Effective Methods in Descriptive Set Theory

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Effective descriptive set theory is not only a refinement of classical descriptive set theory, but also a powerful method able to solve problems of classical type.

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Theme of this talk

Let's apply "*effective*" theory to classical mathematics!

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Let's apply "*effective*" theory to classical mathematics!

Concretely speaking...

By using Computability Theory,
we solve a problem in descriptive set theory proposed by Andretta [1], Semmes [2], Pawlikowski-Sabok [3] and Motto Ros [4].

[1] A. Andretta, *The* **SLO** *principle and the Wadge hierarchy*.

[2] B. Semmes, *A Game for the Borel Functions*.

[3] J. Pawlikowski and M. Sabok, *Decomposing Borel functions and structure at finite levels of the Baire hierarchy*.

[4] L. Motto Ros, *On the structure of finite levels and ω-decomposable Borel functions*.

Main Tools

1. Louveau's separation theorem [5]
2. the Shore-Slaman join theorem [6]

[5] A. Louveau, *A separation theorem for $\mathbf{\Sigma}^1_1$ sets*, **Trans. Amer. Math. Soc.**, **260**, 363–378, 1980.

[6] R. A. Shore and T. A. Slaman, *Defining the Turing jump*, **Math. Res. Lett.**, **6** 711–722, 1999.

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   "*the notion of Borel class is, roughly speaking, an effective notion*" — A. Louveau [5]

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1. Louveau's separation theorem [5]

   "*the notion of Borel class is, roughly speaking, an effective notion*" — A. Louveau [5]

2. the Shore-Slaman join theorem [6]
   a transfinite version of the *Posner-Robinson Join Theorem*, and proved by using *Kumabe-Slaman forcing*. By combining this theorem with the *Slaman-Woodin double jump definability theorem* (obtained from the Slaman and Woodin analysis of automorphisms of the Turing degrees), Shore and Slaman showed that *the Turing jump is definable in $\mathcal{D}_T$*.

[5] A. Louveau, *A separation theorem for $\mathbf{\Sigma}^1_1$ sets*, **Trans. Amer. Math. Soc.**, **260**, 363–378, 1980.

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Some Classical Examples i

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Example

Bourgain [7] considered the following situation: let $(X, \mathcal{M}, \mu)$ be a complete probability space and Y a Polish space. Let $f : X \times Y \to \mathbb{R}$ be a uniformly bounded function such that $x \mapsto f(x, y)$ is $\mathcal{M}$-measurable and $y \mapsto f(x, y)$ is of Baire $\mathbf{1}$. Then $y \mapsto \int_X f(x, y) d\mu(x)$ is of Baire $\mathbf{1}$.

The Baire $\mathbf{2}$ version of this property is false under **CH**, while the Baire $\mathbf{2}$ version can be true if we assume the $\mathcal{M} \otimes \mathcal{B}_Y$-measurability of f.

Louveau [5] used the topology generated by *lightface* Σ^1_1 sets to extend the latter version to any Baire rank (with additional restrictions to spaces.)

[5] A. Louveau, *A separation theorem for $\mathbf{\Sigma}^1_1$ sets*, **Trans. Amer. Math. Soc.**

[7] J. Bourgain, *Decomposition in the product of a measure space and a Polish space*, **Fund. Math.**

Some Classical Examples ii

> *Despite the totally classical descriptive set-theoretic nature of our result, our proof requires the employment of methods of effective descriptive set theory and thus ultimately makes crucial use of computability (or recursion) theory on the integers.*
>
> — Harrington-Kechris-Louveau [8]

Some Classical Examples ii

Despite the totally classical descriptive set-theoretic nature of our result, our proof requires the employment of methods of effective descriptive set theory and thus ultimately makes crucial use of computability (or recursion) theory on the integers.

— Harrington-Kechris-Louveau [8]

Example

In the context of operator algebra, Grimm [9] and Effros [10] showed dichotomy for locally compact group actions and F_σ orbit equivalence relations. Harrington-Kechris-Louveau [8] used the topology generated by *lightface* Σ^1_1 sets to extend the Grimm-Effros dichotomy to any Polish equivalence relations: for every Borel equivalence relation E on a Polish space, either it is smooth, or else $E_0 \sqsubseteq E$.

[8] L. A. Harrington, and A. S. Kechris and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, **J. Amer. Math. Soc., 3**.

[9] J. Glimm, *Type I C^*-algebras*, **Ann. Math. 73**.

