On a Circuit Model proving the existence of Canards in R^{2+2}

27th May, 2014

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tableau

(1) Time-scaled-reduced system

(2) Pseudo-singular point

(3) 3-dimensional Canards

(4) 4-dimensional Canards

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(6) Circuit model

Slow-fast system in R^{2+1}

$\varepsilon dx/dt = h(x,y,\varepsilon)$

$dy/dt = f(x,y,\varepsilon)$

$x \in R$, $y \in R^2$, $\varepsilon > 0$ infinitesimal

$h: R^{2+1} \to R$, $f: R^{2+1} \to R^2$

assumptions

(A1) $S = \{(x,y) \in R^3 \mid h(x,y,0) = 0\}$ is a 2-dim diff manifold, and S intersects

$T = \{(x,y) \in R^3 \mid \partial h(x,y,0)/\partial x = 0\}$ transversely, so that the pli set

$PL = \{(x,y) \in S \cap T\}$ is 1-dim diff manifold

(A2) $f_1(x,y,0) \neq 0$ or $f_2(x,y,0) \neq 0$ on $(x,y) \in PL$

time-scaled-reduced system

On the set S: $h(x,y,0)=0$, differentiating by t,

$$[\partial h/\partial y]\, f(x,y,0)+(\partial h/\partial x)dx/dt=0$$

Then, the above system restricted on S is

$$dy/dt= f(x,y,0)$$
$$dx/dt=-\, [\partial h/\partial y]\, f(x,y,0)/\, (\partial h/\partial x)$$

where $(x,y)\in S\backslash PL$

- To avoid degeneracy, let us consider the following time-scaled-reduced system:

$$dy/dt = -\,(\partial h(x,y,0)/\partial x)\ f(x,y,0)$$

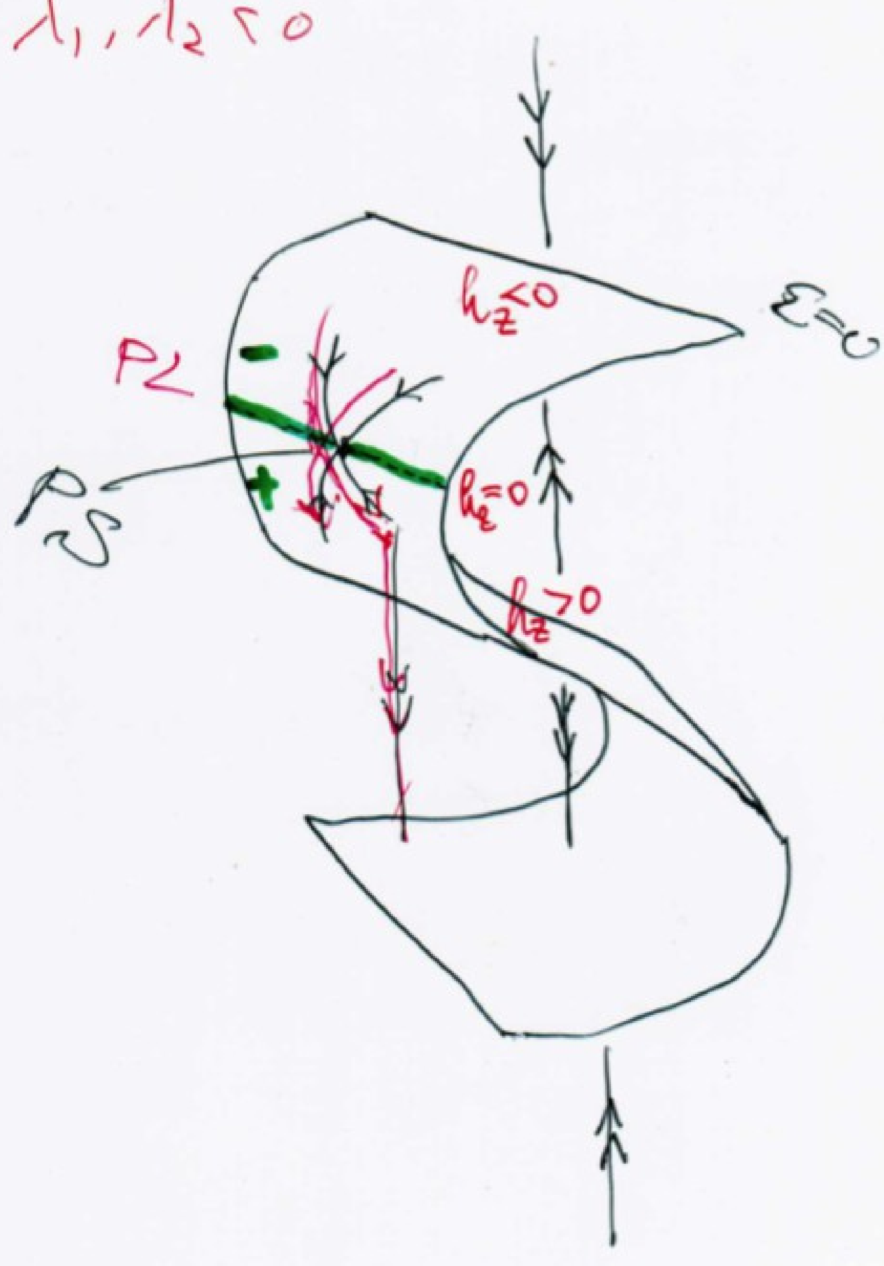
$$dx/dt = [\partial h(x,y,0)/\partial y]\ f(x,y,0)$$

