Classical provability of uniform versions and intuitionistic provability

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(joint work with Ulrich Kohlenbach)

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Many mathematical statements have \( \Pi_2 \) form:

\[
\forall X \ (A(X) \rightarrow \exists Y \ B(X, Y)).
\]

**Intermediate Value Theorem.**

For any continuous function \( f : [0, 1] \rightarrow \mathbb{R} \) s.t. \( f(0) < 0 < f(1) \), then there exists a point \( m \in [0, 1] \) s.t. \( f(m) = 0 \).

For \( \Pi_2 \) statements, we study the relationship between uniform provability in classical reverse mathematics and intuitionistic (constructive) reverse mathematics.
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For $\Pi_2$ statements, we study the relationship between uniform provability in classical reverse mathematics and intuitionistic (constructive) reverse mathematics.
Sequential versions

- Many $\Pi^1_2$ statements are provable in RCA ($\text{RCA}_0 + \text{full induction}$).
- In some of their proofs, however, the construction of the solution $Y$ from given $X$ is not uniform.
- To reveal the non-uniformity, the following sequential version has been investigated.

$$\forall \langle X_n \rangle_{n \in \mathbb{N}} (\forall n A(X_n) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n)).$$

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The following uniform version seems to be rather acceptable than the sequential version as representation of uniformity.

\[ \exists \Phi \forall X (A(X) \rightarrow B(X, \Phi(X))) \].

(Note that uniform version implies sequential version.)

However, for a \( \Pi^1_2 \) sentence, its uniform version is not naturally represented in the language of second-order arithmetic.

To treat uniform versions, the system of arithmetic in all finite types is employed.
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\[ \exists \Phi \forall X (A(X) \rightarrow B(X, \Phi(X))) \].

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However, for a \( \Pi_2^1 \) sentence, its uniform version is not naturally represented in the language of second-order arithmetic.

To treat uniform versions, the system of arithmetic in all finite types is employed.
The following **uniform version** seems to be rather acceptable than the sequential version as representation of uniformity.

$$\exists \Phi \forall X (A(X) \rightarrow B(X, \Phi(X))).$$

(Note that uniform version implies sequential version.)

However, for a $\Pi^1_2$ sentence, its uniform version is not naturally represented in the language of second-order arithmetic.

To treat uniform versions, the system of arithmetic in all finite types is employed.
Hilbert-type system $E$-$HA^\omega$ (resp. $E$-$PA^\omega$) is the finite type extension of HA (resp. PA).

$E$-$PA^\omega := E$-$HA^\omega + \text{LEM}(A \lor \neg A)$.

$RCA^\omega := E$-$PA^\omega + \text{QF-AC}^{1,0}$.

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**Proposition.** (Kohlenbach 2001)

$RCA^\omega$ is a conservative extension of RCA.
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**Proposition.** (Kohlenbach 2001)

$RCA^\omega$ is a conservative extension of RCA.
Strength of Uniform Versions

- **RCA^ω ⊢ UWKL ↔ UACA.** (Kohlenbach 2001)

- **RCA^ω** is too strong as base system for investigating uniform versions!

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UWKL is the uniform version of WKL.

UACA: \(\exists E^2 \forall f^1 (E(f) = 0 \leftrightarrow \exists x^0 (f(x) = 0))\).
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- UWKL is the uniform version of WKL.
- UACA: $\exists E^2 \forall f^1 \ (E(f) = 0 \iff \exists x^0 (f(x) = 0))$.

**Remark.**
RCA$^\omega$ is too strong as base system for investigating uniform versions!
The Systems with Weak Extensionality

- Our systems have only $=_{0}$ as predicate symbol and $s^{\rho} =_{\rho} t^{\rho}$ is the abbreviation for

$$\forall v_{1}^{\rho}, \ldots, v_{k}^{\rho} \ (s(v_{1} \ldots v_{k}) =_{0} t(v_{1} \ldots v_{k}))$$

where $\rho = \rho_{1} \rightarrow \ldots \rightarrow \rho_{k} \rightarrow 0$.

