An application of Weihrauch lattice to constructive reverse mathematics

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June 6, 2014

Background

Background: 1/2

Constructive Reverse Mathematics

- Constructive Math.

  = Intuitionistic Logic + Weak Fragment of Arithmetic
- Constructive Reverse Math.

  = Classifications of theorems or principles on constructive math.
- Example:
  - Borzano-Weierstrass's Thm. implies Heine-Borel's Thm. (not vice versa)
  - Σ^0_1 Law of Excluded Middle implies Σ^0_1 De Morgan's Law

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Separation

- Negative results on implications
- Example:
  - Borzano-Weierstrass's Thm. and Heine-Borel's Thm. are separated

Background: 2/2

Higher Order Arithmetic

- Enough expressiability of higher order objects
  e.g. reals, real functions, closed sets of reals...
- Many theorems or principles can be expressed as a single formula

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Hard and Easy Separations

Hard to refute derivability of $\Gamma \mid \forall y{:}\tau.\psi_0 \vdash \forall x{:}\sigma.\varphi_0$

Easy to refute witnessed derivability
i.e. to show that there is no term $\Gamma \vdash t : \tau$
s.t. $\Gamma, x{:}\sigma \mid \psi_0[t/y] \vdash \varphi_0$ is derivable

My Recent Works

Syntactic Work

- A reduction technic of hard separations into easy separations
- Witness Extraction:
  - Existence Property of Intuitionistic Logic
    i.e. if $\Gamma \mid \Lambda \vdash \exists x{:}\sigma.\varphi_0$ is derivable,
    there is a term $\Gamma \vdash t : \sigma$ s.t. $\Gamma \mid \Lambda \vdash \varphi_0[t/x]$ is derivable
  - Dual Work (impossible in general)
    i.e. if $\Gamma \mid \forall y{:}\tau.\psi_0 \vdash \varphi$ is derivable,
    there is a term $\Gamma \vdash t : \tau$ s.t. $\Gamma \mid \psi_0[t/y] \vdash \varphi$ is derivable

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Semantic Work

- Easy separations by Weihrauch lattice
- Weihrauch lattice: a degree structure, discovered in computable analysis

Syntactic Work : three witness extractions

Typed Lambda Calculus ($\lambda_{(\times,\rightarrow)}$-calculus)

Signature language + axioms for typing judgment ($\Gamma \vdash t : \sigma$)
(i.e. type assignment for function symbols)

Specification signature + axioms for conversion judgment ($\Gamma \vdash t = u : \sigma$)

meta variables

- $x, y, \cdots$ for variables
- $\alpha, \beta, \cdots$ for base types
- $f, g, \cdots$ for function symbols
- $\sigma, \tau, \cdots$ for types
  $$\sigma ::= 1 \mid \alpha \mid \sigma \rightarrow \sigma \mid \sigma \times \sigma$$
- $t, u, \cdots$ for terms
  $$t ::= \langle\rangle \mid f(t, \cdots, t) \mid \lambda x{:}\sigma.t \mid t(t) \mid \langle t, t\rangle \mid \pi t \mid \pi' t$$
- $\Gamma, \Delta, \cdots$ for type contexts, Λ for the empty context
  $$\Gamma \equiv x_1 : \sigma_1, \cdots, x_k : \sigma_k \qquad (x_1, \cdots, x_k : \text{distinct})$$

SIL: 2/6

Extention of Language

Logical Constants : $\bot$

Predicate Symbols : $=, P \in \Pi_{\mathrm{p}}$ (Π_{p}: given)

Logical Connectives: $\wedge, \vee, \rightarrow$

Quantifiers : $\forall, \exists$

Signature for SIL

A specification for typed lambda calculus equipped with a mapping

$$S' : P \mapsto (\sigma_1, \cdots, \sigma_k) \ ({}^{\forall} P \in \Pi_{\mathrm{p}})$$

$P(-, \cdots, -)$: finite symbol sequence with holes as $|S' f|$

Formula

(meta variables: $\varphi, \psi, \chi, \cdots$)

$$\varphi ::= \bot \mid P(t, \cdots, t) \mid t =_\sigma t \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \exists x{:}\sigma.\varphi \mid \forall x{:}\sigma.\varphi$$