(A3) $\forall (x,y) \in S$, $\partial h/\partial y_1 \neq 0$, $\partial h/\partial y_2 \neq 0$

it implies implicit function theorem is applied

(A4) all the singular points of the above system are non-degenerate

$$PS = \{(x,y) \in PL \mid [\partial h/\partial y]\, f(x,y,0)=0\}$$


λ1 , λ2 < 0
hz<0
ε=0
PL
-
+
PS
hz=0
hz>0

Slow-fast system in R^{2+2}

$$\varepsilon dx/dt = h(x,y,\varepsilon)$$

$$dy/dt = f(x,y,\varepsilon)$$

$x \in R^2$, $y \in R^2$, $\varepsilon > 0$ infinitesimal,

$h: R^{4+1} \to R^2$, $f: R^{4+1} \to R^2$

assumptions

(B1) $S = \{(x,y) \in R^4 \mid h(x,y,0) = 0\}$ is a 2-dim diff manifold, and S intersects

$T = \{(x,y) \in R^4 \mid \det[\partial h(x,y,0)/\partial x] = 0\}$ transversely, so that the pli set

$PL = \{(x,y) \in S \cap T\}$ is 1-dim diff manifold

(B2) $f_1(x,y,0) \neq 0$ or $f_2(x,y,0) \neq 0$ at $(x,y) \in PL$

(B3) $\forall (x,y) \in S \setminus PL$, rank $[\partial h/\partial x] = 2$,

$\forall (x,y) \in S$, rank $[\partial h/\partial y] = 2$

on the set PL,

$\partial h_1/\partial x_2 \neq 0$, or $\partial h_2/\partial x_1 \neq 0$

On the set S: $h(x,y,0)=0$, differentiating by t,

$$[\partial h/\partial y]\ (dy/dt)+[\partial h/\partial x]\ (dx/dt)=0$$

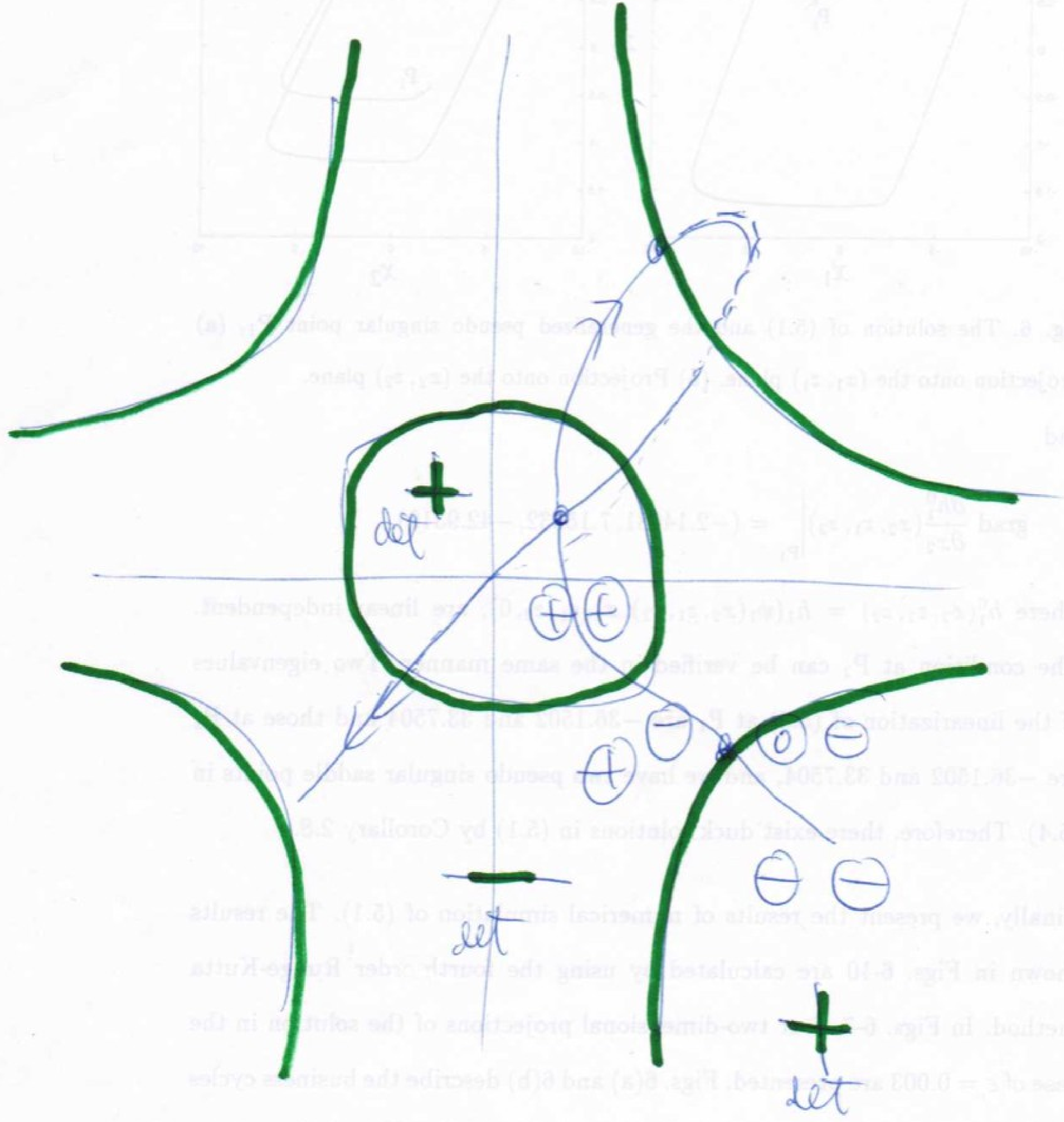
$$dx/dt=-[\partial h/\partial x]^{-1}\ [\partial h/\partial y]\ f(x,y,0)$$

