Determinacy and Turing Determinacy within second-order arithmetic.

Antonio Montalbán
(joint work with Richard A. Shore)

University of California, Berkeley

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Outline

(1) Determinacy

How much determinacy can be proved without using uncountable objects?

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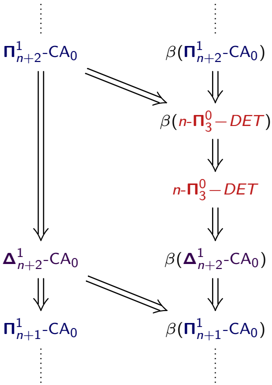
(2) Turing Determinacy

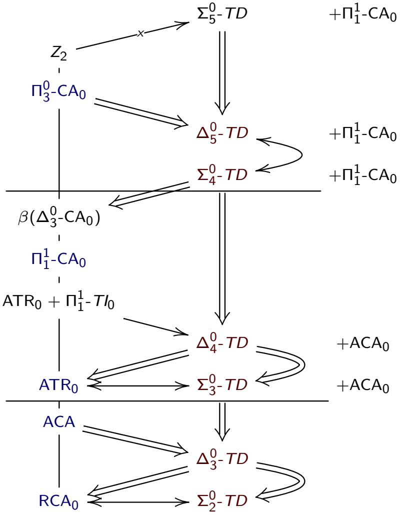
What is the strength of the various levels of Turing determinacy?

A preview

Turing Determinacy

Determinacy





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Second order arithmetic, Z_2, consist of

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Most of classical mathematics can be expressed and proved in Z_2.

Thm: ZFC^- is Π^1_4-conservative over Z_2,
where ZFC^- is ZFC without the Power-set axiom.

(Obs: Borel-DET and $\mathbf{\Pi}^0_k$-DET are Π^1_3-statements.)

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We provide a hierarchy of natural statements
that need axioms all the way up in Z_2.

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For a class of sets of reals $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, let
Γ-DET: Every $A \in \Gamma$ is determined.

Early history

Γ	Γ-DET	remark
Open (Σ^0_1)	[Gale Stwart 53]	
G_δ (Π^0_2)		
$F_{\sigma\delta}$ (Π^0_3)		
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Determinacy, along the Wadge hierarchy,
provides a naturally defined spine of statements

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Equivalently:

How much determinacy can be proved in Z_2?

Previously known results

Γ	strength of Γ-DET		base
Δ^0_1	ATR_0	[Steel 78]	RCA_0
Σ^0_1	ATR_0	[Steel 78]	RCA_0
$\Sigma^0_1 \wedge \Pi^0_1$	Π^1_1-CA_0	[Tanaka 90]	RCA_0
Δ^0_2	Π^1_1-TR_0	[Tanaka 91]	RCA_0
Π^0_2	Σ^1_1-ID_0	[Tanaka 91]	ATR_0
Δ^0_3	$[\Sigma^1_1]^{TR}$-ID_0	[MedSalem, Tanaka 08]	Π^1_1-TI_0
Π^0_3	Π^1_3-$CA_0 \vdash$ ..	Δ^1_3-$CA_0 \nvdash$ .. [Welch 09]	
Π^0_4	$Z_2 \nvdash$ ..	[Martin] [Friedman 71]	

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Theorem ([MS 14] The following are equiconsistent)

- Z_2
- ZFC^-
- *The scheme* $\{$ "Every Boolean combination of n $\mathbf{\Pi}^0_3$ sets is determined.": $n \in \mathbb{N}\}$

Difference hierarchy

Def: $A \subseteq \omega^\omega$ is m-$\mathbf{\Pi}^0_3$ if there are $\mathbf{\Pi}^0_3$ sets $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_m = \emptyset$ s.t.: $A = (\ldots(((A_0 \setminus A_1) \cup A_2) \setminus A_3) \cup \ldots)$

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Q: What is the strength of n-$\mathbf{\Pi}^0_3$-DET?

