Rosser-type Henkin sentences and local reflection principles

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Henkin's problem

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2. **recursively axiomatized;**
3. **an extension of Peano arithmetic PA.**

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Is every sentence asserting its own T-provability provable in T?

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Thus every Henkin sentence of T is provable in T.

Rosser-type Henkin sentences

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A Rosser provability predicate $\mathrm{Pr}_T^R(x)$ of T is defined as $\exists y(\mathrm{Prf}_T(x, y) \land \forall z \leq y \neg \mathrm{Prf}_T(\neg x, z))$.

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Is every Rosser-type Henkin sentence of $\mathrm{Pr}_T^R(x)$ either provable or refutable in T?

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Is every Rosser-type Henkin sentence of $\mathrm{Pr}_T^R(x)$ either provable or refutable in T?

Answer

Whether $\mathrm{Pr}_T^R(x)$ has an independent Rosser-type Henkin sentence is dependent on the choice of $\mathrm{Pr}_T^R(x)$.

Rosser predicate with an independent Rosser-type Henkin sentence

Theorem (K.)

For any Σ_1 sentence φ, T.F.A.E.:

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For any Σ_1 sentence φ, T.F.A.E.:

1. **There is a Σ_1 sentence ψ s.t.**
   - **PA** $\vdash \neg\varphi \vee \neg\psi$,
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2. **There is a proof predicate $\mathrm{Prf}'_T(x, y)$ s.t.**
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Since every negated Rosser sentence of $\mathrm{Pr}^R_T(x)$ satisfies the condition 1 in the statement, we obtained the following corollary.

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Corollary

There is a Rosser provability predicate of T having an independent Rosser-type Henkin sentence.

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On the other hand, we obtained the following theorem.

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There is a Rosser provability predicate $\mathrm{Pr}_T^R(x)$ of T s.t. for any sentence φ,
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Corollary

There is a Rosser provability predicate of T having no independent Rosser-type Henkin sentence.

1. **Rosser-type Henkin sentences**
2. **Rosser-type local reflection principles**

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- This means that T cannot capture the soundness of T.
- This is a generalization of the second incompleteness theorem.

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Question

Is there $\mathsf{Pr}_T^R(x)$ s.t. $T + \mathsf{Rfn}_T^R(x)$ is strictly weaker than $T + \mathsf{Rfn}(T)$?

Shavrukov's problem

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Is there $\mathsf{Prf}_T(x)$ s.t.
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- Shavrukov pointed out that an affirmative answer to his problem gives a Rosser provability predicate $\mathrm{Pr}_T^R(x)$ s.t. $T + \mathrm{Rfn}^R(T)$ is strictly weaker than $T + \mathrm{Rfn}(T)$.

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- $T + \mathsf{Rfn}_{\Pi_1}(T) \nvdash \mathsf{Rfn}_{\Sigma_1}(T)$.

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Theorem (K.)

There is a Rosser provability predicate $\mathsf{Pr}^R_T(x)$ of T s.t. for any $\Gamma \in \{\Sigma_n, \Pi_n : n \geq 1\}$, $T + \mathsf{Rfn}_\Gamma(T)$ and $T + \mathsf{Rfn}^R_\Gamma(T)$ are equivalent.

Open problems

Problem

For $\Gamma \in \{\Sigma_n, \Pi_n : n \geq 1\}$,
is $T + \mathsf{Rfn}^R(T)$ a Γ-conservative extension of $T + \mathsf{Rfn}^R_\Gamma(T)$?

Problem

- Is $T + \mathsf{Rfn}^R_{\Pi_1}(T)$ finitely axiomatizable over T?
- For $\Gamma \in \{\Sigma_n, \Pi_{n+1} : n \geq 1\}$,
  is $T + \mathsf{Rfn}^R_\Gamma(T)$ not finitely axiomatizable over T?