A monad in the combinatory algebras

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Contents

Categories

A category **C** consists of

- a class of objects,
- a class of morphisms; each morphism f has a source object A and a target object B (denoted by $f : A \to B$),
- a composition operator $\circ$ for morphisms $f : A \to B$ and $g : B \to C$ yielding a morphism $g \circ f : A \to C$ such that

  1. if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$,
  2. for each object A there exists a morphism $1_A : A \to A$ such that $1_B \circ f = f \circ 1_A$ for each $f : A \to B$.

Combinatory algebras

A combinatory algebra is a structure $D = \langle \underline{D}, \cdot, \mathtt{k}_D, \mathtt{s}_D \rangle$ with a set $\underline{D}$, a binary operation $\cdot$ on $\underline{D}$ and elements $\mathtt{k}_D$ and $\mathtt{s}_D$ of $\underline{D}$ such that

1. $\neg(\mathtt{k}_D = \mathtt{s}_D)$,
2. $\mathtt{k}_D \cdot x \cdot y = x$,
3. $\mathtt{s}_D \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$.

A homomorphism between combinatory algebras $D = \langle \underline{D}, \cdot, \mathtt{k}_D, \mathtt{s}_D \rangle$ and $D' = \langle \underline{D'}, \cdot', \mathtt{k}_{D'}, \mathtt{s}_{D'} \rangle$ is a mapping $f : \underline{D} \to \underline{D'}$ such that

1. $f(x \cdot y) = f(x) \cdot' f(y)$,
2. $f(\mathtt{k}_D) = \mathtt{k}_{D'}$,
3. $f(\mathtt{s}_D) = \mathtt{s}_{D'}$.

Combinatory algebras

Let $D = \langle \underline{D}, \cdot, \mathtt{k}_D, \mathtt{s}_D \rangle$ be a combinatory algebra, and let

$$\mathtt{i}_D = \mathtt{s}_D \cdot \mathtt{k}_D \cdot \mathtt{k}_D.$$

Then

$$\mathtt{i}_D \cdot x = \mathtt{s}_D \cdot \mathtt{k}_D \cdot \mathtt{k}_D \cdot x = \mathtt{k}_D \cdot x \cdot (\mathtt{k}_D \cdot x) = x.$$

Hence

$$\mathtt{s}_D \cdot (\mathtt{k}_D \cdot a) \cdot \mathtt{i}_D \cdot x = \mathtt{k}_D \cdot a \cdot x \cdot (\mathtt{i}_D \cdot x) = a \cdot x.$$

A category $\mathbf{Calg}^*$ of combinatory algebras

A $*$-homomorphism between combinatory algebtas D and D' is a homomorphism f between D and D' such that for each $a, b \in \underline{D}$ and n if

$$\forall x_1, \ldots, x_n \in \underline{D}(a \cdot x_1 \cdot \ldots \cdot x_n = b \cdot x_1 \cdot \ldots \cdot x_n)$$

then

$$\forall y_1, \ldots, y_n \in \underline{D'}(f(a) \cdot y_1 \cdot \ldots \cdot y_n = f(b) \cdot y_1 \cdot \ldots \cdot y_n).$$

Note that every surjective homomorphism is a $*$-homomorphism.

Let $\mathbf{Calg}^*$ denote the category of combinatory algebras with combinatory algebras as objects and $*$-homomorphisms as morphisms.

Functors

A functor F from a category **C** to a category **D** associates

- to each object A in **C** an object FA in **D**,
- to each morphism $f : A \to B$ in **C** a morphism $Ff : TA \to TB$ in **D**

such that

1. $F1_A = 1_{FA}$ for each object A in **C**,
2. $F(g \circ f) = Fg \circ Ff$ for each $f : A \to B$ and $g : B \to C$ in **C**.

An endofunctor is a functor that maps a category to itself.

The identitiy functor I on a category **C** maps an object to itself and a morphism to itself.
The identitity functor is an endofunctor.

