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A Laplacian Suitable for Diffusion or Delivery on Heterogeneous Structures

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Abstract

We propose a new Laplacian suitable for load balancing in distributed systems on heterogeneous networks. It gives us uniform understanding of the relations in the conventional continuous and discrete operators, as a natural extension for a heterogeneous network with asymmetric weights of different server's performance rates.

1. Introduction

The rapid growth of computer networks provides us with a potential of the cooperative computation powers. For the efficient distributed computing, one of the important issues is load balancing among servers with different performances. Load balancing algorithms have been so far developed for parallel computers consists of homogeneous networks connected with same processors. The fast load balancing depends on not only the calculation scheme but also the network topology and the performance (weight) which are deeply related to the discrete weighted Laplacian [2] for the diffusion or delivery.

On the other hand, it has been shown the discrete (combinatorial) and continuous (geometric) Laplacians have analogous theoretical properties on graphs and manifolds [2] [9]. However, the conventional interests were mainly concerned with the algebraic properties on a regular graph, e.g. the diameter of graph and Isospectral problems, with respect to the eigenvalues of a Laplacian. They were focused on the homogeneous structure. The relations between the discrete and continuous Laplacians are not comprehended, at least, it is unclear what discrete quantities in a distributed system are corresponded to the metric and connection in a manifold.

In this paper, **to uniformly understand the continuous and discrete Laplacians**, we propose a new one induced from the Laplace-Beltrami (partial differential) operator, by introducing the information geometric dually-flat structure [1] instead of the conventional Levi-Civita connection [9]. In particular, for the discrete version, we consider a heterogeneous network with asymmetric weights of different server's performance rates.

Next, we consider the diffusion equation by the proposed Laplacian as **a fundamental modeling for load balancing in distributed systems**, focusing on the differences to parallel computers: loose coupling, independence of processing elements, and heterogeneity [8]. We prove the Green's formula, Maximum-minimum principle, and the conservation of the total weighted load, which are related to the convergence properties. The remarkable point is that the total load itself is changeable. Furthermore, we point out the problem for wasteful migration on cycles, and presented a simpler algorithm based on **efficient message passings on a spanning tree**.

2. Modeling on Heterogeneous Networks

In the relations between continuous and discrete Laplacians, we clarify what discrete quantities are corresponded to the metric and connection which represent a heterogeneous structure in a continuous manifold. In particular, we show our proposed operator includes the conventional discrete Laplacians on a graph, as a natural extension with heterogeneously weighted edges.

2.1. A new Laplacian

For a continuous real function $f(x)$ on a manifold, we propose a new operator:

$$\mathcal{L}f(x) \stackrel{\text{def}}{=} -\sum_{ij} g^{ij}(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j} - \sum_{ik} \partial_i g^{ik}(x)\frac{\partial f(x)}{\partial x^k}, \quad (1)$$

where $\{x^1, \ldots, x^n\}$ is the affine coordinate system [1], $[g^{ij}(x)]$ is the inverse matrix of a Riemannian metric tensor $[g_{ij}(x)]$, and $\partial_i \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i}$. For the conventional Laplacian on a Riemannian manifold, the differences are as follows.

- As shown in Appendix, (1) is derived from the form of Laplace-Beltrami operator [9] by introducing not the usual Levi-Civita connection but the information geometric connections [1].

- $\mathcal{L}$ is self-conjugate in the Green's formula on a measure $dx \stackrel{\text{def}}{=} dx^1 dx^2 \ldots dx^n$ for the inner-product, instead of the volume measure $\sqrt{\det(g_{ij})}dx$. Therefore, it is not also equivalent to the partial differential conjugate operator for advection diffusion.

The operator (1) is naturally corresponded to the discrete version (4) because it is also self-conjugate as shown later in Theorem 1.

Let us consider a finite connected undirected graph (V, E) without multiple edges and self-loop at each vertex. On the graph, the discrete version for (1) is

$$\mathcal{L}f(u) = -\sum_{v \sim u \in E} g^{ii}(u)(f(v) - f(u)) \quad (2)$$

$$- \sum_{v \sim u \in E} \Delta g^{ii}(u)(f(v) - f(u)), \quad (3)$$

$$= -\sum_{v \sim u \in E} g^{ii}(v)\left(f(v) - f(u)\right), \quad (4)$$

$$\Delta g^{ii}(u) \stackrel{\text{def}}{=} g^{ii}(v) - g^{ii}(u),$$

where a directed edge $e_i : u \to v \in E$, the inverse $\bar{e}_i : v \to u \in E$, i denotes an index number in the edge set, $v \sim u$ denotes the set of adjacency vertices $\{v\}$to $u \in V$. Since there is no significant correspondence to the continuous version in the Greens's formula, only the case $i = k$, $g^{ij} = 0$ $(i \neq j)$ is considered in the discrete version. Note that $\mathcal{L}$ in (4) is represented as an asymmetric matrix operator for the vector $(f(1), \ldots, f(m))$, $|V| = m$, $|E| = n$.