SIL: 3/6

Expressions

Typing Judgment: $\Gamma \vdash \varphi : \mathtt{Prop}$

Sequent : $\Gamma \mid \Theta \vdash \varphi$ ($\Theta, \Xi, \cdots$ for finite sequences of formulae)

Specification for SIL

A signature equipped with a set $\mathscr{A}$ of sequents (axiom set) closed under:

$$\frac{\Gamma_0, x_0 : \sigma_0, x_1 : \sigma_1, \Gamma_1 \mid \Theta \vdash \varphi}{\Gamma_0, x_1 : \sigma_1, x_0 : \sigma_0, \Gamma_1 \mid \Theta \vdash \varphi}\ (\mathrm{E})_t \qquad \frac{\Gamma \mid \Theta \vdash \varphi}{\Gamma, x : \sigma \mid \Theta \vdash \varphi}\ (\mathrm{W})_t$$

$$\frac{\Gamma, x_0 : \sigma, x_1 : \sigma \mid \Theta \vdash \varphi}{\Gamma, x_0 : \sigma \mid \Theta[x_0/x_1] \vdash \varphi[x_0/x_1]}\ (\mathrm{C})_t \qquad \frac{\Gamma \vdash t : \sigma \qquad \Gamma, x : \sigma \mid \Theta \vdash \varphi}{\Gamma \mid \Theta[t/x] \vdash \varphi[t/x]}\ (\mathrm{S})$$

We denote by SIL the specification whose axiom set is empty

SIL: 4/6

$$\frac{\Gamma \mid \Theta_0, \psi_0, \psi_1, \Theta_1 \vdash \varphi}{\Gamma \mid \Theta_0, \psi_1, \psi_0, \Theta_1 \vdash \varphi} \text{(E)} \qquad \frac{\Gamma \mid \Theta, \psi, \psi \vdash \varphi}{\Gamma \mid \Theta, \psi \vdash \varphi} \text{(C)} \qquad \frac{\Gamma \mid \Theta \vdash \varphi}{\Gamma \mid \Theta, \psi \vdash \varphi} \text{(W)}$$

$$\frac{\Gamma \mid \Theta \vdash \psi \qquad \Gamma \mid \Xi, \psi \vdash \varphi}{\Gamma \mid \Theta, \Xi \vdash \varphi} \text{(Cut)}$$

"ψ" in (Cut) is called *cut-formula*

$$\frac{\Gamma \vdash \varphi : \mathtt{Prop}}{\Gamma \mid \varphi \vdash \varphi} \text{(Id)} \qquad \frac{\Gamma \vdash \varphi : \mathtt{Prop}}{\Gamma \mid \bot \vdash \varphi} (\bot) \qquad \frac{(\Gamma \mid \Theta \vdash \varphi) \in \mathscr{A}}{\Gamma \mid \Theta \vdash \varphi} (\mathscr{A})$$

$$\frac{\Gamma \vdash t = u : \sigma}{\Gamma \mid \Lambda \vdash t =_\sigma u} \text{(Eq)}$$

$$\frac{\Gamma \vdash t_0, t_1 : \sigma \qquad \Gamma, x : \sigma \vdash \varphi : \mathtt{Prop}}{\Gamma \mid t_0 =_\sigma t_1, \varphi[t_i/x] \vdash \varphi[t_{1-i}/x]} \text{(R)}$$

SIL: 5/6

$$\frac{\Gamma \mid \Theta, \psi_0, \psi_1 \vdash \varphi}{\Gamma \mid \Theta, \psi_0 \wedge \psi_1 \vdash \varphi}\ (\wedge\mathrm{L}) \qquad \frac{\Gamma \mid \Theta \vdash \varphi_0 \qquad \Gamma \mid \Theta \vdash \varphi_1}{\Gamma \mid \Theta \vdash \varphi_0 \wedge \varphi_1}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma \mid \Theta, \psi_0 \vdash \varphi \qquad \Gamma \mid \Theta, \psi_1 \vdash \varphi}{\Gamma \mid \Theta, \psi_0 \vee \psi_1 \vdash \varphi}\ (\vee\mathrm{L}) \qquad \frac{\Gamma \mid \Theta \vdash \varphi_i \qquad \Gamma \vdash \varphi_{1-i} : \mathtt{Prop}}{\Gamma \mid \Theta \vdash \varphi_0 \vee \varphi_1}\ (\vee\mathrm{R})$$

$$\frac{\Gamma \mid \Theta \vdash \psi_0 \qquad \Gamma \mid \Theta, \psi_1 \vdash \varphi}{\Gamma \mid \Theta, \psi_0 \to \psi_1 \vdash \varphi}\ (\to\mathrm{L}) \qquad \frac{\Gamma \mid \Theta, \varphi_0 \vdash \varphi_1}{\Gamma \mid \Theta \vdash \varphi_0 \to \varphi_1}\ (\to\mathrm{R})$$

