Infinite Games in the Cantor Space over Admissible Set Theories

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Introduction

- Axiom of determinacy for infinite games: Set-theoretic statement over second order language stemming from descriptive set theory.
- This work: A fine-grained analysis of $\Delta^0_2$-definable games in the Cantor space over admissible set theories.
  - Why $\Delta^0_2$-games? - The first class for which the different hierarchy makes sense.
  - Why in the Cantor space? - The logical strength of the axiom gets weaker than in the Baire space ($\Pi^1_1$-TR$_0$ to ATR$_0$).
  - Why admissible set theories? - A natural hierarchy reaching ATR$_0$ is known.
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Axiom of determinacy

Two players game $A$: $(x_0, x_1, \ldots \ y_0, y_1, \cdots \in X)$

Player I $\ x_0 \ x_1 \ \cdots$
Player II $\ y_0 \ y_1 \ \cdots$

- A strategy $\sigma$ for Player I is a partial function $X^{<\mathbb{N}} \rightarrow X$ s.t.
  $\sigma(\langle x_0, y_0, \ldots, x_{j-1}, y_{j-1} \rangle) = x_j$.
- A strategy $\sigma$ for Player II is a partial function $X^{<\mathbb{N}} \rightarrow X$ s.t.
  $\sigma(\langle x_0, y_0, \ldots, x_{j-1}, y_{j-1}, x_j \rangle) = y_j$.

Player I wins the game $A \iff \langle x_0, y_0, x_1, y_1, \ldots \rangle \in A$
for any strategy for Player II.

Player II wins the game $A \iff \langle x_0, y_0, x_1, y_1, \ldots \rangle \notin A$
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\[
\begin{array}{c}
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Axiom of determinacy in the Cantor space

Let $\Phi$: class of sets.

**Axiom of determinacy:** Either Player I or II wins the game $A \in \Phi$.

1. $\Phi$-Det: In case $X = \mathbb{N}$.
2. $\Phi$-Det*: In case $X = 2 = \{0, 1\}$.

**Theorem (Nemoto-MedSalem-Tanaka ’07)**

1. $\text{RCA}_0 \vdash \Sigma^0_1$-Det* $\iff$ $\text{WKL}_0$.
2. $\text{RCA}_0 \vdash \Delta^0_2$-Det* $\iff$ $\text{ATR}_0$. 
Shoenfield Limit lemma

Any $\Delta^0_2$ set can be approximated by the symmetric difference of recursively enumerable sets.

**Theorem (Shoenfield)**

For any $\Delta^0_2$-set, there exists a recursive function $f : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ such that $\lim_s f(x, s) = A(x)$. ($A(x) \iff x \in A \iff \chi_A(x) = 1$)

This induces the Ershov hierarchy, the symmetric difference of a recursively enumerable sets for an element $a$ of Klneene’s ordinal notation system $O$. 

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This induces the Ershov hierarchy, the symmetric difference of a recursively enumerable sets for an element $a$ of Kleneene’s ordinal notation system $\mathcal{O}$. 
Kleene’s $\mathcal{O}$

**Definition (Kleene’s $\mathcal{O}$)**

The set $\mathcal{O} \subseteq \mathbb{N}$ of notations, a function $\cdot |_\mathcal{O} : \mathcal{O} \rightarrow \text{Ord}$ and a strict partial order $<_\mathcal{O}$ on $\mathcal{O}$ are defined simultaneously.

1. $1 \in \mathcal{O}$ and $|1|_\mathcal{O} = 0$.
2. If $a \in \mathcal{O}$ and $|a|_\mathcal{O} = \alpha$, then $2^a \in \mathcal{O}$ and $|2^a|_\mathcal{O} = \alpha + 1$.
3. If $e$ is a code of a total recursive function such that $|\{e\}(n)|_\mathcal{O} = \alpha_n$ and $\{e\}(n) <_\mathcal{O} \{e\}(n+1)$ hold for all $n \in \mathbb{N}$, then $3 \cdot 5^e \in \mathcal{O}$ and $|3 \cdot 5^e|_\mathcal{O} = \lim_n \alpha_n$.

