The Order Dimensions of Degree Structures

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1. Order Dimension

A Fact

- Fact: Every partial order $(P, \leq)$ is embeddable into the product order $\prod_{i \in I}(Q_i, \leq_i)$ of linear orders $(Q_i, \leq_i)$, $i \in I$.
- Proof. Consider $F : P \to \prod_{x \in P} 2$ defined via

$$F(y)_x := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{o.w.} \end{cases}$$

Observe that

$$y \leq z \iff \forall x \in P,\ F(y)_x \leq F(z)_x.$$

Order Dimension

- Dfn(Dushnik and Miller 1941, see also Ore 1962):
  The order dimension of a partial order $(P, \leq)$ is defined as $\mathrm{Dim}(P, \leq)$:= (least κ)[$\exists$ a set $\mathcal{S}$ of linear orders, $\kappa = \#(\mathcal{S})$ and $(P, \leq)$ is embeddable into the product order $\prod \mathcal{S}$].
- Prop: $Q \subset P \implies \mathrm{Dim}(Q, \leq \restriction Q \times Q) \leq \mathrm{Dim}(P, \leq)$.
- Prop: $\mathrm{Dim}(P, \leq) \leq \#P$.
  $\because$ $(P, \leq)$ is embeddable into $\prod_{x \in P} 2$ via F defined as

$$F(y)_x = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{o.w.} \end{cases}$$

Some Properties of Order Dimension

- Prop(Ore 1962):
  $\mathrm{Dim}(P, \leq) = $ (least κ)[$\exists$ a set $\{\leq_i\}_{i\in I}$ of linear extensions of $\leq$, $\kappa = \# I$ and $\leq = \bigcap_{i\in I} \leq_i$].
    - Suppose $\leq = \bigcap_{i\in I} \leq_i$. $\prod_{i\in I} \mathrm{Id}$ embeds $(P, \leq)$ into $\prod_{i\in I}(P, \leq_i)$.
    - Suppose F embeds $(P, \leq)$ into $\prod_{i\in I}(Q_i, \leq_i)$. Then $\leq'_i := \leq \cup \{(x, y) : F(x)_i <_i F(y)_i\}$ は $\leq$ is a partial order on P extending $\leq$. Choose a linear extension $\leq_i$ of $\leq'_i$ for each $i \in I$. We have $\leq = \bigcap_{i\in I} \leq_i$.
- Exm:
    - $(P, \leq)$ is linear $\iff$ $\mathrm{Dim}(P, \leq) = 1$.
    - $\leq = \{(x, x) : x \in P\} \implies \mathrm{Dim}(P, \leq) \leq 2$.
      $\because$ Letting $P = \{x_\alpha\}_{\alpha<\lambda}$, define $\leq_0$, $\leq_1$ as

$$x_\alpha \leq_0 x_\beta :\iff \alpha \leq \beta,$$
$$x_\alpha \leq_1 x_\beta :\iff \alpha \geq \beta.$$

Then $\leq = \leq_0 \cap \leq_1$.

2. Degree Structures $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$, $\mathcal{D}_{\mathrm{Much}}$

$\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$, $\mathcal{D}_{\mathrm{Much}}$

- For $f, g \in \omega^\omega$, $f \leq_{\mathrm{T}} g :\iff \exists$ comp. Φ, $\Phi(g) = f$,
  For $M, N \subset \omega^\omega$, $M \leq_{\mathrm{Med}} N :\iff \exists$ comp. $\Phi : N \to M$,
  $M \leq_{\mathrm{Much}} N :\iff \forall g \in N$, $M \leq_{\mathrm{Med}} \{g\}$.
- Prop. $(\omega^\omega, \leq_{\mathrm{T}})$, $(\mathrm{Pow}(\omega^\omega), \leq_{\mathrm{Med}})$, $(\mathrm{Pow}(\omega^\omega), \leq_{\mathrm{Much}})$ are preorders.
- We use $(\mathcal{D}_{\mathrm{T}}, \leq)$, $(\mathcal{D}_{\mathrm{Med}}, \leq)$, $(\mathcal{D}_{\mathrm{Much}}, \leq)$ to denote the naturally induced partial orders via $(\omega^\omega, \leq_{\mathrm{T}})$, $(\mathrm{Pow}(\omega^\omega), \leq_{\mathrm{Med}})$, $(\mathrm{Pow}(\omega^\omega), \leq_{\mathrm{Much}})$, resp.
  They are called Turing degree structure, Medvedev degree structure and Muchnik degree structure.
- What are $\mathrm{Dim}(\mathcal{D}_{\mathrm{T}}, \leq)$, $\mathrm{Dim}(\mathcal{D}_{\mathrm{Med}}, \leq)$ and $\mathrm{Dim}(\mathcal{D}_{\mathrm{Much}}, \leq)$?
  Before investigating the problem, let us talk on our motivation of this study!