In an applicational meaning, $f(u)$ is the amount of load[1] at each server $u \in V$, and the asymmetric (positive) weights $g^{ii}(v) \neq g^{ii}(u)$ represent the server's performance rates for vertices u and v. In the conventional discrete weighted Laplacians for the diffusive load balancing, only symmetric weights have been discussed as parameters [3] [5] or under the assumption of bidirectional same communications on a heterogeneous network topology [4]. The well-known discrete weighted Laplacian [2]: L is corresponded to a special case of $\mathcal{L}$ in (4) with symmetric weights $g^{ii}(u) = g^{ii}(v)$, $\Delta g^{ii} = 0$. We emphasis **that the asymmetric weights in $\mathcal{L}$ is really new for load balancing. Because the diffusive phenomenon itself is different**: the total load is not conserved as shown later.

On the other hand, by considering $g^{ii}(v) = \frac{m_E(e_i)}{m_V(u)} > 0$, the discrete version (4) is equivalent to the following operator in a spring-mass system [6],

$$\Delta_P f(u) \stackrel{\text{def}}{=} -\frac{1}{m_V(u)} \sum_{v \sim u} m_E(e_i)(f(v) - f(u)),$$

different masses: $m_V(u) \neq m_V(v)$, spring coefficient: $m_E(e_i) = m_E(\bar{e}_i)$. Thus, the relation is $\mathcal{L} \Leftrightarrow \Delta_P \to L$.

We can prove the following theorems for the new Laplacian $\mathcal{L}$.

Theorem 1 Green's formula

$$\int_M \mathcal{L}f_1 f_2 dx = \int_M (df_1, df_2)_G dx = \int_M f_1 \mathcal{L} f_2 dx,$$

$$(df_1, df_2)_G = (\mathcal{L}f_1, f_2)_G = (f_1, \mathcal{L}f_2)_G, \quad (5)$$

$$(df_1, df_2)_G \stackrel{\text{def}}{=} \frac{1}{2} \sum_{e_i, \bar{e}_i} df_1(e_i) df_2(e_i) m_E(e_i),$$

$$(\mathcal{L}f_1, f_2)_G \stackrel{\text{def}}{=} \sum_{u \in V} \mathcal{L}f_1(u) f_2(u) m_V(u),$$

$df_1(e_i) \stackrel{\text{def}}{=} f_1(v) - f_1(u)$, $df_2(e_i) \stackrel{\text{def}}{=} f_2(v) - f_2(u)$.

Theorem 2 Maximum-minimum principle
On the graph (V, E), the $\mathcal{L}$-harmonic function $f(u)$, which satisfies $\mathcal{L}f(u) = 0$ for $\forall u \in V$, is constant.

2.2. Load balancing by the diffusion

We consider the load balancing in distributed systems by diffusion methods [3] [4] [5]. The network consists of servers at vertices u and v connected with asymmetric weights $g^{ii}(u) \neq g^{ii}(v)$. The problem is how to migrate what amount of load by using only local diffusion on the edges until the balancing state.

The fundamental properties of $\mathcal{L}$ in Theorem 1,2 are related to the convergence of the diffusion equation:

$$\frac{\partial h(u,t)}{\partial t} = -\mathcal{L}h(u,t). \quad (6)$$

The $\mathcal{L}$-harmonic function is a steady state of (6) as the balancing load $\bar{f}(u) = \bar{f}(v)$ in any pairs of the adjacency vertices. It is

$$\bar{f} = \frac{\sum_{v \in V} m_V(v) h(v,0)}{\sum_{v \in V} m_V(v)}. \quad (7)$$

By substituting $f_2(u) = 1$, $df_2(u) = 0$ into (5), we obtain the following theorem. It means that the trajectory for (6) is constrained on a hyper-plan according to the initial load $h(u, 0)$ in the m-dimensional $\mathbf{h}$-coordinate system. On the contrary, if the conservation law holds for $\mathcal{L}$ as follows (equivalent to $\mathcal{L}^T \mathbf{m}_V = 0$), there exists an unique $m_V(v) > 0$ except for any scalar multiple [7].