To avoid degeneracy, we consider the time-scaled-reduced system:

$$dx/dt=$$

$$-\det[\partial h/\partial x][\partial h/\partial x]^{-1}\ [\partial h/\partial y]\ f(x,y,0),$$

$$\exists g: y = g(x)$$



det
det
det

Remark.1

- All the singular points of the time scaled reduced system are contained in the set

PS =

$\{(x,y)\in PL \mid -\det[\partial h/\partial x]\, [\partial h/\partial x]^{-1}$

$[\partial h/\partial y]\, f(x,y,0)=0\},\ \exists g: y= g(x)$

They are called pseudo-singular points in R^4.

(B4) all the singular points of the time-scaled-reduced system are non-degenerate.

(B5) the invariant manifold Inv(h) intersects the set PL transversely

Definition of Canards in R^4

Def.1

Let $p \in PS$, and let λ_1, λ_2 be

2 eigenvalues associated with the linearized time-scaled-reduced system.

It is called saddle,

if $\lambda_1 < 0 < \lambda_2$ ($\lambda_1 > 0 > \lambda_2$).

It is called node,

if $\lambda_1 > \lambda_2 > 0$ ($\lambda_1 < \lambda_2 < 0$)

and it is called forcus,

if $\lambda = a \pm ib$.

Def.2

Let $p \in PS$ be saddle or node.

If the trajectory follows first the attractive surface before $p \in PS$,

the attractive-repulsive one after PS,

and then it goes along the slow manifold which is not infinitesimally small,

it is called a canard in R^4

Indirect method

Let the assumption (B3) be satisfied, then the following 2 projected systems into R^3 can be reduced under the conditions dx_1/dt, dx_2/dt are limited.

$$\varepsilon dx_1/dt = h_2(x_1, g_2(x_1, y), y, \varepsilon)$$

$$dy/dt = f(x_1, g_2(x_1, y), y, \varepsilon),$$

$$x_2 = g_2(x_1, y), \quad h_1(x, y) = 0$$

$$\varepsilon dx_2/dt = h_1(g_1(x_2, y), x_2, y, \varepsilon)$$

$$dy/dt = f(g_1(x_2, y), x_2, y, \varepsilon),$$

$$x_1 = g_1(x_2, y), \quad h_2(x, y) = 0$$

Def.3

If there exists a canard in the projected system, it is called a partial canard.

If there also exists a canard in the other projected system, it is called a total canard.

Lemmas

Lemma.1

The transversality condition (B1) is established,
iff the transversality condition (A1) is satisfied
at the common pseudo-singular point.

Lemma.2

The both projected systems have the same
pseudo- singular point, if the time-scaled-reduce
system satisfies (B3) with

$\partial h_1/\partial x_2>0, \quad \partial h_2/\partial x_1 >0$

Remark.2

In Def.2, it ensures that only one of the eigenvalues of [$\partial h(x,g(x))/\partial x$] takes zero on PS, that is,

trace [$\partial h(x,g(x))/\partial x$] <0 on PS.