[10] E. G. Effros, *Transformation groups and C^*-algebras*, **Ann. Math. 81**.

Other Examples

Several Other Examples

1. Harrington's alternative proof of Silver's theorem
   - by Gandy-Harrington topology.
2. The Friedman-Stanley theorem: For every prime p, the isomorphism relation on abelian p-groups is complete analytic.
   - by Harrison ordering.
3. Some applications of effective descriptive set theory to Banach space theory (G. Debs, V. Gregoriades, and others)

By employing Louveau's separation theorem and the Shore-Slaman join theorem, we will show the following:

Main Theorem (Gregoriades-K.)

Let X, Y be finite dimensional Polish spaces, and α and β be countable ordinals with $\alpha \leq \beta < \alpha \cdot 2$. Then, the following are equivalent for $f : X \to Y$:

1. If $A \subseteq Y$ is $\utilde{\Sigma}^0_{\alpha+1}$, $f^{-1}[A] \subseteq X$ is $\utilde{\Sigma}^0_{\beta+1}$.
2. There exists a $\utilde{\Pi}^0_\beta$ partition $\{X_i\}_{i\in\omega}$ of X such that for every i, the restriction $f|_{X_i}$ is of Baire γ with $\gamma + \alpha \leq \beta$.

Decomposing a hard function F into easy functions

Decomposing a discontinuous function F into easy functions

Decomposing a discontinuous function F into continuous functions

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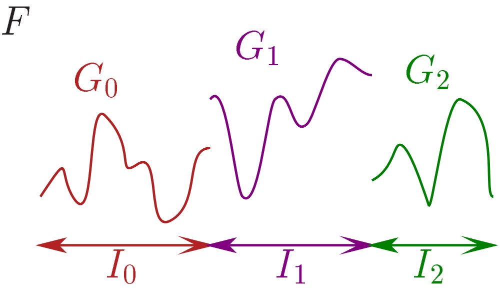
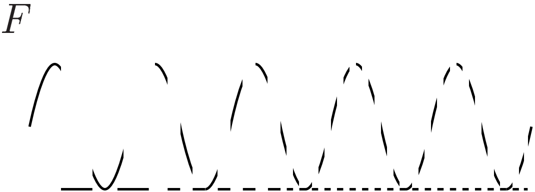


$$F(x) = \begin{cases} G_0(x) & \text{if } x \in I_0 \\ G_1(x) & \text{if } x \in I_1 \\ G_2(x) & \text{if } x \in I_2 \end{cases}$$

Decomposing a discontinuous function into continuous functions



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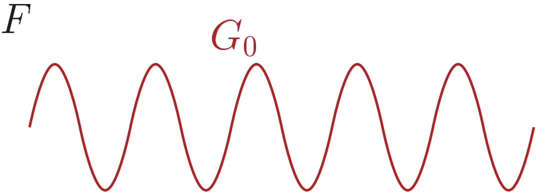

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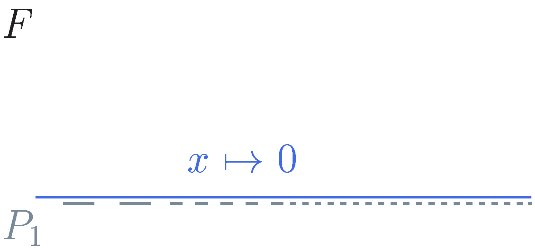

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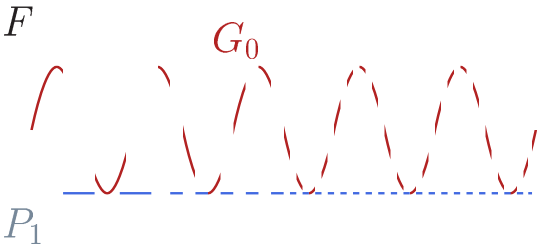

$$F(x) = \begin{cases} G_0(x) & \text{if } x \notin P_1 \\ 0 & \text{if } x \in P_1 \end{cases}$$

Decomposing a discontinuous function into continuous functions

$$\mathrm{Dirichlet}(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x)$$

$$\Downarrow$$

$$\mathrm{Dirichlet}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}. \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

If F is a Borel measurable function on $\mathbb{R}$, then can it be presented by using a countable partition $\{P_n\}_{n\in\omega}$ of $\mathbf{dom}(F)$ and a countable list $\{G_n\}_{n\in\omega}$ of continuous functions as follows?

$$F(x) = \begin{cases} G_0(x) & \text{if } x \in P_0 \\ G_1(x) & \text{if } x \in P_1 \\ G_2(x) & \text{if } x \in P_2 \\ G_3(x) & \text{if } x \in P_3 \\ \vdots & \vdots \end{cases}$$

Luzin's Problem (almost 100 years ago)

Can every Borel function on $\mathbb{R}$ be decomposed into countably many continuous functions?

