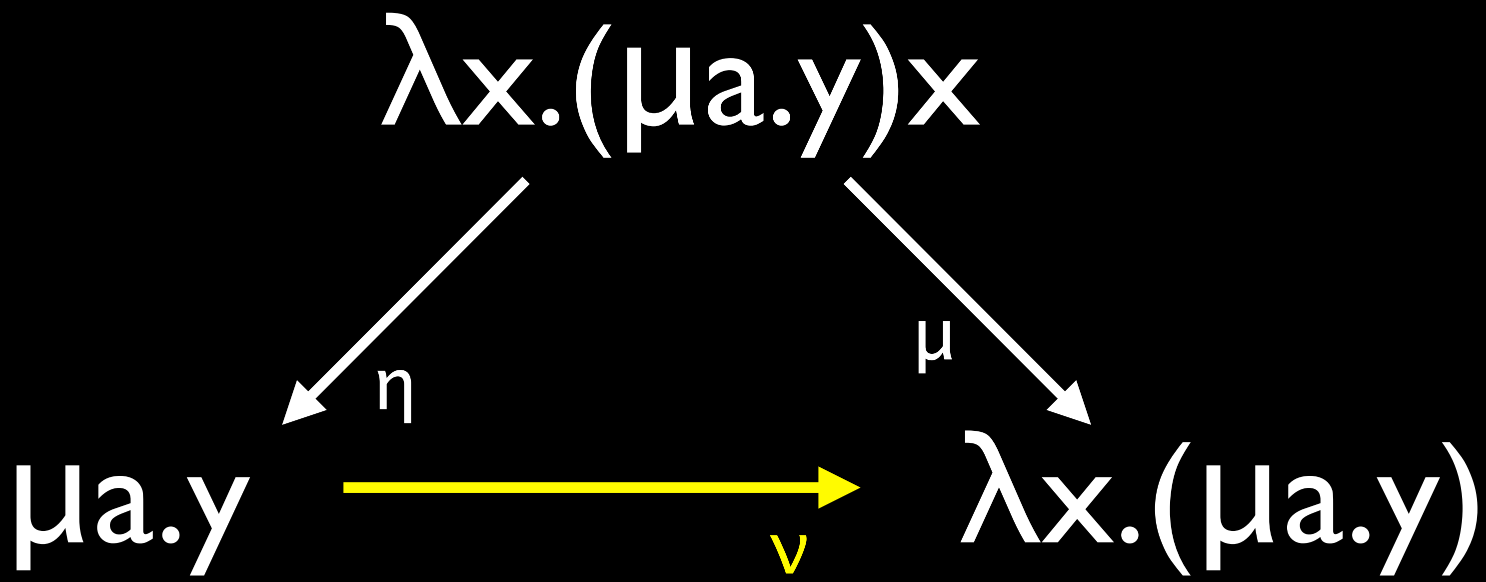

λx.(μa.y)x
η
μ
μa.y
λx.(μb.y)
both are the same
const. function on streams

# A solution [David&Py01]




$$\mu a.t \ \to_\nu \ \lambda x.\mu b.t[\nu a:=(\nu x)b]$$

# A solution [David&Py01]




$a \longmapsto t$

$x{::}b \longmapsto t[a := x{::}b]$

$$\mu a.t \ \rightarrow_\nu \ \lambda x.\mu b.t[\nu a{:=}(\nu x)b]$$

# Λμ [Saurin05]

$t, u ::= x \mid \lambda x.t \mid tu \mid \mu a.t \mid ta$

$$(\lambda x.t)u \rightarrow_\beta t[x:=u]$$
$$(\mu a.t)b \rightarrow_\beta t[a:=b]$$
$$\lambda x.tx \rightarrow_\eta t \quad (x \notin FV(t))$$
$$\mu a.ta \rightarrow_\eta t \quad (a \notin FV(t))$$
$$\mu a.t \rightarrow_\nu \lambda x.\mu b.t[va:=(vx)b]$$

# Properties of Λμ

Theorem (Confluence [Saurin05&10])
Λμ is confluent for stream closed terms

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$\Rightarrow \exists$ E s.t. $E[t] \to^* \text{true}$ & $E[u] \to^* \text{false}$