Note that these 2 eigenvalues are negative when the fast vecor field is attractive. When they have different sign, it is attractive-repulsive. When they have positive, it is repulsive.

Theorems

Thm.1

If the system has a single canard, it has partial canard.

Thm.2

Let $p \in PS$ be saddle or node.

If the system has a total canard with conditions:

(i) $\partial h_1/\partial x_2 > 0$, $\partial h_2/\partial x_1 > 0$

(ii) $\det [\partial h(x,g(x))/\partial x] > 0$ on the slow manifold before PS, and

$\det [\partial h(x,g(x))/\partial x] < 0$ after PS,

it has a canard in R^4.

Direct method

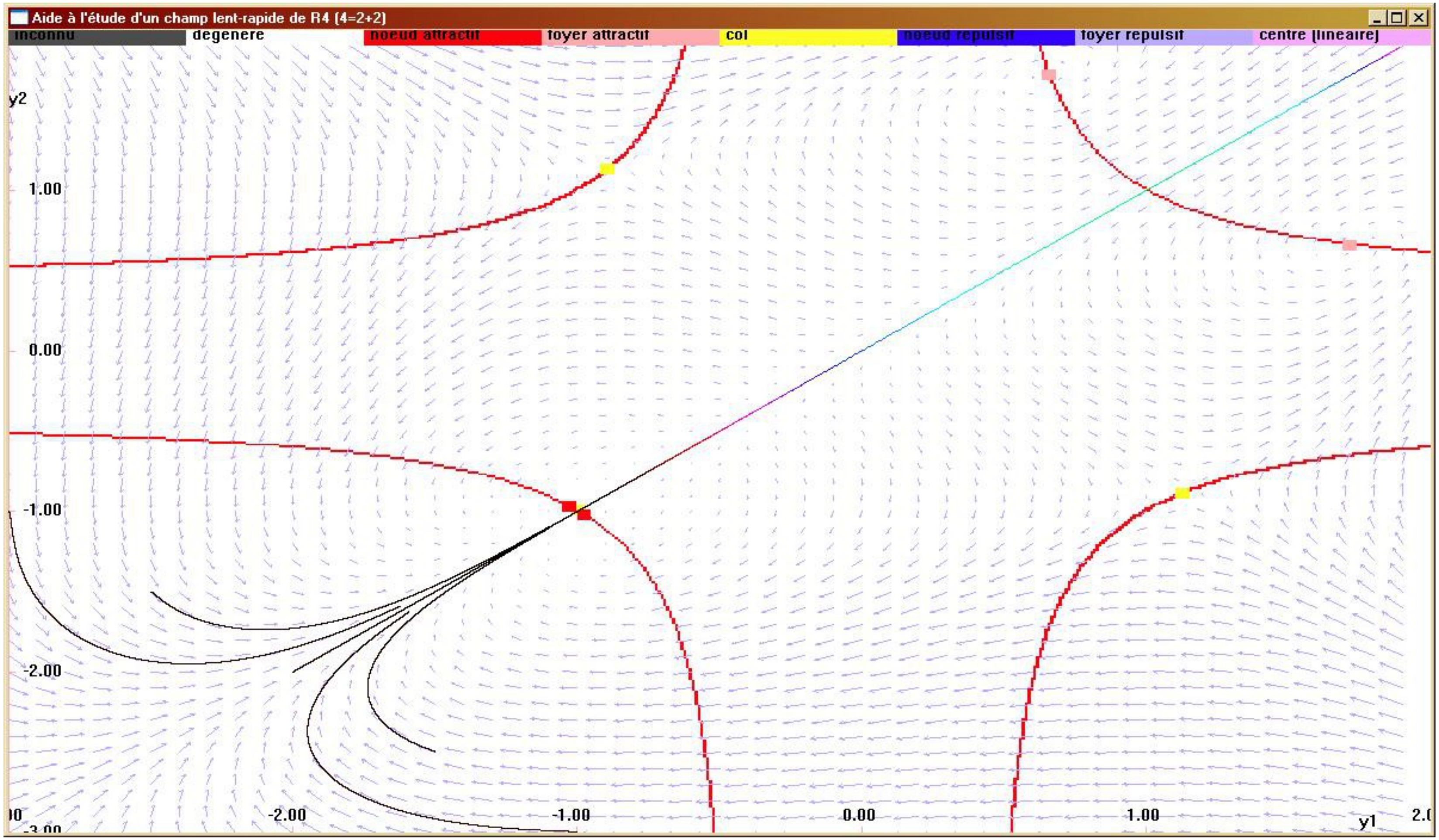
Thm.3

Let $p \in PS$ be saddle or node.