An endofunctor T on $\mathbf{Calg}^*$

Let $D = \langle \underline{D}, \cdot, \mathtt{k}_D, \mathtt{s}_D \rangle$ be a combinatory algebra.

For each $a \in \underline{D}$, define $\mathrm{Fun}_D(a) : \underline{D} \to \underline{D}$ by

$$\mathrm{Fun}_D(a)(x) = a \cdot x.$$

Note that

$$\mathrm{Fun}_D(\mathtt{s}_D \cdot (\mathtt{k}_D \cdot a) \cdot \mathtt{i}_D) = \mathrm{Fun}_D(a).$$

For each $\mathrm{Fun}_D(a)$ and $\mathrm{Fun}_D(b)$, define $\mathrm{Fun}_D(a) \bullet \mathrm{Fun}_D(b)$ by

$$\mathrm{Fun}_D(a) \bullet \mathrm{Fun}_D(b) = \mathrm{Fun}_D(\mathtt{s}_D \cdot a \cdot b).$$

Then $\mathrm{Fun}_D(a) = \mathrm{Fun}_D(a')$ and $\mathrm{Fun}_D(b) = \mathrm{Fun}_D(b')$ imply

$$\mathrm{Fun}_D(a) \bullet \mathrm{Fun}_D(b) = \mathrm{Fun}_D(a') \bullet \mathrm{Fun}_D(b').$$

An endofunctor T on **Calg***

Hence $\bullet$ is a binary operation on

$$\underline{TD} = \{\mathrm{Fun}_D(a) \mid a \in \underline{D}\}.$$

Let

$$\mathtt{k}_{TD} = \mathrm{Fun}_D(\mathtt{k}_D \cdot \mathtt{k}_D), \quad \mathtt{s}_{TD} = \mathrm{Fun}_D(\mathtt{k}_D \cdot \mathtt{s}_D),$$

Then

1. $\neg(\mathtt{k}_{TD} = \mathtt{s}_{TD})$,
2. $\mathtt{k}_{TD} \bullet u \bullet v = u$,
3. $\mathtt{s}_{TD} \bullet u \bullet v \bullet w = u \bullet w \bullet (v \bullet w)$.

Therefore

$$TD = \langle \underline{TD}, \bullet, \mathtt{k}_{TD}, \mathtt{s}_{TD} \rangle$$

is a combinatory algebra.

An endofunctor T on $\mathbf{Calg}^*$

For each $*$-homomorphism $f : D \to D'$, let

$$(Tf)(\mathrm{Fun}_D(a)) = \mathrm{Fun}_{D'}(f(a)).$$

Then, since f is a $*$-homomorphism, we have

$$\mathrm{Fun}_D(a) = \mathrm{Fun}_D(a') \Rightarrow \mathrm{Fun}_{D'}(f(a)) = \mathrm{Fun}_{D'}(f(a')).$$

Hence Tf is a mapping from $\underline{TD}$ into $\underline{TD'}$.

Moreover

1. $(Tf)(u \bullet v) = (Tf)(u) \bullet' (Tf)(v)$,
2. $(Tf)(\mathtt{k}_{TD}) = \mathtt{k}_{TD'}$,
3. $(Tf)(\mathtt{s}_{TD}) = \mathtt{s}_{TD'}$.

Therefore Tf is a homomorphism from TD to TD'.

An endofunctor T on $\mathbf{Calg}^*$

Lemma

Let D be a combinatory algebra. Then for each $a, b \in \underline{D}$ and n

$$\forall u_1, \ldots, u_n \in \underline{TD}(\mathrm{Fun}_D(a) \bullet u_1 \bullet \ldots \bullet u_n = \mathrm{Fun}_D(b) \bullet u_1 \bullet \ldots \bullet u_n)$$

if and only if

$$\forall x_1, \ldots, x_{n+1} \in \underline{D}(a \cdot x_1 \cdot \ldots \cdot x_{n+1} = b \cdot x_1 \cdot \ldots \cdot x_{n+1}).$$

Proof.