SIL: 6/6

$$\frac{\Gamma \vdash \Theta, \exists x{:}\sigma.\psi, \varphi : \mathtt{Prop} \qquad \Gamma, y{:}\sigma \mid \Theta, \psi[y/x] \vdash \varphi}{\Gamma \mid \Theta, \exists x{:}\sigma.\psi \vdash \varphi}\ (\exists\mathrm{L})$$

$$\frac{\Gamma \vdash t{:}\sigma \qquad \Gamma \mid \Theta \vdash \varphi[t/x]}{\Gamma \mid \Theta \vdash \exists x{:}\sigma.\varphi}\ (\exists\mathrm{R})$$

$$\frac{\Gamma \vdash t{:}\sigma \qquad \Gamma \mid \Theta, \varphi[t/x] \vdash \psi}{\Gamma \mid \Theta, \forall x{:}\sigma.\varphi \vdash \psi}\ (\forall\mathrm{L})$$

$$\frac{\Gamma \vdash \Theta, \forall x{:}\sigma.\varphi : \mathtt{Prop} \qquad \Gamma, y{:}\sigma \mid \Theta \vdash \varphi[y/x]}{\Gamma \mid \Theta \vdash \forall x{:}\sigma.\varphi}\ (\forall\mathrm{R})$$

The First Witness Extraction: 1/4

Admissible rules

The following rules are admissible over any specification $\mathscr{A}$:

$$\frac{\Gamma, x_0 : \sigma, x_1 : \sigma \mid \Theta \vdash \varphi}{\Gamma, x_0 : \sigma \mid \Theta[x_0/x_1] \vdash \varphi[x_0/x_1]} \; (\mathrm{C})_t \qquad \frac{\Gamma \mid \Theta \vdash \varphi}{\Gamma, x : \sigma \mid \Theta \vdash \varphi} \; (\mathrm{W})_t$$

$$\frac{\Gamma_0, x_0 : \sigma_0, x_1 : \sigma_1, \Gamma_1 \mid \Theta \vdash \varphi}{\Gamma_0, x_1 : \sigma_1, x_0 : \sigma_0, \Gamma_1 \mid \Theta \vdash \varphi} \; (\mathrm{E})_t \qquad \frac{\Gamma \vdash t : \sigma \qquad \Gamma, x : \sigma \mid \Theta \vdash \varphi}{\Gamma \mid \Theta[t/x] \vdash \varphi[t/x]} \; (\mathrm{S})$$

Cut-Elimination Thm.

Given a pure variable derivation over SIL, one finds an essential-cut free derivation of the same conclusion