If trace $[\partial h(x,g(x))/\partial x] < 0$ on PS,

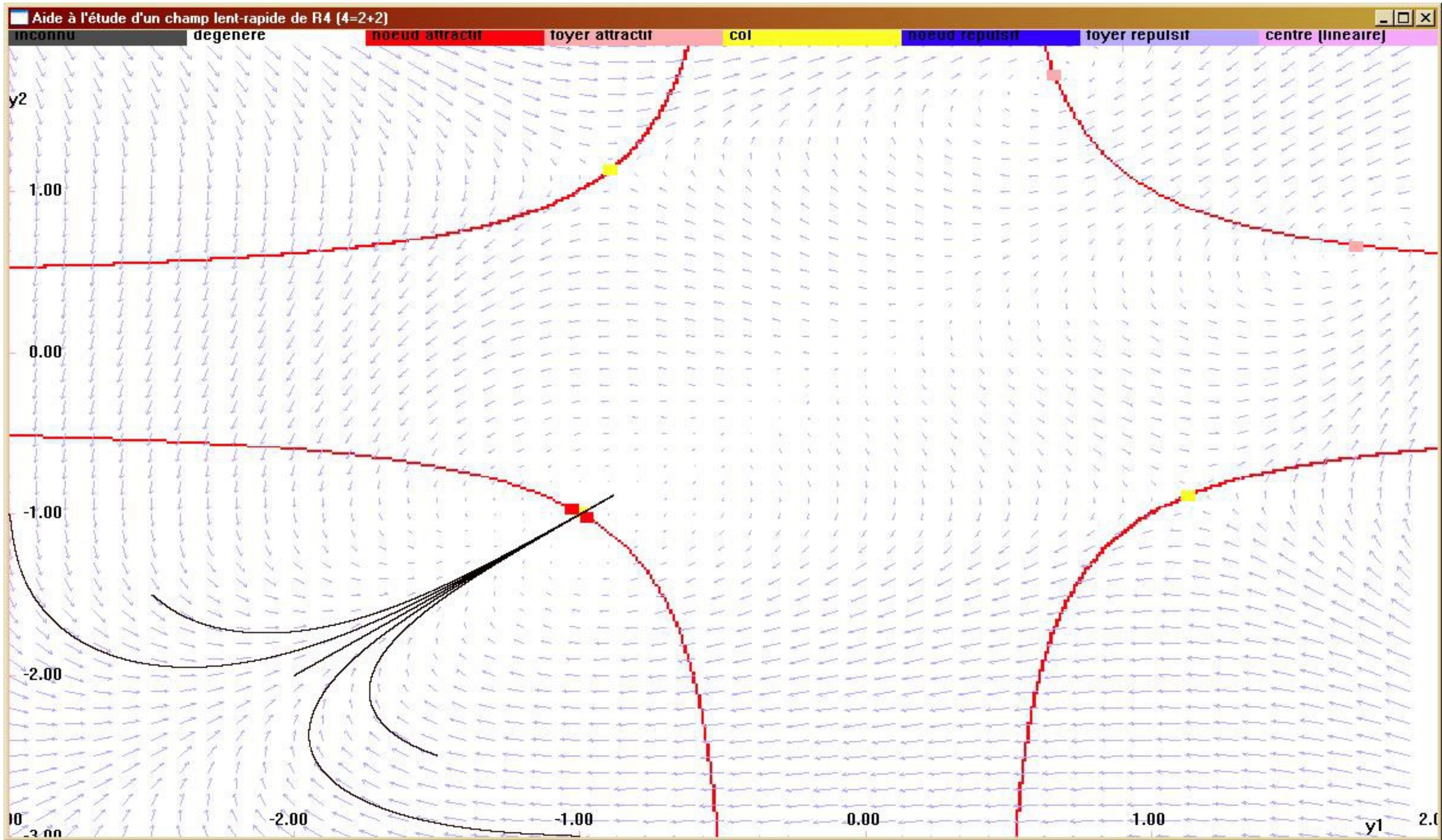
with an efficient local model, the system has a canard in R^4.