Note that

$$(\mathrm{Fun}_D(a) \bullet \mathrm{Fun}_D(\mathtt{k}_D \cdot x_1) \bullet \ldots \bullet \mathrm{Fun}_D(\mathtt{k}_D \cdot x_n))(z) = a \cdot z \cdot x_1 \cdot \ldots \cdot x_n,$$

$$a \cdot z \cdot (x_1 \cdot z) \cdot \ldots \cdot (x_n \cdot z) = (\mathrm{Fun}_D(a) \bullet \mathrm{Fun}_D(x_1) \bullet \ldots \bullet \mathrm{Fun}_D(x_n))(z).$$

□

An endofunctor T on $\mathbf{Calg}^*$

Proposition

Tf is a $*$-homomorphism.

Proof.

Suppose that $\mathrm{Fun}_D(a) \bullet u_1 \bullet \ldots \bullet u_n = \mathrm{Fun}_D(b) \bullet u_1 \bullet \ldots \bullet u_n$ for each $u_1, \ldots, u_n \in \underline{TD}$. Then, by Lemma, we have

$$a \cdot x_1 \cdot \ldots \cdot x_{n+1} = b \cdot x_1 \cdot \ldots \cdot x_{n+1}$$

for each $x_1, \ldots, x_{n+1} \in \underline{D}$, and hence

$$f(a) \cdot y_1 \cdot \ldots \cdot y_{n+1} = f(b) \cdot y_1 \cdot \ldots \cdot y_{n+1}$$

for each $y_1, \ldots, y_{n+1} \in \underline{D'}$. Thus, again by Lemma, we have

$$\mathrm{Fun}_{D'}(f(a)) \bullet v_1 \bullet \ldots \bullet v_n = \mathrm{Fun}_{D'}(f(b)) \bullet v_1 \bullet \ldots \bullet v_n$$

for each $v_1, \ldots, v_n \in \underline{TD'}$. □

Natural transformations

Let F and G be functors between categories $\mathbf{C}$ and $\mathbf{D}$.

A natural transformation τ from F to G, denoted by $\tau : F \stackrel{\cdot}{\to} G$, associates to each object A in $\mathbf{C}$ a morphism $\tau_A : FA \to GA$ such that for each morphism $f : A \to B$ in $\mathbf{C}$ the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\tau_B]{} & GB \end{array}$$

A natural transformation $\eta : I \dot{\to} T$

For each combinatory algebra D, define a mapping $\eta_D : \underline{D} \to \underline{TD}$ by

$$\eta_D(x) = \mathrm{Fun}_D(\mathtt{k}_D \cdot x).$$

Then

1. $\eta_D(x \cdot y) = \eta_D(x) \bullet \eta_D(y)$,
2. $\eta_D(\mathtt{k}_D) = \mathtt{k}_{TD}$,
3. $\eta_D(\mathtt{s}_D) = \mathtt{s}_{TD}$.

Hence η_D is a homomorphism from D to TD.

A natural transformation $\eta : I \stackrel{\cdot}{\to} T$

Proposition

η_D is a $*$-homomorphism.

Proof.

Note that

$$a \cdot (x_1 \cdot z) \cdot \ldots \cdot (x_n \cdot z) = (\mathtt{k}_D \cdot a \cdot z) \cdot (x_1 \cdot z) \cdot \ldots \cdot (x_n \cdot z)$$
$$= (\mathrm{Fun}_D(\mathtt{k}_D \cdot a) \bullet \mathrm{Fun}_D(x_1) \bullet \ldots \bullet \mathrm{Fun}_D(x_n))(z)$$
$$= (\eta_D(a) \bullet \mathrm{Fun}_D(x_1) \bullet \ldots \bullet \mathrm{Fun}_D(x_n))(z).$$

$\square$

A natural transformation $\eta : I \stackrel{\cdot}{\to} T$

Proposition

The following diagram commutes:

$$\begin{array}{ccc} D & \xrightarrow{\eta_D} & TD \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ D' & \xrightarrow[\eta_{D'}]{} & TD' \end{array}$$

Proof.