Natural Structures such as $\mathbb{N}$, $\mathbb{R}$, $2^{<\mathbb{N}}$

- It is well-known that some theories concerning on $\mathbb{N}$, $\mathbb{R}$ and $2^{<\mathbb{N}}$ are decidable.
    - Thm(Presburger): $\mathrm{Th}(\mathbb{N}; +, 0, 1)$ is decidable.
    - Thm(Tarski): $\mathrm{Th}(\mathbb{R}; +, \cdot, 0, 1)$ is decidable.
    - Thm(Rabin): $\mathrm{Th}(2^{<\mathbb{N}}, \mathrm{Pow}(2^{<\mathbb{N}});\ ^\frown 0,\ ^\frown 1)$ is decidable.
- For me, next to $\mathbb{N}$, $\mathbb{R}$ and $2^{<\mathbb{N}}$, the objects $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$ and $\mathcal{D}_{\mathrm{Much}}$ are very natural to be studied.
  Do $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$, $\mathcal{D}_{\mathrm{Much}}$ also have some interesting "decidable aspects" such as $\mathbb{N}$, $\mathbb{R}$, $2^{<\mathbb{N}}$?

Theories of Degree Structures

- Let us see some known facts on $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$ and $\mathcal{D}_{\mathrm{Much}}$.
  - Thm(Steve Simpson): $\mathrm{Th}(\mathcal{D}_{\mathrm{T}}; \leq)$ is recursively isomorphic to $\mathrm{Th}(\mathbb{N}, \mathbb{N}^{\mathbb{N}}; +, \cdot, 0, 1)$.
  - Thm(Paul Shafer): $\mathrm{Th}(\mathcal{D}_{\mathrm{Med}}; \leq)$, $\mathrm{Th}(\mathcal{D}_{\mathrm{Much}}; \leq)$ and $\mathrm{Th}(\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}; +, \cdot, 0, 1)$ are mutually recursively isomorphic.
  - Cor of their proofs: $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$ and $\mathcal{D}_{\mathrm{Much}}$ are strongly undecidable structures.
- It seems very difficult to find interesting "decidable aspects" of them by changing their language.
- How about decompose their orders into other partial orders?

Decomposition of Degree Structures

- As we saw, every partial order $(P, \leq)$ is embeddable into $\prod_{x \in P} 2$. Here $2 = \{0, 1\}$ with the natural order is a decidable structure.
- Thus, we can decompose $\mathcal{D}_{\mathrm{T}}$, $\mathcal{D}_{\mathrm{Med}}$ and $\mathcal{D}_{\mathrm{Much}}$ into very, very easy linear orders.
- Question: are there decomposition of these degree structures into "natural" partially orders defined in terms of concepts relating computability, e.g., complexity, the Turing degree of its jump, etc.
- It is interesting if some or all factors of such decompositions has a decidable theory!
- To determine order dimensions, we know how many at least we need *linear* orders if we decompose them into *linear* orders.