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- An indecomposable Baire **1** function exists!
- Indeed, for every α there is a Baire α function which is not decomposable into countably many Baire $< \alpha$ functions!

Luzin's Problem (almost 100 years ago)

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- An indecomposable Baire **1** function exists!
- Indeed, for every α there is a Baire α function which is not decomposable into countably many Baire $< \alpha$ functions!

In other words,

Theorem (Keldysh 1934)

Let α be a countable ordinal. There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

- If $A \subseteq \mathbb{R}$ is open, $f^{-1}[A] \subseteq \mathbb{R}$ is $\boldsymbol{\Sigma}^0_{\alpha+1}$.
- There exists NO countable partition $\{X_i\}_{i \in \mathbb{N}}$ of $\mathbb{R}^c$ such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is Baire $< \alpha$.

Theorem (Keldysh 1934; $\alpha = 1$)

Let α be a countable ordinal. There exists $f : \mathbb{R} \to \mathbb{R}$ satisfying:

- If $A \subseteq \mathbb{R}$ is open, $f^{-1}[A] \subseteq \mathbb{R}$ is F_σ.
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Theorem (Jayne-Rogers 1982 [11])

X: analytic, Y: separable metrizable.
The following are equivalent for $f : X \to Y$:

1. If $A \subseteq Y$ is F_σ, $f^{-1}[A] \subseteq X$ is F_σ.
2. There exists a closed partition $\{X_i\}_{i \in \mathbb{N}}$ of X such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is continuous.

[11] J. E. Jayne and C. A. Rogers, First level Borel functions and isomorphism, *J. Math. Pure Appl.* (1982).

$\underset{\sim}{\Sigma}_{2,2} = \mathrm{dec}(\underset{\sim}{\Pi}_1; \mathcal{B}_0)$

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Definition

Let $f : X \to Y$ be a function.

1. f is $\underset{\sim}{\Sigma}_{\alpha,\beta}$ if "$A \subseteq Y$ is $\underset{\sim}{\Sigma}^0_\alpha$" implies "$f^{-1}[A]$ is $\underset{\sim}{\Sigma}^0_\beta$".
2. $f \in \mathrm{dec}(\underset{\sim}{\Pi}_\beta; \mathcal{B}_\alpha)$ if there exists a $\underset{\sim}{\Pi}^0_\beta$ partition $\{X_i\}_{i\in\mathbb{N}}$ of X such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is of Baire α.

Remark

1. A function $F : X \to Y$ is Borel if
$$A \in \bigcup_{\alpha<\omega_1} \underset{\sim}{\Sigma}^0_\alpha(Y) \implies F^{-1}[A] \in \bigcup_{\alpha<\omega_1} \underset{\sim}{\Sigma}^0_\alpha(X).$$
2. A function $F : X \to Y$ is $\underset{\sim}{\Sigma}^0_{\alpha+1}$-measurable (Baire α) if
$$A \in \underset{\sim}{\Sigma}^0_1(Y) \implies F^{-1}[A] \in \underset{\sim}{\Sigma}^0_{\alpha+1}(X).$$
3. A function $F : X \to Y$ is $\underset{\sim}{\Sigma}_{\alpha,\beta}$ if
$$A \in \underset{\sim}{\Sigma}^0_\alpha(Y) \implies F^{-1}[A] \in \underset{\sim}{\Sigma}^0_\beta(X).$$

A function $F : X \to Y$ is $\underset{\sim}{\boldsymbol{\Sigma}}_{\alpha,\beta}$ if