**Fact**

1. $<_\mathcal{O}$ and $\mathcal{O}$ are $\Pi^1_1$-definable sets.
2. $<_\mathcal{O}$ is a well-founded partial order on $\mathcal{O}$.
3. $<_\mathcal{O} \upharpoonright a$ is a linear order for any $a \in \mathcal{O}$.
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a-r.e. sets

**Definition (a-r.e. sets)**

Let \( a \in \mathcal{O} \). \( A \subseteq \mathbb{N} \) is a-r.e. if there exist recursive functions \( f : \mathbb{N} \times \mathbb{N} \to \{0, 1\} \) and \( o : \mathbb{N} \times \mathbb{N} \to \mathcal{O} \) s.t.

1. \( f(x, 0) = 0 \) and \( o(x, 0) <_\mathcal{O} a \) for all \( x \).
2. \( o(x, s + 1) \leq_\mathcal{O} o(x, s) \) for all \( x \) and \( s \).
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**Theorem (Stephan-Yang-Yu '10)**

For any \( \Delta^0_2 \) set \( A \subseteq \mathbb{N} \), there exists \( a \in \mathcal{O} \) such that \( |a|_\mathcal{O} = \omega^2 \) and \( A \) is an a-r.e. set.
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**Theorem (Stephan-Yang-Yu ’10)**

*For any $\Delta^0_2$ set $A \subseteq \mathbb{N}$, there exists $a \in \mathcal{O}$ such that $|a|_\mathcal{O} = \omega^2$ and $A$ is an a-r.e. set.*
Remarks

• Original idea: to layer the $\Delta^0_2$-Det* by the Ershov hierarchy.

• Oversight of the speaker: The theorem fails for $A \subseteq 2^\mathbb{N}$ (addressed by T. Kihara).
  - $\Delta^0_2$ subsets of $2^\mathbb{N}$ will not be exhausted at $\omega^2$.
  - The Ershov hierarchy might not be appropriate for fine-grained analysis of determinacy of $\Delta^0_2$-definable games.

• This talk presents very partial results.
**Definition**

Let $a \in \mathcal{O}$. Assume the relation $\langle \mathcal{O}, a \rangle$ can be expressed in an underlying formal system. Then we say a formula is \((\Sigma^0_1)_a\)-formula if it is of the form

$$\exists b < \mathcal{O} a \left[ \varphi(b) \land (\forall c < \mathcal{O} b) \neg \varphi(c) \right]$$

for some $\Sigma^0_1$-formula $\varphi$.

Intuitively, a \((\Sigma^0_1)_a\)-formula expresses:

$$\exists b < \mathcal{O} a \left[ \exists s f(s, b) = 0 \land (\forall c < \mathcal{O} b) \forall s f(s, c) = 1 \right]$$
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A system $K\mathbf{Pu}^0$ of admissible set theory:

Weak subsystem of ZF without (Power) over $\mathcal{L}_{ZF} \cup \{\text{Ad}\}$ s.t.

1. Axiom of Separation is limited to $\Delta_0$-formulas.
2. Axiom of Replacement is limited the axiom of Collection for $\Delta_0$-formulas.
3. Axioms for Ad: $\text{Ad}(z)$ means $z$ is an admissible set, i.e., $z$ satisfies $(\Delta_0\text{-Sep})$ and $(\Delta_0\text{-Col})$.

Note:

- $K\mathbf{Pu}^0 \vdash \Delta_1^1\text{-CA}_0$. Hence $K\mathbf{Pu}^0$ is strong enough for a base system.
- Unlike $K\mathbf{Pu}$ (or KP), transfinite induction holds in $K\mathbf{Pu}$ only for $\Delta_0$-formulas.
Admissible set theory

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Admissible set theory with iterated admissible universes

\[ \text{KPu}^0 + (U_n) \ (\text{over } \mathcal{L}_{ZF} \cup \{\text{Ad}\} \cup \{d_0, \ldots, d_{n-1}\}) : \]

\[ \text{Ad}(d_0) \land \cdots \land \text{Ad}(d_{n-1}) \land d_0 \in d_1 \land \cdots \land d_{n-2} \in d_{n-1} \quad (U_n) \]

The set \(d_0\) could be interpreted as \(L^{\omega_1\text{CK}}\).