# Properties of Λμ

**Theorem** (Confluence [Saurin05&10])
Λμ is confluent for stream closed terms

**Theorem** (Separation [Saurin05])
t, u: distinct canonical nf
$\Rightarrow \exists$ E s.t. $E[t] \rightarrow^* \text{true}$ & $E[u] \rightarrow^* \text{false}$

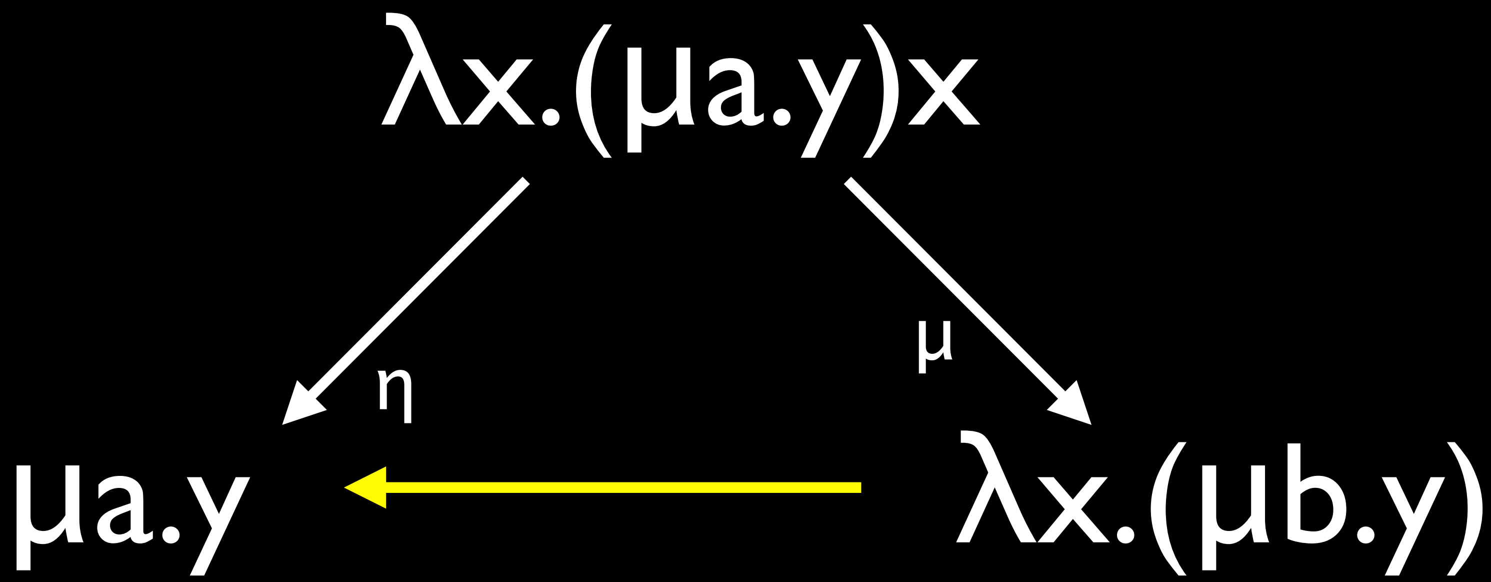
**Theorem** (Strong normalization [Saurin10])
Typable terms are strongly normalizable

# ν is type dependent

$$\mu a^{A\times S}.t \;\to_\nu\; \lambda x^{A}.\mu b^{S}.t[va:=(vx)b]$$

- admissible only when the type of a is of the form "A×S"