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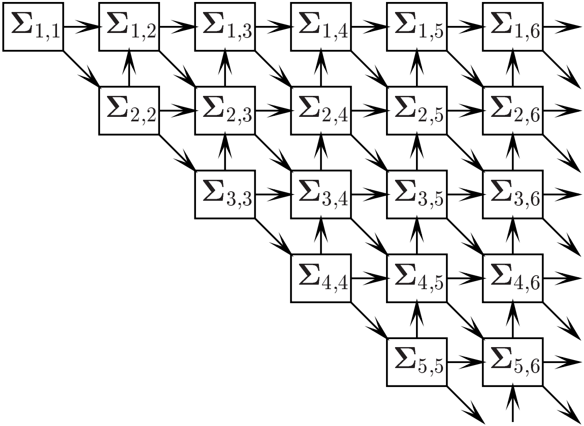
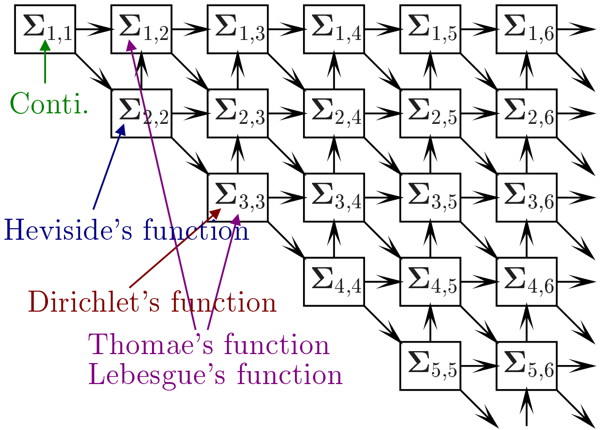


A function $\boldsymbol{F} : \boldsymbol{X} \to \boldsymbol{Y}$ is $\underset{\sim}{\boldsymbol{\Sigma}}_{\alpha,\beta}$ if

$$\boldsymbol{A} \in \underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha(\boldsymbol{Y}) \implies \boldsymbol{F}^{-1}[\boldsymbol{A}] \in \underset{\sim}{\boldsymbol{\Sigma}}^0_\beta(\boldsymbol{X}).$$



A function $F : X \to Y$ is $\utilde{\Sigma}_{\alpha,\beta}$ if

$$A \in \utilde{\Sigma}^0_\alpha(Y) \implies F^{-1}[A] \in \utilde{\Sigma}^0_\beta(X).$$

Remark

- This notion is essentially introduced in Kratowski's book "*Topology I*".
- Jayne (1974) gave a classification of Polish spaces under $\utilde{\Sigma}_{2,2}$-isomorphisms (it has been called the first-level Borel isomorphisms).
- For instance, Jayne (1974) used this notion to show that for realcompact spaces X and Y, X is $\utilde{\Sigma}_{\alpha,\beta}$-isomorphic to Y if and only if the space $\mathbf{B}^*_\alpha(X)$ of bounded Baire α functions on X is linearly isometric to $\mathbf{B}^*_\alpha(Y)$

Borel Functions and Decomposability

	1	2	3	4	5	6
1	$\mathcal{B}_0$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$
2	–	$\Pi_1, \mathcal{B}_0$	?	?	?	?
3	–	–	?	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
6	–	–	–	–	–	?

The Jayne-Rogers Theorem 1982

$\boldsymbol{X}, \boldsymbol{Y}$: metric separable, $\boldsymbol{X}$: analytic

For the class of all functions from $\boldsymbol{X}$ into $\boldsymbol{Y}$,

$$\boxed{\underset{\sim}{\boldsymbol{\Sigma}}_{2,2}} = \boxed{\mathrm{dec}(\underset{\sim}{\boldsymbol{\Pi}}_1; \mathcal{B}_0)}$$

Theorem (Semmes 2009 [2])

X, Y: zero-dimensional Polish spaces. The following are equivalent for $f : X \to Y$:

1. If $A \subseteq Y$ is $G_{\sigma\delta}$, $f^{-1}[A] \subseteq X$ is $G_{\sigma\delta}$.
2. There exists a G_δ partition $\{X_i\}_{i\in\mathbb{N}}$ of X such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is continuous.

[2] B. Semmes, A Game for the Borel Functions, PhD. thesis, 2009.

The second level decomposability of Borel functions

	1	2	3	4	5	6
1	$\mathcal{B}_0$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$
2	–	$\mathbf{\Pi}_1, \mathcal{B}_0$	?	?	?	?
3	–	–	$\mathbf{\Pi}_2, \mathcal{B}_0$	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
6	–	–	–	–	–	?