**Theorem (Jäger ’84)**

\(|T|: \text{maximal order type of recursive well ordering provable in } T.\)

\((\alpha, \beta) \mapsto \varphi(\alpha, \beta): \text{Veblen function.}\)

1. \(|\text{KPu}^0 + (U_1)| = \varphi(\varepsilon_0, 0).\)
2. \(|\text{KPu}^0 + (U_{n+2})| = \varphi(|\text{KPu}^0 + (U_{n+1})|, 0).\)

Therefore \(|\bigcup_{n<\omega} \text{KPu}^0 + (U_n)| = |\text{ATR}_0| = \Gamma_0.|
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**Therefore** \( |\bigcup_{n<\omega} \text{KPu}^0 + (\mathcal{U}_n)| = |\text{ATR}_0| = \Gamma_0 \).
Admissible sets have a closure property: The fixed point axiom for arithmetically definably operators holds in $\bigcup_{n<\omega} \text{KPu} + (\mathcal{U}_n)$.

**Lemma (Jäger '84)**

$\varphi(X, \vec{Y}, x)$: $X$-positive arithmetical formula.

$$\text{KPu} + (\mathcal{U}_{n+1}) \vdash (\forall \vec{Y} \in d_{n-1})(\exists X \in d_n)(\forall x) \left( x \in X \iff \varphi(X, \vec{Y}, x) \right)$$

(Hence at most $n$-fold iterated application of fixed point axiom is possible)

Note: due to absence of transfinite recursion, the leastness of the fixed point is not provable.
Fixed point axiom holds in $\bigcup_{n<\omega} \text{KPu} + (\mathcal{U}_n)$

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**ATR_0** holds in $\bigcup_{n<\omega} \text{KPU}^0 + (\mathcal{U}_n)$

**Lemma (Jäger ’84)**

$\varphi$: arithmetical formula.

\[
(\forall <, \vec{Y} \in d_n) \quad \text{WO}(<) \rightarrow \\
(\exists X \in d_n)(\forall \alpha \in \text{field}(<)) \forall x \left( x \in X_{\alpha} \leftrightarrow \varphi(X_{<\alpha}, \vec{Y}, x) \right)
\]

holds in $\text{KPU}^0 + (\mathcal{U}_{n+1})$. 
Well-ordering of $<_\mathcal{O}$ up to $\omega \cdot n$

**Lemma**

Let $n < \omega$ and $a_n = 3 \cdot 5^{e_n} \in \mathcal{O}$ represent $\omega \cdot (n + 1)$.
1. $<_\mathcal{O} \upharpoonright a_n$ of $<_\mathcal{O}$ is definable in $\text{KP}_u + (\mathcal{U}_{n+1})$.
2. $\text{KP}_u + (\mathcal{U}_{n+1}) \vdash \text{WO}(<_\mathcal{O} \upharpoonright a_n)$.

**Proof.**

By $n$-fold application of FP axiom, define a relation $<_n \in d_n$:

\[
\begin{align*}
  b <_0 a & \iff (b = 1 \land a = 2^1) \lor \exists c (b \leq_0 c \land a = 2^c) \\
  b <_{n+1} a & \iff \begin{cases} 
    b <_n a \lor \exists c (b \leq_{n+1} c \land a = 2^c) \lor \\
    [a = a_n \land \forall m (\{e_n\}(m) <_n \{e_n\}(m + 1)) \land \\
    \exists m (b <_n \{e_n\}(m))] 
  \end{cases}
\end{align*}
\]

See $<_n =<_\mathcal{O} \upharpoonright a_n$. Show $\text{KP}_u + (\mathcal{U}_{n+1}) \vdash \text{WO}(<_n)$ by ind on $n$. \qed
Well-ordering of $<\emptyset$ up to $\omega \cdot n$

**Lemma**

Let $n < \omega$ and $a_n = 3 \cdot 5^e_n \in \emptyset$ represent $\omega \cdot (n + 1)$.