# We propose



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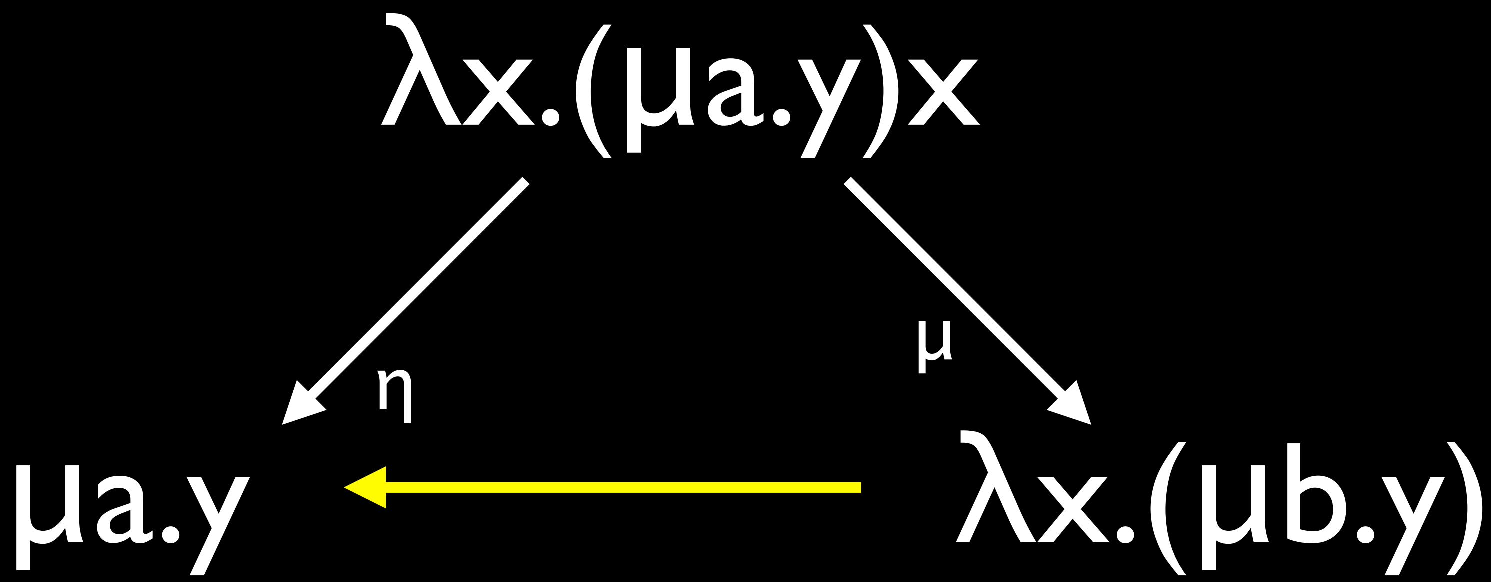


$$\lambda x.t \;\to\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

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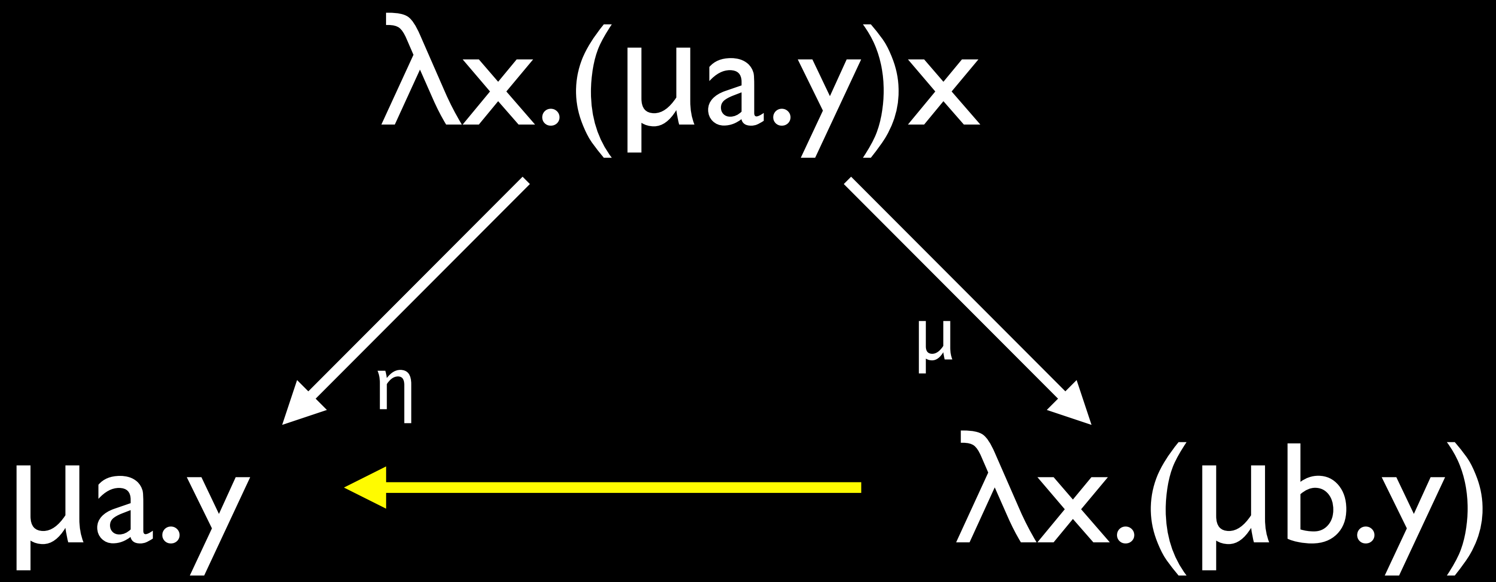