an *essential-cut* :≡ a cut whose cut-formula is not atomic

The First Witness Extraction: 2/4

Notation

$$
\begin{array}{rcll}
\mathrm{Sub}^{+}(a) & = & \{a\} & (a : atomic) \\
\mathrm{Sub}^{+}(\varphi_0 \vee \varphi_1) & = & \mathrm{Sub}^{+}(\varphi_0) \cup \mathrm{Sub}^{+}(\varphi_1) \cup \{\varphi_0 \vee \varphi_1\} & \\
\mathrm{Sub}^{+}(\varphi_0 \wedge \varphi_1) & = & \mathrm{Sub}^{+}(\varphi_0) \cup \mathrm{Sub}^{+}(\varphi_1) \cup \{\varphi_0 \wedge \varphi_1\} & \\
\mathrm{Sub}^{+}(\varphi_0 \to \varphi_1) & = & \mathrm{Sub}^{-}(\varphi_0) \cup \mathrm{Sub}^{+}(\varphi_1) \cup \{\varphi_0 \to \varphi_1\} & \\
\mathrm{Sub}^{+}(\exists x : \sigma.\varphi_0) & = & \mathrm{Sub}^{+}(\varphi_0) \cup \{\exists x : \sigma.\varphi_0\} & \\
\mathrm{Sub}^{+}(\forall x : \sigma.\varphi_0) & = & \mathrm{Sub}^{+}(\varphi_0) \cup \{\forall x : \sigma.\varphi_0\} & \\
\mathrm{Sub}^{-}(a) & = & \emptyset & (a : atomic) \\
\mathrm{Sub}^{-}(\varphi_0 \vee \varphi_1) & = & \mathrm{Sub}^{-}(\varphi_0) \cup \mathrm{Sub}^{-}(\varphi_1) & \\
\mathrm{Sub}^{-}(\varphi_0 \wedge \varphi_1) & = & \mathrm{Sub}^{-}(\varphi_0) \cup \mathrm{Sub}^{-}(\varphi_1) & \\
\mathrm{Sub}^{-}(\varphi_0 \to \varphi_1) & = & \mathrm{Sub}^{+}(\varphi_0) \cup \mathrm{Sub}^{-}(\varphi_1) & \\
\mathrm{Sub}^{-}(\exists x : \sigma.\varphi_0) & = & \mathrm{Sub}^{-}(\varphi_0) & \\
\mathrm{Sub}^{-}(\forall x : \sigma.\varphi_0) & = & \mathrm{Sub}^{-}(\varphi_0) &
\end{array}
$$

The First Witness Extraction: 3/4

Positive Universal Quantification Free

- A formula φ is *p.u.f.*
  iff no formula of the form $\forall x\!:\!\sigma.\psi_0$ belongs to $\mathrm{Sub}^+(\varphi)$
- A specification $\mathscr{A}$ is *p.u.f.*
  iff $(\bigwedge\!\!\!\bigwedge \Theta) \to \varphi$ is p.u.f. for each $(\Gamma \mid \Theta \vdash \varphi) \in \mathscr{A}$

According usage:

p.e.f. (positive existencial quantificatioin free relative to ρ),
n.u.f. (negative universal quantificatioin free relative to ρ),
n.e.f. (negative existencial quantificatioin free relative to ρ),
q.f. (quantification free relative to ρ),...

The First Witness Extraction: 4/4

Fact < Cut-Elimination Thm.

Assume that:

- $\mathscr{A}$ is p.e.f. and n.u.f.
- ψ_0 is p.e.f. and n.u.f.
- φ is p.u.f. and n.e.f.

If $\Gamma \mid \forall y{:}\tau.\psi_0 \vdash \varphi$ is derivable over $\mathscr{A}$, there is a finite sequence ${t_1}_\Gamma^\tau, \cdots, {t_k}_\Gamma^\tau$ of terms s.t.

$$\Gamma \mid \psi_0[{t_1}_\Gamma^\tau/y], \cdots, \psi_0[{t_k}_\Gamma^\tau/y] \vdash \varphi$$

is derivable over $\mathscr{A}$

Abbreviation

$t_\Gamma^\tau \ :\equiv\ (\Gamma, \tau, t)$ if $\Gamma \vdash t : \tau$ is derivable

The Second Witness Extraction: 1/3

ρ: fixed type

Parametrization

Let $\psi :\equiv \forall y\!:\!\tau.\psi_0$. Define:

$$^{\rho}\psi \quad :\equiv \quad \forall y\!:\!(\rho \to \tau).\forall z\!:\!\rho.\psi_0[yz/y]$$

(z:the first flesh variable symbol)

$\Gamma \mid \psi_\Gamma \vdash {}^{\rho}\psi_\Gamma$ is derivable over SIL, and the converse one also derivable whenever ρ is "inhabitant" relative to Γ

Idempotency

ρ is *idempotent* iff ρ is "isomorphic" to $\rho \times \rho$
i.e. there is a pair t and u of terms s.t. the following judgments are derivable

- $\Lambda \vdash tu = (\lambda x\!:\!\rho.x)\!:\!\rho \to \rho$
- $\Lambda \vdash ut = (\lambda x\!:\!\rho \times \rho.x)\!:\!(\rho \times \rho) \to (\rho \times \rho)$

The Second Witness Extraction: 2/3

Positive Universal Quantification Free Relative to ρ

- A formula φ is ${}^{\rho}$*p.u.f.*
  iff no formula of the form $\forall x{:}\sigma.\psi_0$ $(\sigma \not\equiv \rho)$ belongs to $\mathrm{Sub}^{+}(\varphi)$
- An axiom set $\mathscr{A}$ is ${}^{\rho}$*p.u.f.*
  iff $(\bigwedge\!\!\!\bigwedge \Theta) \to \varphi$ is ${}^{\rho}$p.u.f. for each $(\Gamma \mid \Theta \vdash \varphi) \in \mathscr{A}$
- A specification is ${}^{\rho}$*p.u.f.* iff its axiom set is ${}^{\rho}$p.u.f.