Long Canard

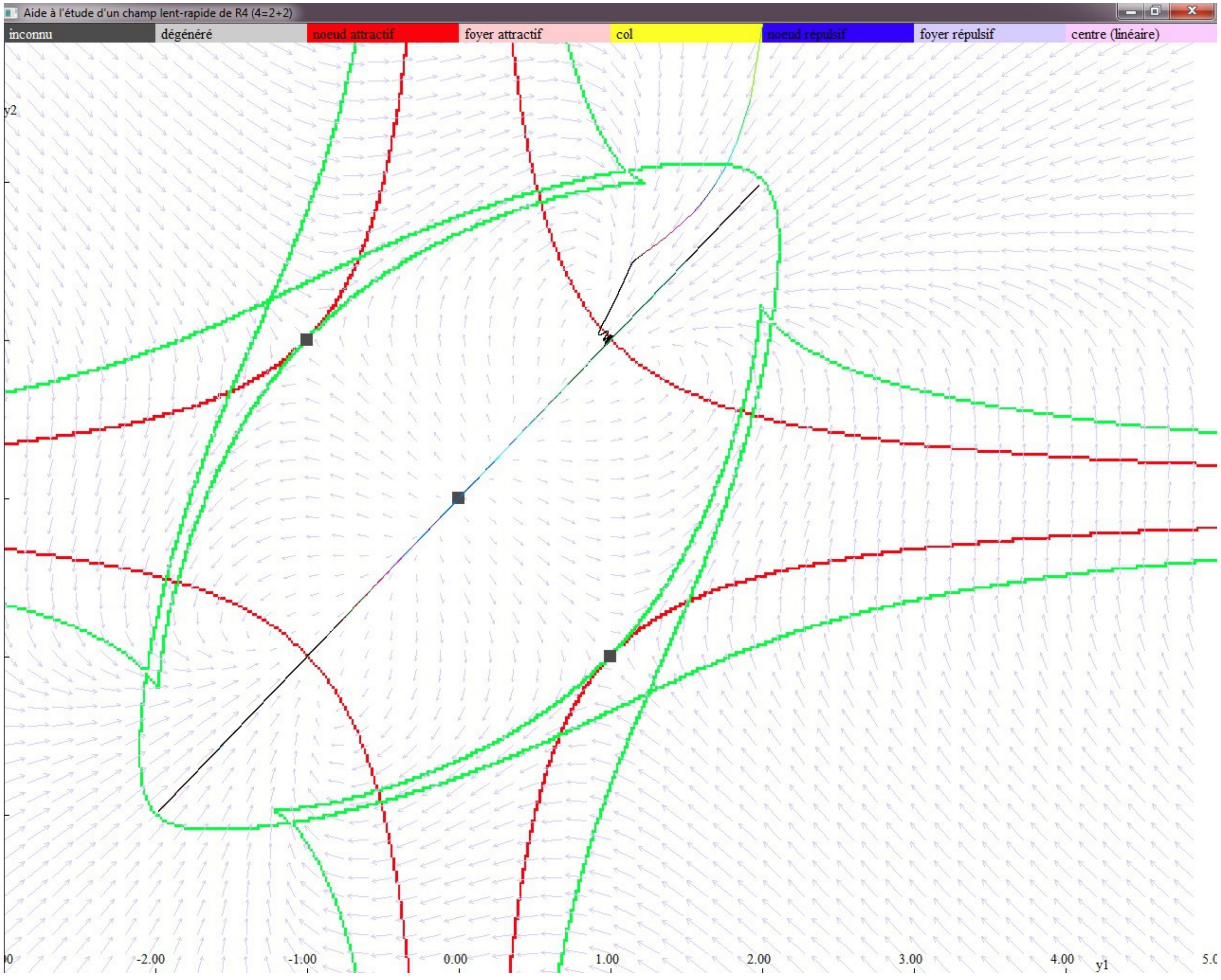


Aide à l'étude d'un champ lent-rapide de R4 (4=2+2)
inconnu
degenere
noeud attractif
foyer attractif
col
noeud repulsif
foyer repulsif
centre (lineaire)
y2
1.00
0.00
-1.00
-2.00
-2.00
-1.00
0.00
1.00
y1