$$\lambda x.t \;\rightarrow\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

with some extension of explicit stream expressions

# This work

- $\Lambda\mu_{cons}$
  - a conservative extension of $\Lambda\mu$
- Type indep. reduction for $\Lambda\mu_{cons}$
  - CR (not only for closed) and SN

# $\Lambda\mu_{cons}$

equational logic

# $\Lambda\mu_{cons}$

Terms: $t,u ::= x \mid \lambda x.t \mid tu \mid \mu a.t \mid tS \mid \text{car } S$

Streams: $S ::= a \mid t::S \mid \text{cdr } S$

Axioms:

$$(\lambda x.t)u = t[x:=u]$$

$$(\mu a.t)S = t[a:=S]$$

$$\lambda x.tx = t \qquad (x \notin FV(t))$$

$$\mu a.ta = t \qquad (a \notin FV(t))$$

$$t\ u\ S = t\ (u::S)$$

$$\text{car}(t::S) = t$$

$$\text{cdr}(t::S) = S$$

$$(\text{car } S)::(\text{cdr } S) = S$$

# $\Lambda\mu_{cons}$

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$t\ u\ S = t\ (u::S)$

$\text{car}(t::S) = t$

$\text{cdr}(t::S) = S$

$(\text{car } S)::(\text{cdr } S) = S$

$t\ t_1 \ldots t_n\ a = t(t_1:: \ldots ::t_n::a)$

# $\Lambda\mu_{cons}$

Terms: $t,u ::= x \mid \lambda x.t \mid tu \mid \mu a.t \mid tS \mid car\ S$

Streams: $S ::= a \mid t::S \mid cdr\ S$

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$t\ u\ S = t\ (u::S)$

$car(t::S) = t$

$cdr(t::S) = S$

$(car\ S)::(cdr\ S) = S$

$t\ t_1 \ldots t_n\ a = t(t_1:: \ldots ::t_n::a)$

(::) is surjective

# Example

Y = fix pt operator in λ

nth = Y(λf.μa.λn.
ifzero n then (car a)
else (f (n-1) (cdr a))

$$\text{nth } S\ n = \text{car } (\text{cdr}^n\ S)$$

# Conservation

**Theorem** (Conservation)

For any t,u $\in \Lambda\mu$,

$$t = u \text{ in } \Lambda\mu_{cons} \quad \Leftrightarrow \quad t = u \text{ in } \Lambda\mu$$

# $\Lambda\mu_{cons}$ reduction system

# Reduction for $\Lambda\mu_{cons}$

- "complete" w.r.t. the equality
  - equivalence closure of $\rightarrow$ is $=$
- confluent
- SR and SN
  - type independent