According usage:

${}^{\rho}$p.e.f. (positive existencial quantificatioin free relative to ρ),
${}^{\rho}$n.u.f. (negative universal quantificatioin free relative to ρ),
${}^{\rho}$n.e.f. (negative existencial quantificatioin free relative to ρ),
${}^{\rho}$q.f. (quantification free relative to ρ),...

The Second Witness Extraction: 3/3

Fact < Cut-Elimination Thm.

Assume that:

- $\mathscr{A}$ is ${}^{\rho}$p.e.f. and ${}^{\rho}$n.u.f.
- ψ_0 is ${}^{\rho}$p.e.f. and ${}^{\rho}$n.u.f.
- φ is ${}^{\rho}$p.u.f. and ${}^{\rho}$n.e.f.
- ρ is idempotent

If $\Gamma \mid \forall y{:}\tau.\psi_0 \vdash \varphi$ is derivable over $\mathscr{A}$, there is a finite sequence ${t_1}_{\Gamma}^{\rho\to\tau}, \cdots, {t_k}_{\Gamma}^{\rho\to\tau}$ of terms s.t.

$$\Gamma \mid (\forall z{:}\rho.\psi_0[yz/y])[{t_1}_{\Gamma}^{\rho\to\tau}/y], \cdots, (\forall z{:}\rho.\psi_0[yz/y])[{t_k}_{\Gamma}^{\rho\to\tau}/y] \vdash \varphi$$

is derivable over $\mathscr{A}$

Remark

From the resulting "witnessed" sequent, $\Gamma \mid {}^{\rho}(\forall y{:}\tau.\psi_0) \vdash \varphi$ is derivable

The Third Witness Extraction: 1/3

Signature of $\mathrm{HA}^{\lambda+}$

Base Type Symbol: N for natural number system
2 for two elements boolean

Function Symbol : S for successor function
0_N for constants of type N
$0_2, 1_2$ for constants of type 2
E for embedding of 2 into N
R^σ for recursors

$\Delta_0(\Gamma)$-formula

$$\delta \ ::= \ \bot \mid t^N_\Gamma =_N t^N_\Gamma \mid t^2_\Gamma =_2 t^2_\Gamma \mid \delta \lor \delta \mid \delta \land \delta \mid \delta \to \delta \mid \exists n \le t^N_\Gamma.\delta \mid \forall n \le t^N_\Gamma.\delta$$

The Third Witness Extraction: 2/3

Axioms of $\mathrm{HA}^{\lambda+}$

- Axioms for $S, 0_N$
- Axioms for E (and $0_2, 1_2$) as an embedding of 2 into N
- Axioms for R^σ as a recursor
- Induction Scheme:

$$\Gamma \mid \varphi[0/n],\ \forall n{:}N.(\varphi \to \varphi[n+1/n]) \vdash \forall n{:}N.\varphi$$

(where φ : ^{N}q.f.)

- Δ_0-Comprehension Scheme:

$$\Gamma \mid \Lambda \vdash \exists p{:}(N \to 2).\forall n{:}N.(\delta \leftrightarrow pn =_2 1_2)$$

(where δ : $\Delta_0(\Gamma, n{:}N)$-formula, p : the first flesh variable)

- Extensionality Scheme w.r.t. N:

$$\Gamma \mid \forall n{:}N.(t_0 =_\sigma t_1) \vdash (\lambda n{:}N.t_0) =_{N\to\sigma} (\lambda n{:}N.t_1)$$

The Third Witness Extraction: 3/3

Main Lemma < Cut-Elimination Thm.