3. The order dimensions of degree structures

$\mathrm{Dim}(\mathcal{D}_{\mathrm{Much}},\leq) = 2^{\aleph_0}$

- Thm(Pouzet 1969):
  $\mathrm{Dim}(\mathrm{EndSeg}(P),\subset) =$ the Chain Covering Number of P, where $\mathrm{EndSeg}(P)$ is the set of all end segments of P, and the chain covering number of P is the least cardinal κ s.t. $\exists$ a set $\mathcal{C}$ of chains of P, $\#\mathcal{C} = \kappa$ and $\bigcup\mathcal{C} = P$.
- Prop: $(\mathcal{D}_{\mathrm{Much}},\leq) \simeq (\mathrm{EndSeg}(\mathcal{D}_{\mathrm{T}}),\subset)$.
- Prop: The chain covering number of $\mathcal{D}_{\mathrm{T}}$ is $2^{\aleph_0}$.
  $\because$ At most $2^{\aleph_0}$ since $\#\mathcal{D}_{\mathrm{T}} = 2^{\aleph_0}$.
  At least $2^{\aleph_0}$ since $\mathcal{D}_{\mathrm{T}}$ has an antichain of cardinality $2^{\aleph_0}$.
- Cor: $\mathrm{Dim}(\mathcal{D}_{\mathrm{Much}},\leq) = 2^{\aleph_0}$.

Bounds of $\mathrm{Dim}(\mathcal{D}_{\mathrm{Med}}, \leq)$

- Recall that $\mathrm{Dim}(P, \leq) \leq \#P$. Thus, $\mathrm{Dim}(\mathcal{D}_{\mathrm{Med}}, \leq) \leq 2^{2^{\aleph_0}}$.
- Recall that if $Q \subset P$, then $\mathrm{Dim}(Q, \leq) \leq \mathrm{Dim}(P, \leq)$. Thus, $\mathrm{Dim}(\mathcal{D}_{\mathrm{Much}}, \leq) \leq \mathrm{Dim}(\mathcal{D}_{\mathrm{Med}}, \leq)$.
  $\because M \leq_{\mathrm{Much}} N \iff \mathrm{Up}(M) \leq_{\mathrm{Med}} \mathrm{Up}(N)$, where $\mathrm{Up}(M) := \{g : \exists f \in M, f \leq_{\mathrm{T}} g\}$.
- Cor. $2^{\aleph_0} \leq \mathrm{Dim}(\mathcal{D}_{\mathrm{Med}}, \leq) \leq 2^{2^{\aleph_0}}$.

Bounds of $\mathrm{Dim}(\mathcal{D}_{\mathrm{T}},\leq)$

- Recall that $\mathrm{Dim}(P,\leq) \leq \#P$. Thus, $\mathrm{Dim}(\mathcal{D}_{\mathrm{T}},\leq) \leq 2^{\aleph_0}$.
- Thm: Suppose that a poset P contains an uncountable subset U s.t. $\forall$ cntb $C \subset U$, $\forall x \in U \setminus C$, $\exists$ upper bound $y \in P$ of C, $y \not\geq x$. Then $\mathrm{Dim}(P,\leq) \geq \aleph_1$.
- Thm: $(\mathcal{D}_{\mathrm{T}},\leq)$ satisfies the above property.
  - Fact: $\exists$ $U \subset \mathcal{D}_{\mathrm{T}}$ of the cardinality $\mathfrak{c}$, $\forall$ fin. $F \subset U$, $\forall x \in U \setminus F$, $\sup(F) \not\geq x$.
  - Fact: Suppose $\forall$ ctbl $C \subset \mathcal{D}_{\mathrm{T}}$, $\forall x \in \mathcal{D}_{\mathrm{T}} \setminus C$, $\forall$ fin. $F \subset C$, $\sup(F) \not\geq x$. Then, $\exists y \in \mathcal{D}_{\mathrm{T}}$, y is an upper bound of C and $y \not\geq x$.

Final Comments

- Question: $\mathrm{Dim}_{\leq}(\mathcal{D}_{\mathrm{T}}) = \aleph_1$? $\mathrm{Dim}_{\leq}(\mathcal{D}_{\mathrm{T}}) = 2^{\aleph_0}$?
  What is $\mathrm{Dim}(\mathcal{D}_{\mathrm{Med}}, \leq)$?
  How about other degree structures?
- Since $\mathrm{Dim}(\mathcal{D}_{\mathrm{T}}, \leq) \geq \aleph_1$, I feel it is impossible to find "natural" linear orders s.t. $\mathcal{D}_{\mathrm{T}}$ is embeddable into its product order and it is better to investigate its decomposition into partial orders.