# μ-reduction in $\Lambda\mu_{cons}$

$$(\mu a.t)u \rightarrow_{\mu} \mu b.t[va:=(vu)b]$$

# μ-reduction in $\Lambda\mu_{cons}$

$$(\mu a.t)u \rightarrow_{\mu} \mu b.t[a{:=}u{::}b]$$

# μ-reduction in $\Lambda\mu_{cons}$

$$(\mu a.t)u \;\to_\mu\; \mu b.t[a{:=}u{::}b]$$

(μa. … va …)u

# μ-reduction in $\Lambda\mu_{cons}$

$$(\mu a.t)u \;\to_\mu\; \mu b.t[a{:=}u{::}b]$$

$(\mu a.\ \ldots\ va\ \ldots)u$

$\quad\to_\mu\ \mu b.\ \ldots\ v(u{::}b)\ \ldots$

# μ-reduction in $\Lambda\mu_{cons}$

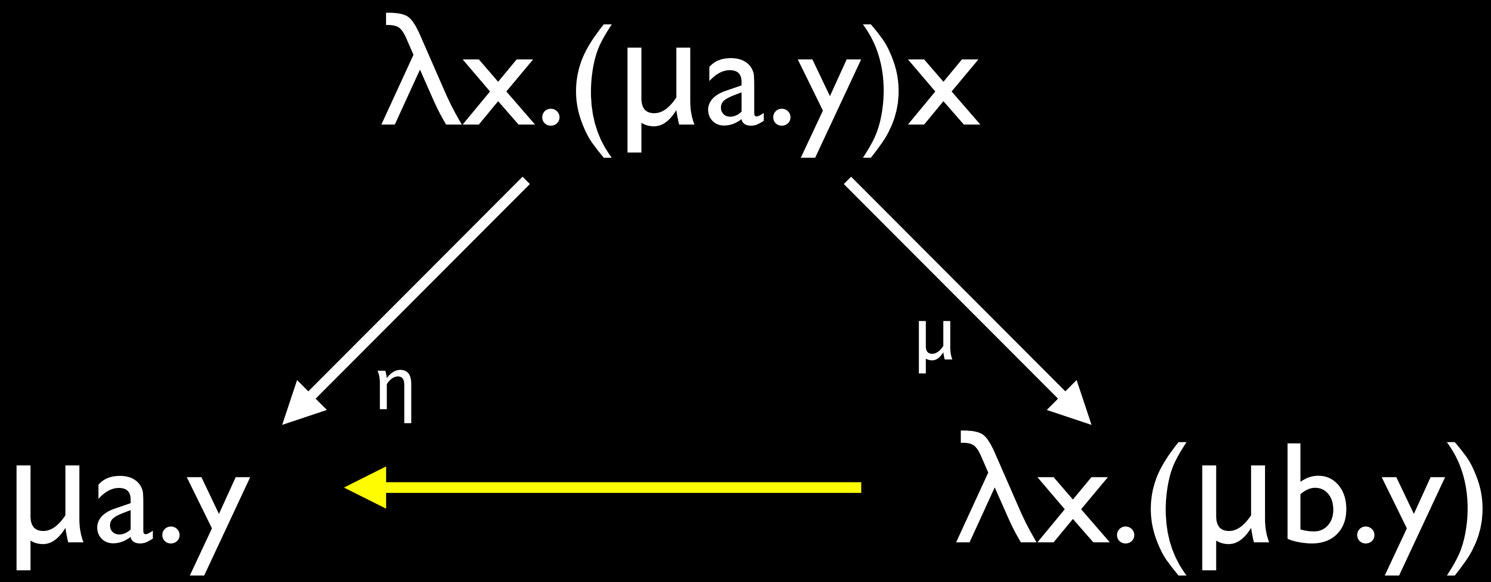
$$(\mu a.t)u \;\to_\mu\; \mu b.t[a{:=}u{::}b]$$

$$(\mu a.\ \ldots\ va\ \ldots)u$$

$$\to_\mu\ \mu b.\ \ldots\ v(u{::}b)\ \ldots$$

$$=\ \mu b.\ \ldots\ (vu)b\ \ldots$$

# New rule




$$\lambda x.t \;\rightarrow\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

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$\lambda x.t \rightarrow \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$

$(\lambda x.t)u$

# New rule

$$\lambda x.t \to \mu a.t[x{:=}\text{car } a](\text{cdr } a)$$

$(\lambda x.t)u$

$\to (\mu a.t[x{:=}\text{car } a](\text{cdr } a))u$

# New rule

$$\lambda x.t \;\rightarrow\; \mu a.t[x:=\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$(\lambda x.t)u$

$\rightarrow\;(\mu a.t[x:=\mathrm{car}\ a](\mathrm{cdr}\ a))u$

$\rightarrow_{\mu}\;\mu b.t[x:=\mathrm{car}\ (u{::}b)](\mathrm{cdr}\ (u{::}b))$

# New rule

$$\lambda x.t \;\to\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$(\lambda x.t)u$