Theorem (Semmes 2009)

For the class of functions on a zero dim. Polish space,

$$\boxed{\underset{\sim}{\mathbf{\Sigma}}_{3,3}} = \boxed{\mathbf{dec}(\underset{\sim}{\mathbf{\Pi}}_2; \mathcal{B}_0)}$$

Theorem (Semmes 2009 [2])

X, Y: zero-dimensional Polish spaces. The following are equivalent for $f : X \to Y$:

1. If $A \subseteq Y$ is F_σ, $f^{-1}[A] \subseteq X$ is $G_{\sigma\delta}$.
2. There exists a G_δ partition $\{X_i\}_{i\in\mathbb{N}}$ of X such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is Baire 1.

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The second level decomposability of Borel functions

	1	2	3	4	5	6
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2	–	$\boldsymbol{\Pi}_1, \mathcal{B}_0$	$\boldsymbol{\Pi}_2, \mathcal{B}_1$	?	?	?
3	–	–	$\boldsymbol{\Pi}_2, \mathcal{B}_0$	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
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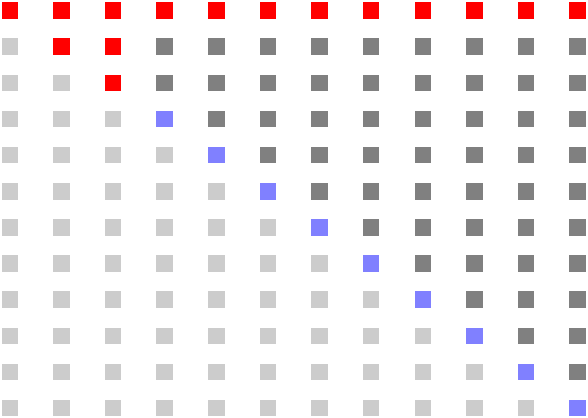
The Decomposability Problem

	1	2	3	4	5	6
1	$\mathcal{B}_0$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$
2	–	$\Pi_1, \mathcal{B}_0$	$\Pi_2, \mathcal{B}_1$	$\Pi_3, \mathcal{B}_2$	$\Pi_4, \mathcal{B}_3$	$\Pi_5, \mathcal{B}_4$
3	–	–	$\Pi_2, \mathcal{B}_0$	$\Pi_3, \mathcal{B}_1$	$\Pi_4, \mathcal{B}_2$	$\Pi_5, \mathcal{B}_3$
4	–	–	–	$\Pi_3, \mathcal{B}_0$	$\Pi_4, \mathcal{B}_1$	$\Pi_5, \mathcal{B}_2$
5	–	–	–	–	$\Pi_4, \mathcal{B}_0$	$\Pi_5, \mathcal{B}_1$
6	–	–	–	–	–	$\Pi_5, \mathcal{B}_0$

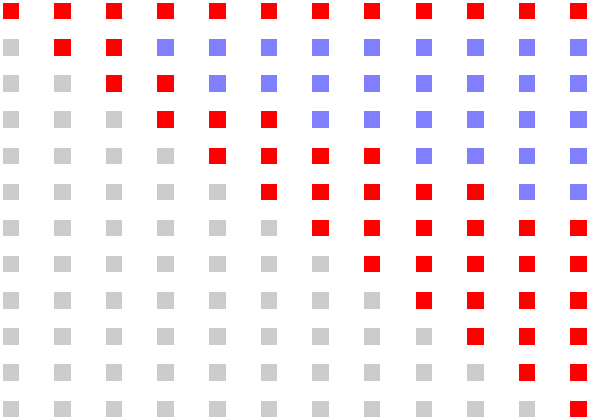
Decomposability Problem (Andretta, Motto Ros et al.)

$$\underset{\sim}{\Sigma}_{m+1,n+1} = \mathrm{dec}(\underset{\sim}{\Pi}_n; \mathcal{B}_{n-m})?$$

Overview of Previous Research



Main Theorem



The decomposability of Borel functions

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2	–	$\Pi_1, \mathcal{B}_1$	$\Pi_2, \mathcal{B}_2$	?	?	?
3	–	–	$\Pi_2, \mathcal{B}_1$	$\Pi_3, \mathcal{B}_2$	?	?
4	–	–	–	$\Pi_3, \mathcal{B}_1$	$\Pi_4, \mathcal{B}_2$	$\Pi_5, \mathcal{B}_3$
5	–	–	–	–	$\Pi_4, \mathcal{B}_1$	$\Pi_5, \mathcal{B}_2$
6	–	–	–	–	–	$\Pi_5, \mathcal{B}_1$

Main Theorem (Gregoriades-K.)