Assume that:

- $\mathscr{A}$ is an extension of $\mathrm{HA}^{\lambda+} \setminus \{\Delta_0\text{-comprehensions}\}$
- $\mathscr{A}$ is ${}^{\rho}$p.e.f. and ${}^{\rho}$n.u.f.
- ψ_0 is ${}^{\rho}$p.e.f. and ${}^{\rho}$n.u.f.
- φ is ${}^{\rho}$p.u.f. and ${}^{\rho}$n.e.f.

If $\Gamma \mid \forall y\!:\!\tau.\psi_0 \vdash \varphi$ is derivable over $\mathscr{A}$, there is a term $t_{\Gamma}^{N\to\tau}$ s.t.

$$\Gamma \mid (\forall z\!:\!\rho.\psi_0[yz/y])[t_{\Gamma}^{N\to\tau}/y] \vdash \varphi$$

is derivable over $\mathscr{A}$

Remark

From the resulting "witnessed" sequent, $\Gamma \mid {}^{N}(\forall y\!:\!\tau.\psi_0) \vdash \varphi$ is derivable

Semantic Work : an interpretation into Weihrauch lattice

Extended Weihrauch Lattice

Notation

(meta variable: $F, G, \cdots \subseteq \omega^\omega \times \omega^\omega$)

- $F[\alpha] := \{\beta \in \omega^\omega : (\alpha, \beta) \in F\}$
- $\mathrm{supp}(F) := \{\alpha \in \omega^\omega : F[\alpha] \neq \emptyset\}$

Weihruach Reducibility

Let (u, F) and (v, G) be two pairs such that:

- $\mathrm{supp}(F) \subseteq u \subseteq \omega^\omega$
- $\mathrm{supp}(G) \subseteq v \subseteq \omega^\omega$

Define:

$$\begin{aligned} & (u, F) \leq_W (v, G) \\ \iff & {}^\exists k, l : \text{computable}. {}^\forall \alpha \in u. \\ & \quad k\alpha\downarrow \in v \ \&\ (\alpha \in \mathrm{supp}(F) \Rightarrow k\alpha \in \mathrm{supp}(G) \ \&\ l\langle\alpha, \beta\rangle\downarrow \in F[\alpha]) \end{aligned}$$

$\mathfrak{Y}$: the induced degree structure w.r.t. $\leq_W$

Main Theorem: 1/5

Main Theorem

There is an interpretation $[\![-]\!]$ from $\mathrm{HA}^{\lambda+}$ into extended Weihrauch lattice $\mathfrak{Y}$
s.t. if $\Lambda \mid \forall y{:}\tau.\psi_0 \vdash \forall x{:}\sigma.\varphi_0$ is derivable over $\mathrm{HA}^{\lambda+}$,
then $[\![\varphi_0]\!]_{x:\sigma} \geq_W [\![\varphi_0]\!]_{x:\sigma}$ in $\mathfrak{Y}$

whenever:

- ψ_0 is ^{N}p.e.f. and ^{N}n.u.f.
- φ_0 is ^{N}n.e.f. and ^{N}p.u.f.

Proof:1/4

We use the standard semantics of SIL by f.o.f. (first order fibrations) with Cartesian closed base category.

$$\mathrm{Fib}_{\mathrm{SIL}} \underset{\mathcal{T}}{\overset{\mathrm{Int}}{\underset{\top}{\rightleftarrows}}} \mathrm{Spec}_{\mathrm{SIL}}$$

where:

- $\mathrm{Spec}_{\mathrm{SIL}}$: metacategory of specifications of SIL
- $\mathrm{Fib}_{\mathrm{SIL}}$: metacategory of f.o.f. with Cartesian closed base category
- Int: constructions of internal theories
- $\mathcal{T}$: constructions of term models

In particular counit ϵ of the above (pseudo) adjunction is a natural equivalence

We denote by $\mathtt{Rep}$ the type-2 realizability model

Proof: 2/4

Define a translation $^{\dagger}(-)$ from $\mathrm{HA}^{\lambda+}$ to Int($\mathtt{Rep}$) by:

base type : $N \mapsto \overline{\omega}$ where $\omega = (\omega, \delta_\omega),\ \delta_\omega : ip \mapsto i$,
$2 \mapsto \overline{2}$ where $2 = (2, \delta_2), \delta_2 : 0p \mapsto 0,\ 1p \mapsto 1$

function symbol: $0_N \mapsto \overline{(0 : 1 \to \omega)}, S \mapsto \overline{(- + 1)}, 0_2 \mapsto \overline{(0 : 1 \to 2)}$,
$1_2 \mapsto \overline{(1 : 1 \to 2)}, E \mapsto \overline{(\iota : 2 \to \omega)}, \cdots$ (Omit)