- E-PA$^\omega$ have the **extensionality** axiom (E):

$$\forall z^{\rho \rightarrow \tau}, x^{\rho}, y^{\rho} (x =_{\rho} y \rightarrow z(x) =_{\tau} z(y))$$

- WE-PA$^\omega$ (resp. WE-HA$^\omega$) is the subsystem of E-PA$^\omega$ (resp. E-HA$^\omega$) where (E) is replaced by the **weak extensionality** rule:

$$\frac{A_{qf} \rightarrow s =_{\rho} t}{A_{qf} \rightarrow r^{\tau}[s/x^{\rho}] =_{\tau} r[t/x^{\rho}]}.$$ 

- WRCA$^\omega :=$ WE-PA$^\omega + \text{QF-AC}^{1,0}$.

- WRCA$^\omega$ is a conservative extension of RCA.
By comparing the provably recursive functions, we have

\[ \text{WRCA}^\omega + \text{UWKL} \nsubseteq \text{UACA}. \]

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By comparing the provably recursive functions, we have

$$\text{WRCA}^\omega + \text{UWKL} \not\models \text{UACA}.$$
For the statements non-uniformly provable in RCA, the shift of the strength by uniformization seems to be caused from the use of LEM : $A \lor \neg A$ for undecidable $A$. 
The following result expresses the informal idea that if a \( \Pi_2 \) statement is provable without the use of \( \text{LEM} \), then it has a uniform proof.

**Theorem. (Hirst-Mummert 2011)**

For a \( \Pi_2 \) sentence \( S := \forall x^\rho (A(x) \rightarrow \exists y^\tau B(x, y)) \) where \( A \) is purely universal and \( B \) has the suitable syntactical form, if

\[
\text{WE-HA}^\omega + \text{AC}^\omega + \text{IP}_\forall^\omega + \text{M}^\omega \vdash S,
\]

then

\[
\text{WRCA}^\omega \vdash \text{Uni}(S).
\]

- \( \text{IP}_\forall^\rho,^\tau : (\forall z^\rho A_{qf} \rightarrow \exists x^\tau B(x)) \rightarrow \exists x^\tau (\forall z^\rho A_{qf} \rightarrow B(x^\rho)) \).
- \( \text{M}^\rho : \neg\neg\exists x^\rho A_0(x) \rightarrow \exists x^\rho A_0(x) \).

The proof is straightforward by the usual Dialectica interpretation (which extracts the term constructing \( y \) from \( x \)).
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- \( \text{IP}_\forall^{\rho, \tau} : (\forall z^\rho A_{qf} \rightarrow \exists x^\tau B(x)) \rightarrow \exists x^\tau (\forall z^\rho A_{qf} \rightarrow B(x^\rho)) \).
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Corollary.

For a $\Pi_2$ sentence $S$ of the previous syntactical form, if

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then

$$\text{RCA} \vdash \text{Seq}(S).$$

Application.

IVT, IPP, JD are not provable in $\text{E-HA}^\omega + \text{AC}^\omega + \text{IP}_\forall^\omega + \text{M}^\omega$. 
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Motivating Results

Hierarchy of **LEM** over HA (Akama et al., 2004)

- **M⁰**: \( \neg\neg\exists x^0 A_{qf} \rightarrow \exists x^0 A_{qf} \)
- **Σ¹-LEM**: \( \exists x^0 A_{qf} \lor \neg\exists x^0 A_{qf} \)
- **Π¹-DML**: \( \neg(\exists x^0 A_{qf} \land \exists y^0 B_{qf}) \rightarrow (\neg\exists x^0 A_{qf} \lor \neg\exists y^0 B_{qf}) \)

Some equivalences over intuitionistic systems (like WE-HA\(^\omega\)) have been established.

**Proposition.** (Ishihara, 2005)

1. ACA \(\leftrightarrow\) \(\Sigma^0_1\)-LEM + \(\Pi^0_1\)-AC\(^0\cdot^0\).
2. WKL \(\leftrightarrow\) \(\Sigma^0_1\)-DML + \(\Pi^0_1\)-AC\(^\lor\).
Motivating Results

Hierarchy of LEM over HA (Akama et al., 2004)

\[ \Sigma_1^0 - \text{LEM} \]

\[ \Pi_1^0 - \text{LEM} \]

\[ \Sigma_1^0 - \text{DML} \]

\[ M^0 : \neg \exists x^0 A_{qf} \rightarrow \exists x^0 A_{qf} \]

\[ \Sigma_1^0 - \text{LEM} : \exists x^0 A_{qf} \lor \neg \exists x^0 A_{qf} \]

\[ \Sigma_1^0 - \text{DML} : \neg (\exists x^0 A_{qf} \land \exists y^0 B_{qf}) \rightarrow (\neg \exists x^0 A_{qf} \lor \neg \exists y^0 B_{qf}) \]

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Proposition. (Ishihara, 2005)

1. ACA \leftrightarrow \Sigma_1^0 - \text{LEM} + \Pi_1^0 - \text{AC}^{0,0}.
2. WKL \leftrightarrow \Sigma_1^0 - \text{DML} + \Pi_1^0 - \text{AC}^{\vee}.
Question.
Can we extract stronger unprovability for the statement whose sequential version implies ACA rather than only WKL?