If $2 \leq m \leq n < 2m$ then

$$\boxed{\underset{\sim}{\Sigma}_{m+1,n+1}} = \boxed{\mathrm{dec}(\underset{\sim}{\Pi}_n; \mathcal{B}_{n-m})}$$

Question ([1,2,3,4])

X, Y: Polish spaces. Are the following equivalent for $f : X \to Y$?

1. If $A \subseteq Y$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{m+1}$, $f^{-1}[A] \subseteq X$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{n+1}$.
2. There exists a $\underset{\sim}{\boldsymbol{\Pi}}^0_n$ partition $\{X_i\}_{i\in\mathbb{N}}$ of X such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is Baire $n - m$.

Main Theorem (Gregoriades-K. 201x)

Let X, Y be finite dimensional Polish spaces, and $\alpha \leq \beta < \alpha \cdot 2$. Then, the following are equivalent for $f : X \to Y$:

1. If $A \subseteq Y$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\alpha+1}$, $f^{-1}[A] \subseteq X$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{\beta+1}$.
2. There exists a $\underset{\sim}{\boldsymbol{\Pi}}^0_\beta$ partition $\{X_i\}_{i\in\omega}$ of X such that for every i, the restriction $f|_{X_i}$ is of Baire γ with $\gamma + \alpha \leq \beta$.

Louveau's Theorem (1980)

If a $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ set $A \subseteq \omega^\omega$ has a hyperarithmetical Borel code then A also has a hyperarithmetical $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$-code.

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Remark

Moreover, one can find such a code in a uniform way:
there exists a Borel measurable function h such that
if c is a Borel code of a $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ set A, then $h(c)$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$-code of A.

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Moreover, one can find such a code in a uniform way: there exists a Borel measurable function h such that if c is a Borel code of a $\undertilde{\Sigma}^0_\xi$ set A, then $h(c)$ is a $\undertilde{\Sigma}^0_\xi$-code of A.

As a corollary:

Borel-Uniformization Lemma (G.-K.)

Assume that $f^{-1}\undertilde{\Sigma}^0_{\alpha+1} \subseteq \undertilde{\Sigma}^0_{\beta+1}$ (equivalently, $f^{-1}\undertilde{\Sigma}^0_{\alpha} \subseteq \undertilde{\Delta}^0_{\beta+1}$). Then, there exists a Borel measurable function h such that if c is a $\undertilde{\Sigma}^0_\alpha$-code of A, then $h(c)$ is a $\undertilde{\Delta}^0_{\beta+1}$-code of $f^{-1}[A]$.

Borel-Uniformization Lemma (G.-K.)

Assume that $f^{-1}\underset{\sim}{\Sigma}^0_{\alpha+1} \subseteq \underset{\sim}{\Sigma}^0_{\beta+1}$ (equivalently, $f^{-1}\underset{\sim}{\Sigma}^0_{\alpha} \subseteq \underset{\sim}{\Delta}^0_{\beta+1}$).
Then, there exists a Borel measurable function $h : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that
if c is a $\underset{\sim}{\Sigma}^0_{\alpha}$-code of A, then $h(c)$ is a $\underset{\sim}{\Delta}^0_{\beta+1}$-code of $f^{-1}[A]$.

Then, we can extract a degree-theoretic content from Borel-Uniformization Lemma as follows:

Borel-Uniformization Lemma (G.-K.)

Assume that $f^{-1}\boldsymbol{\Sigma}^0_{\alpha+1} \subseteq \boldsymbol{\Sigma}^0_{\beta+1}$ (equivalently, $f^{-1}\boldsymbol{\Sigma}^0_{\alpha} \subseteq \boldsymbol{\Delta}^0_{\beta+1}$).
Then, there exists a Borel measurable function $h : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that
if c is a $\boldsymbol{\Sigma}^0_{\alpha}$-code of A, then $h(c)$ is a $\boldsymbol{\Delta}^0_{\beta+1}$-code of $f^{-1}[A]$.

Then, we can extract a degree-theoretic content from Borel-Uniformization Lemma as follows:

Key Lemma (G.-K.)