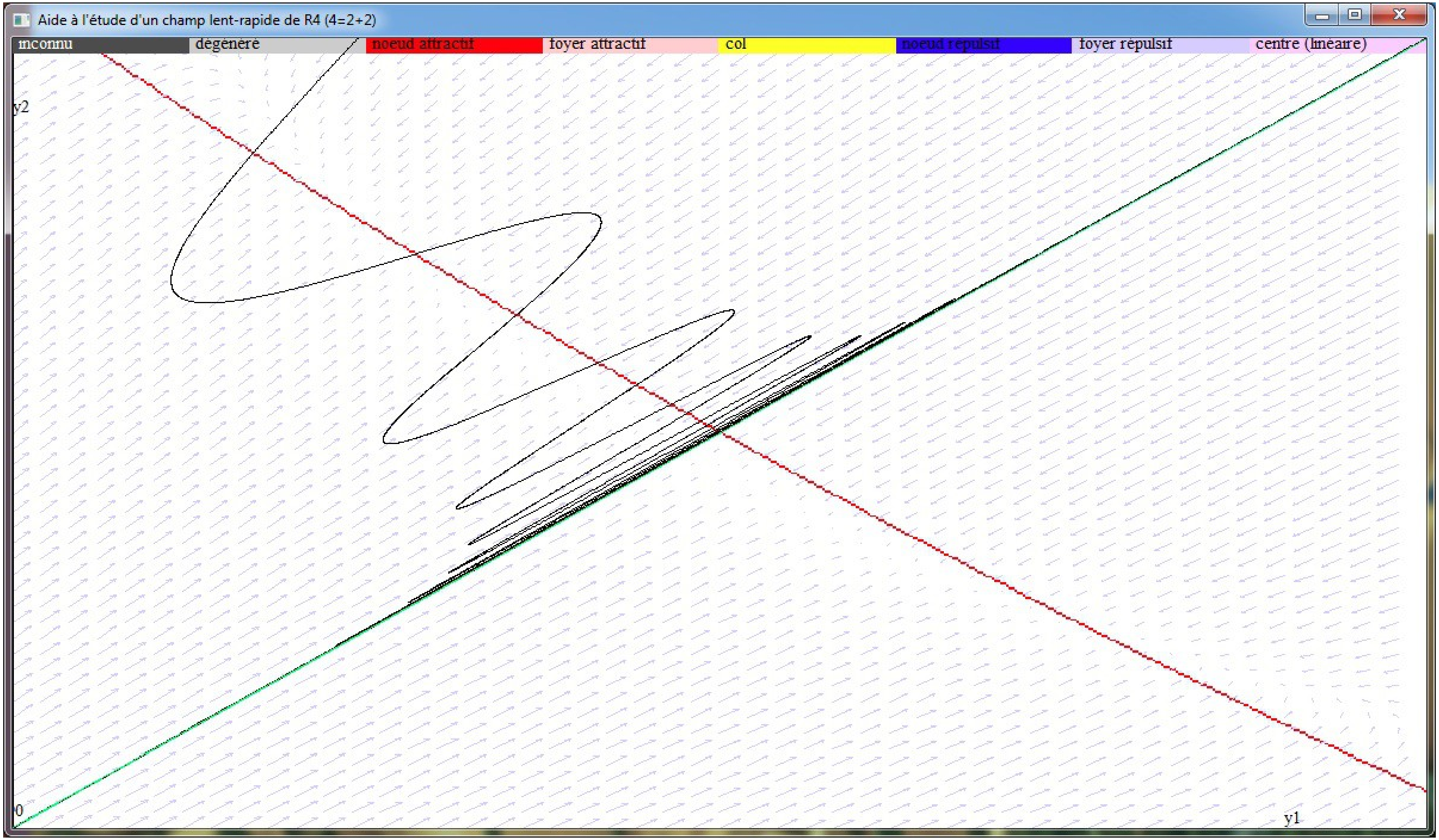
Short Canard



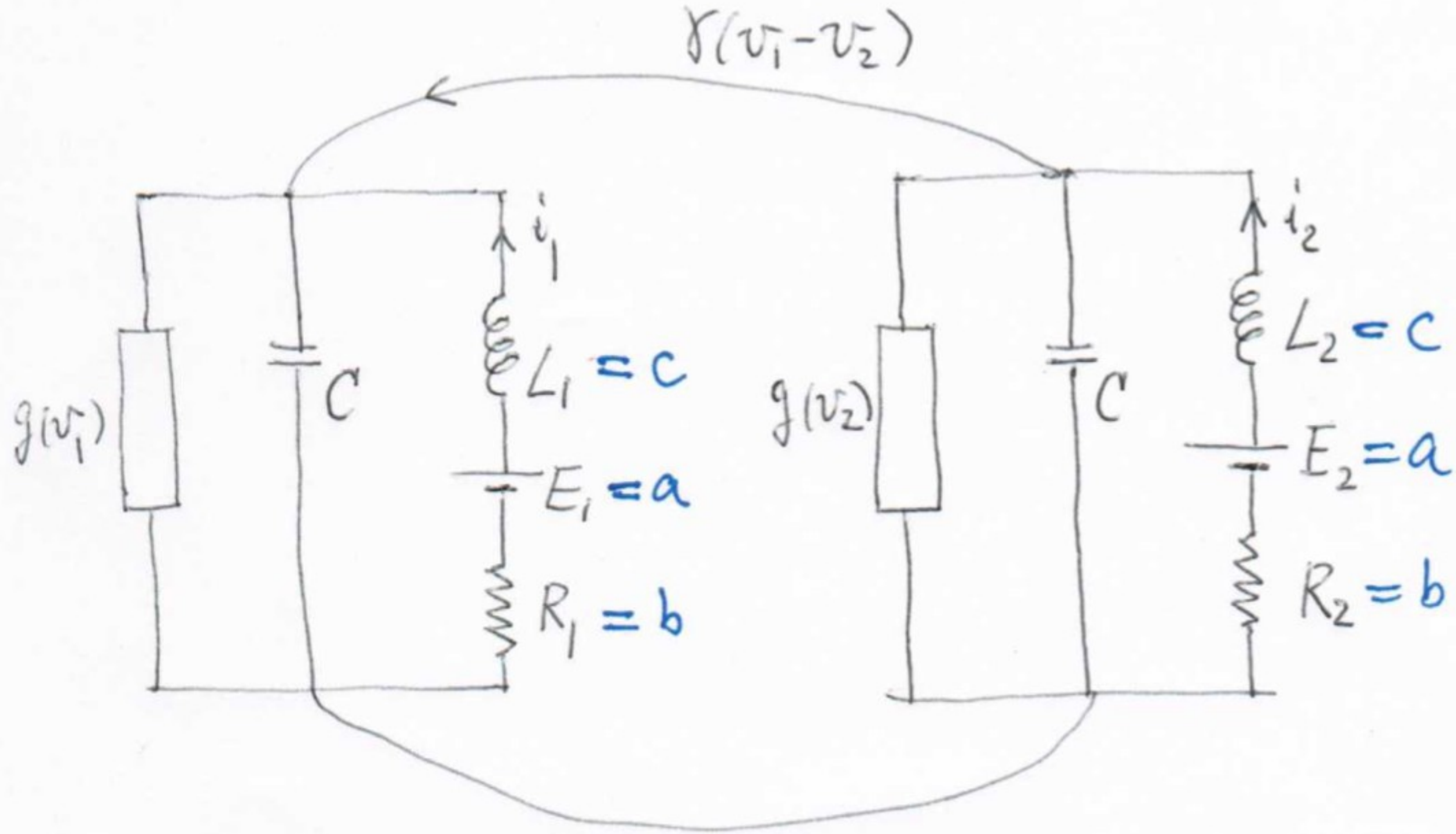
Relative Stability 1



Relative Stability 2



Circuit model



Circuit Model

$L_1\, di_1/dt = E_1 - R_1\, i_1 - v_1$

$C_1\, dv_1/dt = i_1 - g(v_1) + \gamma(v_1 - v_2)$

$g(v_1) = -v_1 + v_1^3/3$

$L_2\, di_2/dt = E_2 - R_2\, i_2 - v_2$

$C_2\, dv_2/dt = i_2 - g(v_2) + \gamma(v_2 - v_1)$

$g(v_2) = -v_2 + v_2^3/3$

$i_k \rightarrow y_k,\ v_k \rightarrow x_k,\ i_k \rightarrow y_k,\ \ k=1,2$

$C \rightarrow \varepsilon$

$C_1=C_2=a,\ \ E_1=E_2=0,\ \ R_1=R_2=b,\ \ L_1=L_2=c,$

$$\varepsilon dx/dt = h(x,y,\varepsilon)$$

$$dy/dt = f(x,y,\varepsilon)$$

$$\varepsilon/a \rightarrow \varepsilon$$

$$\varepsilon dx_1/dt = y_1 + x_2 - x_1^3/3$$

$$\varepsilon dx_2/dt = y_2 + x_1 - x_2^3/3$$

$$dy_1/dt = -x_1 - by_1$$