$$\begin{aligned} (Tf \circ \eta_D)(x) &= Tf(\mathrm{Fun}_D(\mathtt{k}_D \cdot x)) = \mathrm{Fun}_{D'}(f(\mathtt{k}_D \cdot x)) \\ &= \mathrm{Fun}_{D'}(f(\mathtt{k}_D) \cdot' f(x)) = \mathrm{Fun}_{D'}(\mathtt{k}_{D'} \cdot' f(x)) \\ &= (\eta_{D'} \circ f)(x). \end{aligned}$$

$\square$

A natural transformation $\mu : T^2 \dot{\rightarrow} T$

For each combinatory algebra D and $c \in \underline{D}$, define a mapping $\theta_D^c : \underline{TD} \rightarrow \underline{D}$ by

$$\theta_D^c(\mathrm{Fun}_D(x)) = x \cdot c.$$

Then

1. $\theta_D^c(u \bullet v) = \theta_D^c(u) \cdot \theta_D^c(v)$,
2. $\theta_D^c(\mathtt{k}_{TD}) = \mathtt{k}_D$,
3. $\theta_D^c(\mathtt{s}_{TD}) = \mathtt{s}_D$.

Hence θ_D^c is a homomorphism from TD to D.

Let

$$\mathtt{id}_D = \mathrm{Fun}_D(\mathtt{i}_D).$$

Then

$$\theta_D^c(\mathtt{id}_D) = c.$$

A natural transformation $\mu : T^2 \xrightarrow{\cdot} T$

Lemma

$\theta_D^c \circ \eta_D = 1_D$.

Proof.

$$(\theta_D^C \circ \eta_D)(x) = \theta_D^c(\mathrm{Fun}_D(\mathtt{k}_D \cdot x)) = \mathtt{k}_D \cdot x \cdot c = x.$$

□

Proposition

θ_D^c *is a* $*$*-homomorphism.*

Proof.

Note that θ_D^c is a surjective homomorphism. □

A natural transformation $\mu : T^2 \stackrel{\cdot}{\to} T$

For each combinatory algebra D, define $*$-homomorphism $\mu_D : T^2 D \to TD$ by

$$\mu_D = \theta_{TD}^{\mathtt{id}_D}.$$

Lemma

$Tf(\mathtt{id}_D) = \mathtt{id}_{D'}$ for each $*$-homomorphism $f : D \to D'$.

Proof.

$$Tf(\mathtt{id}_D) = Tf(\mathrm{Fun}_D(\mathtt{i}_D)) = \mathrm{Fun}_{D'}(f(\mathtt{i}_D)) = \mathrm{Fun}_{D'}(\mathtt{i}_{D'}) = \mathtt{id}_{D'}.$$

□

A natural transformation $\mu : T^2 \stackrel{\cdot}{\to} T$

Proposition

The following diagram commutes:

$$\begin{array}{ccc} T^2 D & \xrightarrow{\mu_D} & TD \\ {\scriptstyle T^2 f}\downarrow & & \downarrow{\scriptstyle Tf} \\ T^2 D' & \xrightarrow[\mu_{D'}]{} & TD' \end{array}$$

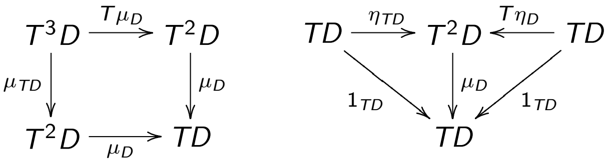
Proof.

$$(\mu_{D'} \circ T^2 f)(\mathrm{Fun}_{TD}(u)) = \mu_{D'}(T^2 f(\mathrm{Fun}_{TD}(u)))$$
$$= \mu_{D'}(\mathrm{Fun}_{TD'}(Tf(u))) = Tf(u) \bullet' \mathtt{id}_{D'} = Tf(u) \bullet' Tf(\mathtt{id}_D)$$
$$= Tf(u \bullet \mathtt{id}_D) = Tf(\mu_D(\mathrm{Fun}_{TD}(u))) = (Tf \circ \mu_D)(\mathrm{Fun}_{TD}(u)).$$

□

Monads

A monad $\langle T, \eta, \mu \rangle$ in a category $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\eta : I \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ such that the following diagrams commute:



A monad $\langle T, \eta, \mu \rangle$ in **Calg***

Theorem

$\langle T, \eta, \mu \rangle$ is a monad in **Calg****.*

Proof.