Assume that $f^{-1}\boldsymbol{\Sigma}^0_{\alpha+1} \subseteq \boldsymbol{\Sigma}^0_{\beta+1}$. Then,

$$(\exists p)(\exists \xi < \omega_1^p)(\forall x, z)\ (f(x) \oplus z)^{(\alpha)} \leq_T (x \oplus (z \oplus p)^{(\xi)})^{(\beta)}.$$

We have: $(\forall x, z)\ (f(x) \oplus z)^{(\alpha)} \leq_T (x \oplus (z \oplus p)^{(\xi)})^{(\beta)}$

The following lemma is the heart of our proof.

Cancellation Lemma (G.-K.)

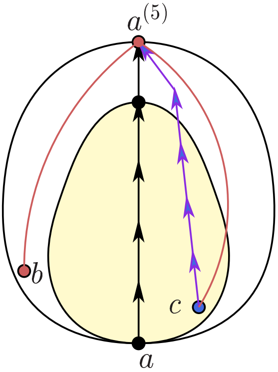
Assume that the above formula holds. Then,

$$(\forall x)(\exists \gamma)\ \gamma + \alpha \leq \beta\ \&\ f(x) \leq_T (x \oplus p^{(\xi)})^{(\gamma)}.$$

To show Cancellation Lemma, we use the following theorem:

Shore-Slaman Join Theorem (1999)

Let $\eta < \omega_1^{CK}$. If $B \not\leq_T A^{(\delta)}$ for every $\delta < \eta$, there exists $C \geq_T A$ such that $C^{(\eta)} \leq_T B \oplus C$.



Cancellation Lemma (Gregoriades-K.)

$$(\forall x,z)\ (f(x)\oplus z)^{(\alpha)} \leq_T (x\oplus (z\oplus p)^{(\xi)})^{(\beta)}$$
$$\Longrightarrow\ (\forall x)(\exists \gamma)\ \gamma+\alpha\leq\beta\ \&\ f(x)\leq_T (x\oplus p^{(\xi)})^{(\gamma)}.$$

Proof (Simplified by Andrew Marks)

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- Otherwise, there exists x such that $f(x) \not\leq_T (x \oplus p^{(\xi)})^{(\delta)}$ for every $\delta < \eta := \min\{\gamma : \gamma + \alpha > \beta\}$.

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- Then, $f(x) \not\leq_T y^{(\xi+\delta)}$ for every $\delta < \eta$.

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- Note that $z \geq_T y$ implies $z \geq_T p$ and $z^{(\xi)} \geq_T x$.

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- Note that $z \geq_T y$ implies $z \geq_T p$ and $z^{(\xi)} \geq_T x$.
- $(f(x) \oplus z)^{(\alpha)} \geq_T z^{(\xi+\eta+\alpha)} >_T z^{(\xi+\beta)} \geq_T (x \oplus (z \oplus p)^{(\xi)})^{(\beta)}$.

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- $(f(x) \oplus z)^{(\alpha)} \geq_T z^{(\xi+\eta+\alpha)} >_T z^{(\xi+\beta)} \geq_T (x \oplus (z \oplus p)^{(\xi)})^{(\beta)}$.
- This contradicts our assumption.

Cancellation Lemma

Assume that $f^{-1}\underset{\sim}{\Sigma}^0_{\alpha+1} \subseteq \underset{\sim}{\Sigma}^0_{\beta+1}$. Then,

$$(\forall x)(\exists \gamma)\ \gamma + \alpha \leq \beta\ \&\ f(x) \leq_T (x \oplus p^{(\xi)})^{(\gamma)}.$$

Now, we consider the following function g_e^γ:

- Input: x.
- Simulate the computation of e-th Turing machine with oracle $(x \oplus p^{(\xi)})^{(\gamma)}$.

Note that the function g_e^γ is of Baire class γ.

Lemma

Assume that $f^{-1}\underset{\sim}{\Sigma}^0_{\alpha+1} \subseteq \underset{\sim}{\Sigma}^0_{\beta+1}$. Then,

$$(\forall x \in \mathrm{dom}(f))(\exists \gamma, e)\ \gamma + \alpha \leq \beta\ \&\ f(x) = g_e^\gamma(x).$$

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(g^γ_e is of Baire γ, for every e, γ).

Put $X^\gamma_e = \{x \in \mathrm{dom}(f) : f(x) = g^\gamma_e(x)\}$.

Main Theorem (Gregoriades-K.)

Let $f : \mathbb{R} \to \mathbb{R}$. Assume that

- if $A \subseteq \mathbb{R}$ is $\underset{\sim}{\Sigma}^0_{\alpha+1}$, $f^{-1}[A] \subseteq \mathbb{R}$ is $\underset{\sim}{\Sigma}^0_{\beta+1}$.