$\to\ (\mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a))u$

$\to_{\mu}\ \mu b.t[x{:=}\mathrm{car}\ (u{::}b)](\mathrm{cdr}\ (u{::}b))$

$\to^{*}\ \mu b.t[x{:=}u]b$

# New rule

$$\lambda x.t \to \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$(\lambda x.t)u$

$\to (\mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a))u$

$\to_\mu \mu b.t[x{:=}\mathrm{car}\ (u{::}b)](\mathrm{cdr}\ (u{::}b))$

$\to^* \mu b.t[x{:=}u]b$

$\to_\eta t[x{:=}u]$

# New rule

$$\lambda x.t \rightarrow \mu a.t[x:=\text{car } a](\text{cdr } a)$$

# New rule

$\lambda x.t \;\rightarrow\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$

$\lambda x.tx$

# New rule

$$\lambda x.t \;\rightarrow\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$\lambda x.tx$

$$\rightarrow\; \mu a.(tx)[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

# New rule

$$\lambda x.t \rightarrow \mu a.t[x:=\text{car } a](\text{cdr } a)$$

$\lambda x.tx$

$\rightarrow \mu a.(tx)[x:=\text{car } a](\text{cdr } a)$

$\equiv \mu a.t(\text{car } a)(\text{cdr } a)$

# New rule

$$\lambda x.t \rightarrow \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$\lambda x.tx$

$$\rightarrow \mu a.(tx)[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$$\equiv \mu a.t(\mathrm{car}\ a)(\mathrm{cdr}\ a)$$

$$= \mu a.t(\mathrm{car}\ a :: \mathrm{cdr}\ a)$$

# New rule

$$\lambda x.t \;\to\; \mu a.t[x:=\text{car } a](\text{cdr } a)$$

$\lambda x.tx$

$\to \quad \mu a.(tx)[x:=\text{car } a](\text{cdr } a)$

$\equiv \quad \mu a.t(\text{car } a)(\text{cdr } a)$

$= \quad \mu a.t(\text{car } a :: \text{cdr } a)$

$\to \quad \mu a.ta$

# New rule

$$\lambda x.t \;\rightarrow\; \mu a.t[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$\lambda x.tx$

$$\rightarrow \quad \mu a.(tx)[x{:=}\mathrm{car}\ a](\mathrm{cdr}\ a)$$

$$\equiv \quad \mu a.t(\mathrm{car}\ a)(\mathrm{cdr}\ a)$$

$$= \quad \mu a.t(\mathrm{car}\ a :: \mathrm{cdr}\ a)$$

$$\rightarrow \quad \mu a.ta$$

$$\rightarrow_{\eta} \quad t$$

# Reduction for $\Lambda\mu_{cons}$

$$
\begin{aligned}
(\mu\alpha.t)u &\to t[\alpha := u :: \alpha] && (\beta_T)\\
(\mu\alpha.t)S &\to t[\alpha := S] && (\beta_S)\\
\lambda x.t &\to \mu\alpha.t[x := \mathsf{car}\alpha](\mathsf{cdr}\alpha) && (\mathsf{exp})\\
t(u :: S) &\to tuS && (\mathsf{assoc})\\
\mathsf{car}(u :: S) &\to u && (\mathsf{car})\\
\mathsf{cdr}(u :: S) &\to S && (\mathsf{cdr})\\
\mu\alpha.t\alpha &\to t \qquad (\alpha \notin FV(t)) && (\eta_S)\\
(\mathsf{car}S) :: (\mathsf{cdr}S) &\to S && (\eta_{::})\\
t(\mathsf{car}S)(\mathsf{cdr}S) &\to tS && (\eta'_{::})
\end{aligned}
$$