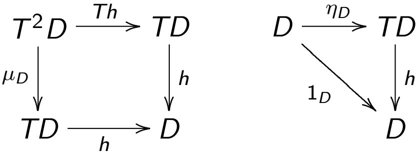
$$(\mu_D \circ T\eta_D)(\mathrm{Fun}_D(x)) = \mu_D(T\eta_D(\mathrm{Fun}_D(x)))$$
$$= \mu_D(\mathrm{Fun}_{TD}(\eta_D(x))) = \eta_D(x) \bullet \mathtt{id}_D = \mathrm{Fun}_D(\mathtt{k}_D \cdot x) \bullet \mathrm{Fun}_D(\mathtt{i}_D)$$
$$= \mathrm{Fun}_D(\mathtt{s}_D \cdot (\mathtt{k}_D \cdot x) \cdot \mathtt{i}_D) = \mathrm{Fun}_D(x).$$

$$(\mu_D \circ T\mu_D)(\mathrm{Fun}_{T^2D}(\mathrm{Fun}_{TD}(u))) = \cdots$$
$$= u \bullet \mathtt{id}_D \bullet \mathtt{id}_D = \cdots$$
$$= (\mu_D \circ \mu_{TD})(\mathrm{Fun}_{T^2D}(\mathrm{Fun}_{TD}(u))).$$

□

T-algebras

A T-algebra $\langle D, h \rangle$ is a pair consisting of an object D (the underlying object of the algebra) and an arrow $h : TD \to D$ such that the following diagrams commute:



A T-algebras $\langle D, \theta^c_D \rangle$

Proposition

$\langle D, \theta^c_D \rangle$ is a T-algebra for each $c \in \underline{D}$.

Proof.

$$
\begin{aligned}
&(\theta^c_D \circ T\theta^c_D)(\mathrm{Fun}_{TD}(u)) = \theta^c_D(T\theta^c_D(\mathrm{Fun}_D(u))) \\
&= \theta^c_D(\mathrm{Fun}_D(\theta^c_D(u))) = \theta^c_D(u) \cdot c = \theta^c_D(u) \cdot \theta^c_D(\mathtt{id}_D) \\
&= \theta^c_D(u \bullet \mathtt{id}_D) = \theta^c_D(\mu_D(\mathrm{Fun}_{TD}(u))) = (\theta^c_D \circ \mu_D)(\mathrm{Fun}_{TD}(u)).
\end{aligned}
$$

□

A bijection

Theorem

For any object D in $\mathbf{Calg}^*$, there exists a bijection between $\underline{D}$ and the set of T-algebras with the underlying object D.

Proof.

Let

$$\varphi : c \mapsto \theta^c_D, \quad \psi : h \mapsto h(\mathtt{id}_D).$$

Then $(\psi \circ \varphi)(c) = \theta^c_D(\mathtt{id}_D) = c$. Furthermore

$$\begin{aligned} &((\varphi \circ \psi)(h))(\mathrm{Fun}_D(x)) = \theta_D^{h(\mathtt{id}_D)}(\mathrm{Fun}_D(x)) = x \cdot h(\mathtt{id}_D) \\ &= h(\eta_D(x)) \cdot h(\mathtt{id}_D) = h(\eta_D(x) \bullet \mathtt{id}_D) \\ &= h(\mu_D(\mathrm{Fun}_{TD}(\eta_D(x)))) = h(\mu_D(T\eta_D(\mathrm{Fun}_D(x)))) \\ &= h((\mu_D \circ T\eta_D)(\mathrm{Fun}_D(x))) = h(\mathrm{Fun}_D(x)). \end{aligned}$$

□