Theorem. (Kohlenbach-F.)
For a $\Pi_2$ sentence $S$ of the previous syntactical form, if

$$\text{WE-HA}^\omega + \text{AC}^\omega + \text{IP}_\forall^\omega + \text{M}^\omega + \text{UWKL} + \text{KL} \vdash S,$$

then

$$\text{WRCA}^\omega + \text{UWKL} \vdash \text{Uni}(S).$$

Application. (Note that $\text{WRCA}^\omega + \text{UWKL} \nvdash \text{ACA}$.)
IPP, JD are not provable in
WE-HA$^\omega + \text{AC}^\omega + \text{IP}_\forall^\omega + \text{M}^\omega + \text{UWKL} + \text{KL}$. 
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\[
\text{WE-HA}^\omega + \text{AC}^\omega + \text{IP}^\omega + M^\omega + \text{UWKL} + \text{KL}.
\]
However, we can extract further stronger unprovability if each uniform version implies $\text{UACA}$ over $\text{WRCA}^{\omega}$.

That is the merit to investigate uniform versions rather than sequential versions!
Main Theorem. (Kohlenbach-F.)

For a $\Pi_2$ sentence $S$ of the previous syntactical form, if

$$WRCA^\omega + \text{Uni}(S) \vdash UACA,$$

then

$$WE-HA^\omega + AC^\omega + IP_\forall^\omega + M^\omega + UWKL + KL + BI^\omega \nvdash S.$$ 

- $BI^\omega$ is the bar induction scheme in all finite type.

Application.

IPP, JD are not provable in

$$WE-HA^\omega + AC^\omega + IP_\forall^\omega + M^\omega + UWKL + KL + BI^\omega.$$ 

Remark.

1. WRCA$^\omega$ cannot be replaced by RCA$^\omega$.
2. UACA cannot be replaced by ACA.
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Remark.

1. $WRCA^\omega$ cannot be replaced by $RCA^\omega$.
2. $UACA$ cannot be replaced by $ACA$. 

Tools for the proof of main theorem.

- $\text{WE-HA}^\omega + \text{AC}^\omega + \text{IP}^\omega + M^\omega + \text{BR}^\omega \text{(bar recursion)} \vdash \text{BI}^\omega$ (Howard 1968).
- Negative translation.
- The Dialectica interpretation without extracting terms.
- A non-standard principle $F^-$ related to the fan principle.
- The model $M^\omega$ of all strongly majorizable functionals.
Corollary. (due to Luckhard’s technique)

For a $\Pi^1_2$ sentence $S$ of the previous syntactical form, if

$$\text{WRCA}^\omega + \text{Uni}(S) \vdash \text{UACA}, \text{ then}$$

$$\text{E-HA}^\omega + \text{AC}^\omega * + \text{IP}_{\forall}^{1,1} + M^1 + \text{KL} + \text{BI}^1 \not
\vdash S.$$

- $\text{AC}^\omega * := \text{AC}^{1,\tau} + \text{AC}^{0,\tau}$.
- $\text{BI}^1$ is the restriction of $\text{BI}^\omega$ to type 1 objects.
Roughly speaking, our meta-theorem allows one to detect using classical reasoning on Uni(S) that S implies at least the $\Pi^0_1$-LEM rather than only the strictly weaker principle $\Sigma^0_1$-DML.
Future Work.

- In intuitionistic reverse mathematics, a lot of relationships between non-constructive principles still remain to be open.
- Theorems of this kind might be strong tools to analyze the structure of hierarchy between non-constructive principles for constructive reverse mathematics.

⇒ Analyze relationships between non-constructive principles by using theorems of this kind and uniform reverse mathematics!
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⇒ Analyze relationships between non-constructive principles by using theorems of this kind and uniform reverse mathematics!
Thank you for your attention!