Theorem 3 Conservation of the total weighted load
The diffusion equation (6) conserves the total weighted load at any time $t > 0$,

$$\sum_{u \in V} m_V(u) \times \frac{\partial h(u,t)}{\partial t} = -\sum_{u \in V} m_V(u) \times \mathcal{L}h(u,t) = 0,$$

[1] We assume the load is infinitely divisible as similar to [3] [4] [5] because of the independence of processing elements.

$$\sum_{u \in V} m_V(u) \times h(u,0) = \sum_{u \in V} m_V(u) \times h(u,t).$$

The continuous version $\int \frac{\partial h}{\partial t} dx = 0$ also holds. Unlike this, the total load $\sum_{u \in V} h(u,t)$ is conserved for $\frac{\partial h(u,t)}{\partial t} = -Lh(u,t)$ by $\sum_{u \in V} Lh(u,t) = 0$. The trajectory for L is constrained on a simplex $\sum_{u \in V} h(u,t) = const.$ in the $\mathbf{h}$-coordinate system, while the hyper-plain for $\mathcal{L}$ is split from the simplex by the effect of $\mathbf{m}_V$.

3. Efficient message passings on a tree

In this section, we point out the problem for wasteful migration on cycles, and presented a simpler algorithm based on efficient message passings on a spanning tree.

For the origin u on a directed edge e_i, the migration flow is given by $g^{ii}(v)(f(u) - f(v))$ in the right-side of (4). While it is $g^{ii}(u)(f(v) - f(u))$ for the terminal v. Thus, the sum of outer and inner flow at each vertex is given by $\mathcal{B}\mathbf{z}$, and the balancing condition is

$$\mathcal{B}\mathbf{z} = \mathbf{f}^0 - \bar{\mathbf{f}}, \qquad (8)$$

where $\mathbf{z}$ denotes n-dimensional flow vector, $\mathbf{f}^0$ m-dimensional initial load vector, $\bar{\mathbf{f}} = (\bar{f}, \ldots, \bar{f})$ the balancing solution vector, $\mathcal{B}$ is a $m \times n$ modified incidence matrix:

$$(\mathcal{B})_{ji} \stackrel{\text{def}}{=} \begin{cases} +1 & j = u : \textit{ the origin of } e_i \\ -\frac{g^{ii}(u)}{g^{ii}(v)} & j = v : \textit{ the terminal of } e_i \\ 0 & \textit{otherwise.} \end{cases}$$

Note that, for the balancing condition (8), as any cyclic flow Δz such that $\mathcal{B}\Delta z = 0$ is added to a feasible solution, it also becomes a balancing load, but obviously wasteful. This problem is essential on a network with cycles. To avoid it, we present a simpler algorithm based on message passings to directly (not iteratively solve (6)) calculate the flow on a spanning tree because of no-cycles.

First, we find a spanning tree with the least connectedness. Next, the amounts of weighted load and the vertex weight are accumulated to the parent (from leaves to the root). Then, the root broadcasts the balancing value $\bar{f}$ as in (7) on the inverse directions. After the reaching, the flow $z_e = f(u) - \bar{f}$ is uniquely determined at a leaf $u \in V$, e: $u \to v$. Recursively, the flow on e': $v \to w$ from an articulation vertex to the parent is obtained by $z_{e'} = f(v) - \bar{f} + \sum_e \hat{z}_e$. Because of the asymmetry of $\mathcal{L}$, we must apply the inner flow $\hat{z}_e \stackrel{\text{def}}{=} \frac{g^{ii}(u)}{g^{ii}(v)} z_e$ to the parent, instead of the previously determined outer flow z_e from the children. In other words, the inner and outer flow are different on a same edge. The sign of flow is corresponded to the direction of migration. After the calculation of flow immediately, the migration of load is executed. Thus, the algorithm with $O(m)$ message complexity is asynchronously performed on a distributed system.

4. Conclusion

We have proposed a new Laplacian $\mathcal{L}$ as a natural extension for a heterogeneous space: continuous manifold or discrete graph. The obtained results for the proposed asymmetric Laplacian are

- It is clarified what discrete quantities are corresponded to the Riemannian metric and connection in a continuous manifold.

- The discrete version of $\mathcal{L}$ is equivalent to the operator Δ_P in a mass-spring system [6]. As a special symmetric case, it includes the well-known discrete weighted Laplacian [2].

Thus, the relations of Laplacians in the state of the art are uniformly understood.