$$dy_2/dt = -x_2 - by_2$$

- invariant manifold:

$$Inv = \{(x_1, x_2, y_1, y_2) |\ x_1 - x_2 = 0,\ y_1 - y_2 = 0\}$$

- time scaled reduced system:

$$dx_1/dt = - x_2^2 (x_1 + b(-x_2 + x_1^3/3))$$
$$- (x_2 + b(-x_1 + x_2^3/3))$$
$$dx_2/dt = - (x_1 + b(-x_2 + x_1^3/3))$$
$$- x_1^2 (x_2 + b(-x_1 + x_2^3/3))$$

- pseudo singular point:

$$P_0 = (\pm(3/b + 0.5\sqrt{(9/b^2 - 4)})/2)^{1/2},$$
$$\pm(3/b - 0.5\sqrt{(9/b^2 - 4)})/2)^{1/2}$$

when $b \approx 3/2$, $P_0 \approx (\pm 1, \pm 1)$ is saddle slow manifold on the set Inv:

$$y_1 = - x_1 + x_1^3/3$$

the intersection with $dy_1/dt = 0$, ie., by$_1$ = x_1

$$P_0 = (1, 1) \text{ when } b=3/2 \qquad \begin{pmatrix} -1 & -3+4b/3 \\ -3+4b/3 & -1 \end{pmatrix}$$

- trace [∂h(1,1)/∂x] = -2
- det [∂h(1,1)/∂x] =0

References

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