Then, there exists a countable cover $\{X^\gamma_e\}^{\gamma+\alpha\leq\beta}_{e\in\omega}$ of $\mathbb{R}$ such that for every such e, γ, the restriction $f|_{X^\gamma_e}$ agrees with g^γ_e.

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Let X, Y be finite dimensional Polish spaces, and $f : X \to Y$. Assume that

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Main Theorem (Gregoriades-K.)

Let X, Y be finite dimensional Polish spaces, $f : X \to Y$ and $\alpha \leq \beta < \alpha \cdot 2$. Then, the following are equivalent:

1. If $A \subseteq Y$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha+1}$, $f^{-1}[A] \subseteq X$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta+1}$.
2. There exists a $\underset{\sim}{\mathbf{\Pi}}^0_{\beta}$ partition $\{X_i\}_{i\in\omega}$ of X such that for every i, the restriction $f|_{X_i}$ is of Baire γ with $\gamma + \alpha \leq \beta$.

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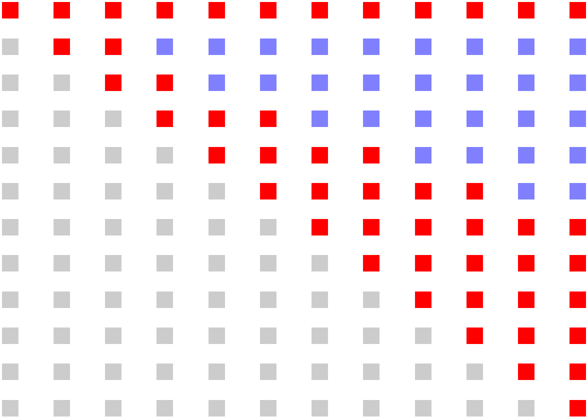
6. The degree structures of ∞-dim. spaces seem quite complex! To show our theorem for ∞-dim. cases, we need further researches for computability theory for ∞-dim. spaces!

Main Theorem



The decomposability of Borel functions

	1	2	3	4	5	6
1	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$
2	–	$\Pi_1, \mathcal{B}_1$	$\Pi_2, \mathcal{B}_2$	?	?	?
3	–	–	$\Pi_2, \mathcal{B}_1$	$\Pi_3, \mathcal{B}_2$	?	?
4	–	–	–	$\Pi_3, \mathcal{B}_1$	$\Pi_4, \mathcal{B}_2$	$\Pi_5, \mathcal{B}_3$
5	–	–	–	–	$\Pi_4, \mathcal{B}_1$	$\Pi_5, \mathcal{B}_2$
6	–	–	–	–	–	$\Pi_5, \mathcal{B}_1$

Main Theorem (Gregoriades-K.)

If $2 \leq m \leq n < 2m$ then

$$\boxed{\underset{\sim}{\Sigma}_{m+1,n+1}} = \boxed{\mathrm{dec}(\underset{\sim}{\Pi}_n; \mathcal{B}_{n-m})}$$

Question ([1,2,3,4])

X, Y: Polish spaces. Are the following equivalent for $f : X \to Y$?

1. If $A \subseteq Y$ is $\boldsymbol{\Sigma}^0_{m+1}$, $f^{-1}[A] \subseteq X$ is $\boldsymbol{\Sigma}^0_{n+1}$.
2. There exists a $\boldsymbol{\Pi}^0_n$ partition $\{X_i\}_{i \in \mathbb{N}}$ of X such that for every $i \in \mathbb{N}$, the restriction $f|_{X_i}$ is Baire $n - m$.

Main Theorem (Gregoriades-K. 201x)

Let X, Y be finite dimensional Polish spaces, and $\alpha \leq \beta < \alpha \cdot 2$. Then, the following are equivalent for $f : X \to Y$:

1. If $A \subseteq Y$ is $\boldsymbol{\Sigma}^0_{\alpha+1}$, $f^{-1}[A] \subseteq X$ is $\boldsymbol{\Sigma}^0_{\beta+1}$.
2. There exists a $\boldsymbol{\Pi}^0_\beta$ partition $\{X_i\}_{i \in \omega}$ of X such that for every i, the restriction $f|_{X_i}$ is of Baire γ with $\gamma + \alpha \leq \beta$.