# Confluence

# Reduction for Λμ$_{\text{cons}}$

B

$$(\mu\alpha.t)u \to t[\alpha := u :: \alpha] \qquad (\beta_T)$$

$$(\mu\alpha.t)S \to t[\alpha := S] \qquad (\beta_S)$$

$$\lambda x.t \to \mu\alpha.t[x := \mathsf{car}\alpha](\mathsf{cdr}\alpha) \qquad (\mathsf{exp})$$

$$t(u :: S) \to tuS \qquad (\mathsf{assoc})$$

$$\mathsf{car}(u :: S) \to u \qquad (\mathsf{car})$$

$$\mathsf{cdr}(u :: S) \to S \qquad (\mathsf{cdr})$$

$$\mu\alpha.t\alpha \to t \quad (\alpha \notin FV(t)) \qquad (\eta_S)$$

$$(\mathsf{car}S) :: (\mathsf{cdr}S) \to S \qquad (\eta_{::})$$

$$t(\mathsf{car}S)(\mathsf{cdr}S) \to tS \qquad (\eta'_{::})$$

E

# Confluence

B

| | |
|---|---|
| $(\mu\alpha.t)u \to t[\alpha := u :: \alpha]$ | $(\beta_T)$ |
| $(\mu\alpha.t)S \to t[\alpha := S]$ | $(\beta_S)$ |
| $\lambda x.t \to \mu\alpha.t[x := \mathsf{car}\alpha](\mathsf{cdr}\alpha)$ | (exp) |
| $t(u :: S) \to tuS$ | (assoc) |
| $\mathsf{car}(u :: S) \to u$ | (car) |
| $\mathsf{cdr}(u :: S) \to S$ | (cdr) |
| $\mu\alpha.t\alpha \to t \qquad (\alpha \notin FV(t))$ | $(\eta_S)$ |
| $(\mathsf{car}S) :: (\mathsf{cdr}S) \to S$ | $(\eta_{::})$ |
| $t(\mathsf{car}S)(\mathsf{cdr}S) \to tS$ | $(\eta'_{::})$ |

E

- Confluence of B
- Confluence of E
- B and E commute

# Confluence

B

$$
\begin{aligned}
(\mu\alpha.t)u &\to t[\alpha := u :: \alpha] && (\beta_T)\\
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\mathsf{cdr}(u :: S) &\to S && (\mathsf{cdr})\\
\mu\alpha.t\alpha &\to t \qquad (\alpha \notin FV(t)) && (\eta_S)\\
(\mathsf{car}\,S) :: (\mathsf{cdr}\,S) &\to S && (\eta_{::})\\
t(\mathsf{car}\,S)(\mathsf{cdr}\,S) &\to tS && (\eta'_{::})
\end{aligned}
$$

E

easy
by Newman's lemma

- Confluence of B
- Confluence of E
- B and E commute

# Confluence

B

$$(\mu\alpha.t)u \to t[\alpha := u :: \alpha] \qquad (\beta_T)$$
$$(\mu\alpha.t)S \to t[\alpha := S] \qquad (\beta_S)$$
$$\lambda x.t \to \mu\alpha.t[x := \mathsf{car}\alpha](\mathsf{cdr}\alpha) \qquad (\mathsf{exp})$$
$$t(u :: S) \to tuS \qquad (\mathsf{assoc})$$
$$\mathsf{car}(u :: S) \to u \qquad (\mathsf{car})$$
$$\mathsf{cdr}(u :: S) \to S \qquad (\mathsf{cdr})$$
$$\mu\alpha.t\alpha \to t \quad (\alpha \notin FV(t)) \qquad (\eta_S)$$
$$(\mathsf{car}S) :: (\mathsf{cdr}S) \to S \qquad (\eta_{::})$$
$$t(\mathsf{car}S)(\mathsf{cdr}S) \to tS \qquad (\eta'_{::})$$

E

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- Confluence of E
- B and E commute

easy
by Newman's lemma

a little complicated
due to non-linear rules

# Confluence

B

| | |
|---|---|
| $(\mu\alpha.t)u \to t[\alpha := u :: \alpha]$ | $(\beta_T)$ |
| $(\mu\alpha.t)S \to t[\alpha := S]$ | $(\beta_S)$ |
| $\lambda x.t \to \mu\alpha.t[x := \mathsf{car}\alpha](\mathsf{cdr}\alpha)$ | (exp) |
| $t(u :: S) \to tuS$ | (assoc) |
| $\mathsf{car}(u :: S) \to u$ | (car) |
| $\mathsf{cdr}(u :: S) \to S$ | (cdr) |
| $\mu\alpha.t\alpha \to t \qquad (\alpha$ | |
| $(\mathsf{car}S) :: (\mathsf{cdr}S) \to S$ | |
| $t(\mathsf{car}S)(\mathsf{cdr}S) \to tS$ | |