Define a subsystem $\mathscr{A}$ of Int($\mathtt{Rep}$) by:

- Add tranlations (via $^{\dagger}(-)$) of axioms for $S, 0_N, E, 0_2, 1_2, R^\sigma$, Induction Scheme and Extensionality Scheme
- Add $^{\dagger}\Gamma \mid \Lambda \vdash {}^{\dagger}(\forall n{:}N.(\delta \leftrightarrow pn =_2 1_2))[t_{^{\dagger}\Gamma}^{\dagger(N\to 2)}/p]$ iff it is an axiom of Int($\mathtt{Rep}$) and δ is $\Delta_0(\Gamma, n{:}N)$-formula

Then:

- $\mathscr{A}$ is $^{\overline{\omega}}$p.e.f. and $^{\overline{\omega}}$n.u.f.
- if $\Gamma \mid \Theta \vdash \varphi$ is deriv. over $\mathrm{HA}^{\lambda+}$, then $^{\dagger}\Gamma \mid {}^{\dagger}\Theta \vdash {}^{\dagger}\varphi$ is deriv. over $\mathscr{A}$

Main Theorem: 4/5

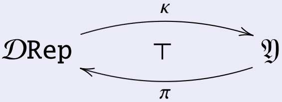
Proof: 3/4

For two formulae $\varphi_{v_0:\sigma}$ and $\psi_{v_0:\tau}$ over Int(Rep), define:

$$\begin{aligned} & \varphi_{v_0:\sigma} \leq^1 \psi_{v_0:\tau} \\ \Longleftrightarrow \quad & {}^{\exists} t^{\tau}_{v_0:\sigma} \text{ s.t. } v_0 : \sigma \mid \psi_{v_0:\tau}[t^{\tau}_{v_0:\sigma}/v_0] \vdash \varphi_{v_0:\sigma} \text{ is derivable over Int(Rep)} \end{aligned}$$

$\mathcal{D}\mathrm{Rep}$: the induced degree structure w.r.t. $\leq^1$

Then $\mathfrak{Y}$ has an embedding π into $\mathcal{D}\mathrm{Rep}$ which has a right adjoint κ



Define $[\![\varphi]\!]_\Gamma := \kappa^{\dagger N}(\varphi_\Gamma)$ for each formula φ_Γ over $\mathrm{HA}^{\lambda+}$

Main Theorem: 5/5

Proof: 4/4

We obtain:

$$
\begin{array}{lll}
 & \Lambda \mid \forall y{:}\tau.\psi_0 \vdash \forall x{:}\sigma.\varphi_0 & \text{is derivable over } \mathrm{HA}^{\lambda+} \\
\Longleftrightarrow & \Lambda \mid \forall y{:}\tau.\psi_0 \vdash {}^{N}\forall x{:}\sigma.\varphi_0 & \text{is derivable over } \mathrm{HA}^{\lambda+} \\
\Longrightarrow & \Lambda \mid {}^{\dagger}\forall y{:}\tau.\psi_0 \vdash {}^{\dagger N}\forall x{:}\sigma.\varphi_0 & \text{is derivable over } \mathscr{A} \\
\Longleftrightarrow & \multicolumn{2}{l}{x{:}{}^{\dagger}(N \to \sigma) \mid ({}^{\dagger}\forall w{:}N.\psi_0[yw/y])[t^{\dagger(N\to\tau)}_{x:{}^{\dagger}\sigma}/y] \vdash {}^{\dagger}\forall z{:}N.\varphi_0[xz/x]} \\
 & \text{is derivable over } \mathscr{A} \text{ for some } t^{\dagger(N\to\tau)}_{x:{}^{\dagger}\sigma} & (\text{Witness Extraction}) \\
\Longleftrightarrow & {}^{\dagger N}(\psi_0)_{y:\tau} \ {}^{1}\!\geq\ {}^{\dagger N}(\varphi_0)_{x:\sigma} & \\
\Longrightarrow & \kappa^{\dagger N}(\psi_0)_{y:\tau} \geq_W \kappa^{\dagger N}(\varphi_0)_{x:\sigma} & \\
\Longleftrightarrow & [\![\psi_0]\!]_{y:\tau} \geq_W [\![\varphi_0]\!]_{x:\sigma} &
\end{array}
$$