Next, we have considered the diffusion equation by $\mathcal{L}$ as a fundamental modeling for load balancing in distributed systems, focusing on the differences to parallel computers: loose coupling, independence of processing elements, and heterogeneity. We have proved the Green's formula, Maximum-minimum principle, and the conservation of the total weighted load, which are related to the convergence properties. The remarkable point is that the total load itself is changeable for migrations between the heterogeneous measures. Although such phenomenon has been not discussed in literatures, it is rather natural as similar to the inconsistency e.g. in the circulation of financial market prices or subjective evaluations of information on various rates or criterions. Furthermore, we have pointed out the essential problem for wasteful migration on cycles, and presented a simpler algorithm based on efficient message passings. Such message passing is a common approach to the Bethe approximation (also known as "belief propagation algorithm") in statistical physics or graphical modeling [10].

Since a Riemannian manifold is by nature a representation model for heterogeneous structures with a different metric at each point, the corresponded discrete Laplacian $\mathcal{L}$ gives us a natural modeling for the diffusion or delivery on heterogeneous networks. Thus, the modeling will construct a theoretical foundation of the distributed computing. It will be applicable to a random walk [2] [6] or circulation of evaluations for information contents on the WWW.

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Appendix

We derive the new operator $\mathcal{L}$ from the form of Laplace-Beltrami operator by introducing a dually-flat structure of information geometry [1].

For a real C^∞-function $f(x)$ over a compact Riemannian manifold M (corresponded to a finite graph), we consider the form of Laplace-Beltrami operator [9],

$$\mathcal{L}_B f(x) \stackrel{\text{def}}{=} -\sum_{ij} g^{ij}(x)\left(\frac{\partial^2 f(x)}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij}(x)\frac{\partial f(x)}{\partial x^k}\right), \tag{9}$$

where $\Gamma^k_{ij}(x) = \sum_h g^{hk}(x)\Gamma_{ijh}(x)$ is a coefficient for the connection between tangent spaces in the manifold.

Instead of the usual Levi-Civita connection $\hat{\Gamma}_{ijh}$, we introduce a pair of primal-dual connections Γ_{ijh} and Γ^*_{ihj} with a dually-flat structure[2] in information geometry [1]: $\Gamma^*_{ihj} = 0$ into the form (9). Then, we have

$$\partial_i g_{jh} = \Gamma_{ijh} + \Gamma^*_{ihj} = \Gamma_{ijh}. \tag{10}$$

From $\sum_h \partial_i \left(\sum_j g_{jh} g^{ij}\right) = \sum_h \partial_i \delta^i_h = 0$, we have

$$\partial_i g^{ik} = -\sum_j \Gamma^k_{ij} g^{ij}. \tag{11}$$

Thus, we obtain the new operator (1).

In general [1], given a metric g and a connection ∇ on M, there exists a unique dual connection ∇^* with respect to g. However, given some connection, a corresponding affine coordinate system does not in general exist. If the dualistic structure of quaternary (M, g, ∇, ∇^*) is a dually-flat, then there exists a pair of ∇- and ∇^*-affine coordinate systems. Therefore, only such coordinate systems (with an implicit dual coordinate system) is considered for $\mathcal{L}$, the restriction is significant in applications, e.g. in the case of an exponential family in statistical manifolds [1].

On the other hand, when the manifold is not dually-flat,

$$\partial_i g^{ik} = -\sum_{j,h} g^{hk} g^{ij}(\hat{\Gamma}_{ijh} + \hat{\Gamma}_{ihj}),$$

$$\partial_i g^{ik} = -\sum_{j,h} g^{hk} g^{ij}(\Gamma_{ijh} + \Gamma^*_{ihj}),$$

is obtained from the above relations. It is not $\mathcal{L}_B$ in (9) by comparing with (11), and seems to be the conjugate operator A with advection which is corresponded to the second-term in the above right-sides, respectively. However it is not naturally corresponded to the discrete version (4) as self-conjugate, because there is the Green's formula (not self-conjugate):

$$\int_M A f_1 f_2 \sqrt{\det(g_{ij})} dx = \int_M f_1 A^* f_2 \sqrt{\det(g_{ij})} dx,$$

where A^* is another conjugate pair of A. Thus, in the only case of dually-flat, $\mathcal{L}$ is represented by the form of Laplace-Beltrami operator.

[2] For a familiar form in differential geometry, the duality (10) is represented by $Z < X, Y > = < \nabla_Z X, Y > + < X, \nabla^*_Z Y >$, where X, Y, Z denote vector-fields on the tangent space.