E

by “generalized complete development”

- Confluence of B
- Confluence of E
- B and E commute

easy
by Newman’s lemma

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due to non-linear rules

# Confluence of B

- By "generalized complete development"

**Theorem** [Dehornoy+08, Komori+13]
(A, →): abstract rewriting system
if there exists $(\cdot)^+ : A \to A$ s.t.

$$a \to b \quad \Rightarrow \quad b \to^* a^+ \to^* b^+$$

then (A, →) is confluent

# Confluence of B

$$x^\dagger = x$$

$$(\lambda x.t)^\dagger = \mu\alpha.t^\dagger[x := \mathsf{car}\alpha](\mathsf{cdr}\alpha)$$

$$(\mu\alpha.t)^\dagger = \mu\alpha.t^\dagger$$

$$((\mu\alpha.t)u)^\dagger = \mu\alpha.t^\dagger[\alpha := u^\dagger :: \alpha]$$

$$(tu)^\dagger = t^\dagger u^\dagger \quad (\text{otherwise})$$

$$((\mu\alpha.t)S)^\dagger = t^\dagger[\alpha := S^\dagger]$$

$$((\mu\alpha.t)uS)^\dagger = t^\dagger[\alpha := u^\dagger :: S^\dagger]$$

$$(t(u :: S))^\dagger = t^\dagger u^\dagger S^\dagger \quad (t \neq \mu\text{-abst.})$$

$$(tS)^\dagger = t^\dagger S^\dagger \quad (\text{otherwise})$$

$$(\mathsf{car}(t :: S))^\dagger = t^\dagger$$

$$(\mathsf{car}S)^\dagger = \mathsf{car}S^\dagger \quad (\text{otherwise})$$

$$\alpha^\dagger = \alpha$$

$$(\mathsf{cdr}(t :: S))^\dagger = S^\dagger$$

$$(\mathsf{cdr}S)^\dagger = \mathsf{cdr}S^\dagger \quad (\text{otherwise})$$

$$(t :: S)^\dagger = t^\dagger :: S^\dagger$$

# Confluence of $\Lambda\mu_{cons}$

**Theorem** (Confluence)

$\Lambda\mu_{cons}$ is confluent

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not restricted to stream closed terms

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**Theorem** (Confluence)

$\Lambda\mu_{cons}$ is confluent

**Theorem** (Separation)

t, u: distinct nf

$\Rightarrow \exists$ E s.t. $E[t] \to^* true$ & $E[u] \to^* false$

# Typed $\Lambda\mu_{cons}$

**Theorem** (Subject reduction)

$$\Gamma \mid \Delta \vdash t : A \;\&\; t \to u \quad \Rightarrow \quad \Gamma \mid \Delta \vdash u : A$$

# Typed $\Lambda\mu_{cons}$

**Theorem** (Subject reduction)

$\Gamma \mid \Delta \vdash t : A \;\&\; t \rightarrow u \quad \Rightarrow \quad \Gamma \mid \Delta \vdash u : A$

**Theorem** (Strong normalization)

any typable term is strongly normalizable

# Conclusion

# Result

- $\Lambda\mu_{cons}$
  - a conservative extension of $\Lambda\mu$
- reduction for $\Lambda\mu_{cons}$
  - CR, SR, and SN

# Other topics

- Stream models [N&Katsumata12]
  - $S = D \times S$ & $D = S \to D$
  - $\Lambda\mu_{cons}$ is sound and complete

# Other topics

- Stream models [N&Katsumata12]
  - $S = D \times S$ & $D = S \to D$
  - $\Lambda\mu_{cons}$ is sound and complete
- Friedman's theorem for typed $\Lambda\mu_{cons}$
  - extensional equality is characterized by any "full" stream model

# Further direction

- Categorical stream models
- Logical aspect of typed $\Lambda\mu_{cons}$
- Combinatory calculus corresponding to $\Lambda\mu_{cons}$
- Application for program with streams