A closer look at our first paper

Recall:

Π^1_n-CA_0 is Z_2 with induction and comprehension restricted to Π^1_n formulas.

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Corollary: $(\forall n)$, $Z_2 \vdash n\text{-}\mathbf{\Pi}^0_3 - DET$, but
$Z_2 \nvdash (\forall n)\ n\text{-}\mathbf{\Pi}^0_3 - DET$.

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Obs: Borel-DET and m-$\mathbf{\Pi}^0_3$−DET are Π^1_3 theorems of ZFC.

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For every $n \in \mathbb{N}$ we have the following consistency strength relations:

$$\cdots \quad \mathbf{\Pi}^1_{n+1}\text{-CA}_0 \quad <_c \quad \mathbf{\Delta}^1_{n+2}\text{-CA}_0 \quad <_c \quad n\text{-}\mathbf{\Pi}^0_3\text{-}DET \quad <_c \quad \mathbf{\Pi}^1_{n+2}\text{-CA}_0 \quad \cdots.$$

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$$\cdots \quad \mathbf{\Pi}^1_{n+1}\text{-CA}_0 \quad <_c \quad \mathbf{\Delta}^1_{n+2}\text{-CA}_0 \quad <_c \quad n\text{-}\mathbf{\Pi}^0_3\text{-}DET \quad <_c \quad \mathbf{\Pi}^1_{n+2}\text{-CA}_0 \quad \cdots.$$

Theorem ([MS 14])

$$Z_2 \quad <_c \quad \omega\text{-}\mathbf{\Pi}^0_3\text{-}DET.$$

On our second paper

But if we look at consitency strength, reversals are possible:

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Thm: [Friedman] $\mathsf{RCA}_0 \vdash \beta(\mathsf{ATR}_0) \iff \Pi^1_1\text{-}\mathsf{CA}_0$.

The precise theorem

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$$\begin{array}{ccc}
\vdots & & \vdots \\
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\Downarrow & \searrow & \Downarrow \\
 & & \beta(n\text{-}\mathbf{\Pi}^0_3-DET) \\
 & & \Downarrow \\
 & & n\text{-}\mathbf{\Pi}^0_3-DET \\
 & & \Downarrow \\
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First theorem (repeated): Δ^1_{n+2}-$\mathsf{CA}_0 \nvdash n$-$\mathbf{\Pi}^0_3$−DET.

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Lemma: [Friedman 71]

The least ordinal such that $L_\beta \models \Pi^0_{4+\beta}$-DET

is greater than or equal

the least ordinal such that $L_\beta \models$ZFC+ β-iterates of Power set.

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Hence $2^\omega \cap L_{\alpha_n} \models \Delta^1_{n+1}\text{-}CA_0$ & $\neg(n\text{-}1)\text{-}\mathbf{\Pi}^0_3\text{-}DET$

Turing Determinacy

What is the strength of the various levels of Turing determinacy?

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For a class of sets of reals $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, let
Γ-TD: Every degree-invariant $A \in \Gamma$ either contains or is disjoint from a cone.

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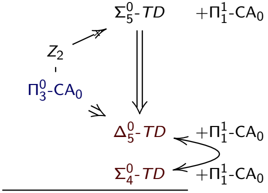

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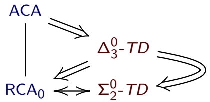

Σ_3^0 and Δ_3^0-TD

Theorem ([MS 15])

$ACA_0 + \mathbf{\Sigma}_3^0\text{-}TD \vdash ATR_0.$

Pf: Uses $P = \{X | (\exists\beta < \alpha)(0^\beta \oplus X \equiv_T W^X)\}$, Posner-Robinson, JS-Pseudo-jump inversion.