Application: 1/3

LPO

$\mathrm{LPO} :\equiv \Gamma \mid \Lambda \vdash \exists n{:}N.\delta \vee \neg\exists n{:}N.\delta \qquad (\delta : \Delta_0(\Gamma, n{:}N)\text{-formula})$

LLPO

$\mathrm{LLPO} :\equiv \Gamma \mid \neg(\exists n{:}N.\delta_0 \wedge \exists n{:}N.\delta_1) \vdash \neg\exists n{:}N.\delta_0 \vee \neg\exists n{:}N.\delta_1$
$(\delta_0, \delta_1 : \Delta_0(\Gamma, n{:}N)\text{-formula})$

Proposition

Over $\mathrm{HA}^{\lambda+}$, LLPO does not imply LPO

Application: 2/3

Proof

Define:

$$\mathrm{LPO}_0 \;:\equiv\; \exists n\!:\!N.pn =_2 1_2 \vee \neg\exists n\!:\!N.pn =_2 1_2$$

$$\mathrm{LLPO}_0 \;:\equiv\; \neg(\exists n\!:\!N.p(2n) =_2 1_2 \wedge \exists n\!:\!N.p(2n+1) =_2 1_2) \rightarrow \neg\exists n\!:\!N.p(2n) =_2 1_2 \vee \neg\exists n\!:\!N.p(2n+1) =_2 1_2$$

We obtain:

$$\begin{aligned} & \text{LLPO implies LPO over } \mathrm{HA}^{\lambda+} \\ \Longleftrightarrow\quad & \Lambda \mid \forall p\!:\!(N\to 2).\mathrm{LLPO}_0 \vdash \forall p\!:\!(N\to 2).\mathrm{LPO}_0 \text{ is derivable over } \mathrm{HA}^{\lambda+} \\ \Longrightarrow\quad & [\![\mathrm{LLPO}_0]\!]_{p:N\to 2} \geq_W [\![\mathrm{LPO}_0]\!]_{p:N\to 2} \qquad \text{(Main Theorem)} \end{aligned}$$

However $[\![\mathrm{LLPO}_0]\!]_{p:N\to 2} \not\geq_W [\![\mathrm{LPO}_0]\!]_{p:N\to 2}$ (V. Brattka & G. Gherardi, 2011).

Application: 3/3

Remark

The separation of LLPO and LPO is valid even if we add the following axioms to $\mathrm{HA}^{\lambda+}$

- $\mathrm{MP} :\equiv \Gamma \mid \neg\neg\exists n{:}N.\delta \vdash \exists n{:}N.\delta$ $\qquad$ ($\delta :\ \Delta_0(\Gamma, n{:}N)$-formula)
- $\mathrm{CP} :\equiv \Gamma \mid \Lambda \vdash \forall f{:}(N\to N)\to N.\forall\alpha{:}N\to N.\exists k{:}N.$
$\qquad \forall\beta{:}N\to N.(\forall i{:}N.(i \le k \to \alpha i = \beta i) \to f\alpha = f\beta)$
- $\mathrm{CK} :\equiv \Gamma \mid \Lambda \vdash \forall f{:}(N\to 2)\to N.\exists k{:}N.\forall\alpha{:}N\to 2.f\alpha \le k$

However the following axiom can NOT be added

- $\Delta_0\text{-CCA} :\equiv \Gamma \mid \forall i{:}N.\exists j{:}2.\delta \vdash \exists f{:}N\to 2.\forall i{:}N.\delta[fi/j]$
$\qquad$ ($\delta :\ \Delta_0(\Gamma, n{:}N)$-formula)

Conclusion

Conclusion

Syntactic and Semantic Works

- Reductions of hard separations into easy separations
- Uses of Weihrauch lattice for easy separations
- A combination of the above two yields a separation technic

Main Argument

- Fix a semantic structure
- Take its internal theory
- Use proof theoretic technics
- Conclude a structure or a property of the term model
- Conclude a structure or a property of the original structure
  via equivalence theorem

Thank you for listening