1. $<\emptyset \upharpoonright a_n$ of $<\emptyset$ is definable in $\text{KP}_u + (\mathcal{U}_{n+1})$.
2. $\text{KP}_u + (\mathcal{U}_{n+1}) \vdash \text{WO}(<\emptyset \upharpoonright a_n)$.

**Proof.**

By $n$-fold application of FP axiom, define a relation $<_{n} \in d_n$:

$$b <_0 a \iff (b = 1 \land a = 2^1) \lor \exists c (b \leq_0 c \land a = 2^c)$$

$$b <_{n+1} a \iff \begin{cases} b <_n a \lor \exists c (b \leq_{n+1} c \land a = 2^c) \lor \\ [a = a_n \land \forall m (e_n(m) <_n \{e_n\}(m + 1)) \land \\ \exists m (b <_n \{e_n\}(m))] \end{cases}$$

See $<_n = <\emptyset \upharpoonright a_n$. Show $\text{KP}_u + (\mathcal{U}_{n+1}) \vdash \text{WO}(<_n)$ by ind on $n$. □
Theorem

Let $1 \leq n$. Suppose that $a \in \mathcal{O}$ is a notation for $\omega \cdot n$. Then $\text{KPU}^0 + (\mathcal{U}_n) \vdash (\Sigma^0_1)_{a-\text{Det}^*}$.

Outline of Proof.

Given a $(\Sigma^0_1)_a$ formula $\varphi(f)$, define a set $W_b \in d_{n-1} (b < \mathcal{O} \upharpoonright a)$ of winning positions $s \in 2^{\leq \mathbb{N}}$ by (ATR):

$$s \in W_b \iff \psi(s, W_{<\mathcal{O}b}),$$

where $\psi \in \Pi^1_0$ is defined from $\varphi$. Define a new $\Sigma^0_1$ game $\varphi'(f) : \equiv \exists m(\exists b < \mathcal{O} a)\langle f(0), \ldots, f(2m-1) \rangle \in W_b$.

1. If Player I wins $\varphi'(f)$, then I wins $\varphi(f)$.
2. If Player II wins $\varphi'(f)$, then II wins $\varphi(f)$.

Note: $\Sigma^0_1$-$\text{Det}^*$ holds in $\text{KPU}^0 + (\mathcal{U}_n)$. 

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Infinite Games over Admissible Set Theories
Theorem

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Outline of Proof.

Given a $(\Sigma^0_1)_a$ formula $\varphi(f)$, define a set $W_b \in d_{n-1}$ ($b < \mathcal{O} \upharpoonright a$) of winning positions $s \in 2^{<\mathbb{N}}$ by (ATR):

$$s \in W_b \leftrightarrow \psi(s, W_{<\mathcal{O}b}),$$

where $\psi \in \Pi^1_0$ is defined from $\varphi$. Define a new $\Sigma^0_1$ game $\varphi'(f) := \exists m (\exists b < \mathcal{O} \upharpoonright a) \langle f(0), \ldots, f(2m-1) \rangle \in W_b$.

1. If Player I wins $\varphi'(f)$, then I wins $\varphi(f)$.
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Summary

- Aiming fine-grained analysis of determinacy of $\Delta^0_2$-definable games in the Cantor space.
- Layering based on the Ershov hierarchy, which turns out to be questionable.
- Obtained partial results strongly rely on the definability and provability of the well ordering of $\langle O \upharpoonright a \rangle$.
- This observation is consistent with the results about $\left( \Sigma^0_1 \right)_\alpha$-Det* ($\alpha < \Gamma_0$) by Nemoto-Sato.

Thank you for your listening!

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Conclusions

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- Aiming fine-grained analysis of determinacy of \( \Delta^0_2 \)-definable games in the Cantor space.
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