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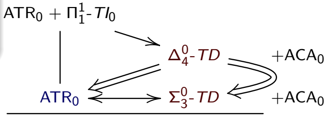

Σ^0_4-TD

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Lemma ([MS 15])

Π^1_1-CA$_0$ + Σ^0_4-TD $\vdash \alpha_2$ exists.

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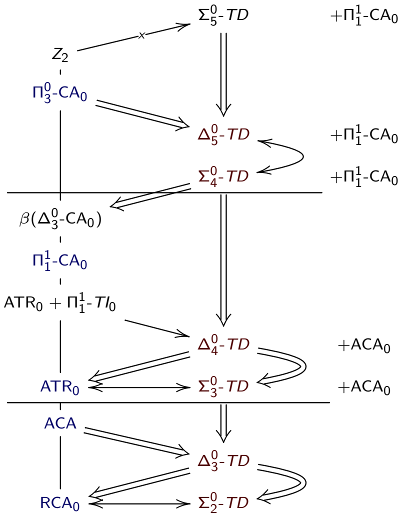
Lemma ([MS 15])

Π^1_1-CA_0 + Σ^0_4-TD $\vdash \alpha_2$ *exists.*

Corollary ([MS 15])

Π^1_1-CA_0 + Σ^0_4-TD $\vdash \beta(\Delta^0_3$-$CA_0)$.

The Picutre



Σ^0_5-TD
$+\Pi^1_1$-CA_0
Z_2
x
Π^0_3-CA_0
Δ^0_5-TD
$+\Pi^1_1$-CA_0
Σ^0_4-TD
$+\Pi^1_1$-CA_0
$\beta(\Delta^0_3$-$CA_0)$
Π^1_1-CA_0
$ATR_0 + \Pi^1_1$-TI_0
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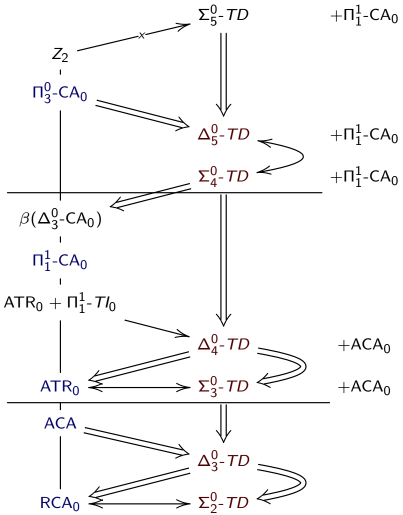

Q: $WKL_0 + \Delta^0_3$-$TD \vdash ACA_0$?

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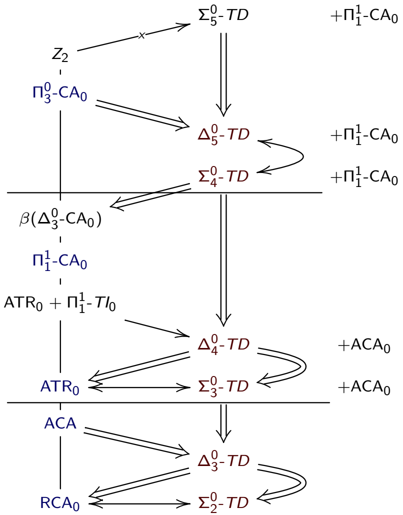


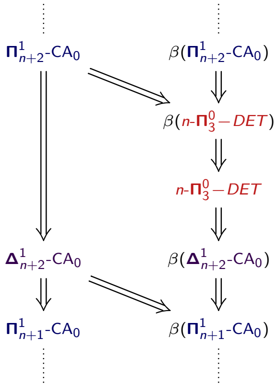
Q: $WKL_0 + \Delta^0_3$-$TD \vdash ACA_0$?

Q: $ATR_0 + \Sigma^1_1$-$IND \vdash \Delta^0_4$-TD?

Q: $ACA_0 + \Delta^0_4$-$TD \vdash \Sigma^1_1$-IND?

Thank you

Determinacy



Turing Determinacy