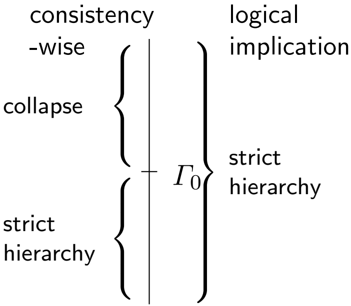
The strength of determinacy between Σ_1^0 and Δ_2^0

Takako Nemoto (JAIST)
joint work with Dr. Kentaro Sato, Univ. Bern

Highlights

- Complete description of the strengths of all the “reasonably defined” determinacy schemata below Δ^0_2



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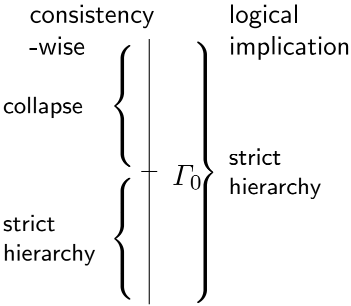

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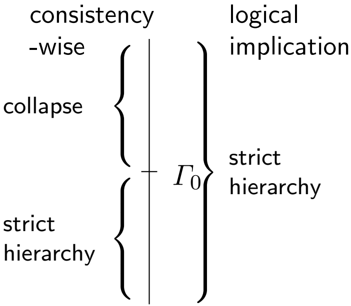

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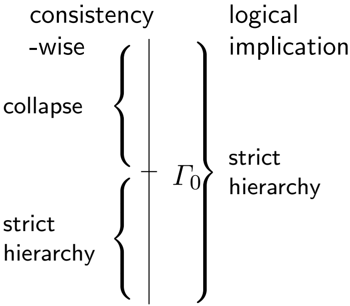

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  - strict in the sense of logical implication
  - but collapses consistency-wise.
- The hierarchy of determinacy statements might be "better" than that of transfinite recursion (jump statements), as a measure:
  - $(\Sigma^0_1)_\alpha\text{-Det}^*$ is always below $\Delta^0_2\text{-Det}^*$, whereas
  - $(\Sigma^0_1\text{-CA}_0)_\alpha$ is sometimes beyond $\Sigma^1_1\text{-CA}_0$ and more.

Infinite games?

Let X be either $\mathbb{N}$ or $\{0, 1\}$. For a $\mathcal{L}_2$-formula $\psi(f)$,

- Players I and II alternately choose $x \in X$ to form $f \in X^{\mathbb{N}}$.

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- Γ *determinacy* asserts that every $\psi(f) \in \Gamma$ is determinate.

Base theory RCA_0

An $\mathcal{L}_2$-theory RCA_0 consists of:

Basic arithmetic

$$\begin{aligned} \text{Successor} \quad & n+1 \neq 0, \quad n+1 = m+1 \to n = m, \\ \text{Addition} \quad & n+0 = n, \quad n+(m+1) = (n+m)+1, \\ \text{Multiplication} \quad & n \cdot 0 = 0, \quad n \cdot (m+1) = n \cdot m + n, \\ \text{Order} \quad & \neg m < 0, \quad m < n+1 \leftrightarrow m \leq n, \end{aligned}$$

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Σ^0_1 induction

$\psi(0) \land \forall n(\psi(n) \to \psi(n+1)) \to \forall n \psi(n)$, for $\psi \in \Sigma^0_1$.

Reverse mathematical results of determinacy

We had the following equivalences over RCA_0^* (except †: $+\Sigma^1_3$-Ind.):

	Systems	determinacy in $2^{\mathbb{N}}$ (-Det*)	determinacy in $\mathbb{N}^{\mathbb{N}}$(-Det)	
strong	Π^1_3-CA_0			
$\Uparrow$		Σ^0_3	Σ^0_3	
	$[\Sigma^1_1]^{\mathrm{TR}}$-$\mathsf{ID}_0$	Δ^0_3	Δ^0_3	†
	$[\Sigma^1_1]^2$-ID_0	$(\Sigma^0_2)_3$	$(\Sigma^0_2)_2$	
	Π^1_1-ID_0	$(\Sigma^0_2)_2$	Σ^0_2	
	Π^1_1-TR_0	Bisep(Δ^0_2, Σ^0_2)	Δ^0_2	
	Π^1_1-CA_0	Bisep(Σ^0_1, Σ^0_2)	$(\Sigma^0_1)_2$	
	Π^0_1-TR_0	Δ^0_2, Σ^0_2	Δ^0_1, Σ^0_1	
	$\vdots$			
	$(\Pi^0_1$-$\mathsf{CA}_0)_{\omega^\alpha}$			
	$\vdots$			
	ACA_0^+	$(\Sigma^0_1)_\omega$		
	ACA_0'	$(\Sigma^0_1)_{<\omega}$		
$\Downarrow$	Π^0_1-CA_0	$(\Sigma^0_1)_2$		
weak	WKL_0^*	Δ^0_1, Σ^0_1		

(Steel, Tanaka, MedSalem, Welch and N)

Hausdorf's difference hierarchy of $(\Sigma^0_1)_\alpha$

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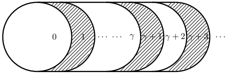
- $(\Sigma^0_1)_2$ is the class of formulas of the form $\psi_1(f) \wedge \neg\psi_0(f)$, where $\psi_i \in \Sigma^0_1$.
- For any α, $(\Sigma^0_1)_\alpha$ is the class of all formulas of the form

$$\exists \text{ odd } \beta \prec \alpha(\psi(\beta, f) \wedge \neg(\exists\gamma \prec \beta)\psi(\gamma, f)), \quad \text{(for even } \alpha)$$
$$\exists \text{ even } \beta \prec \alpha(\psi(\beta, f) \wedge \neg(\exists\gamma \prec \beta)\psi(\gamma, f)), \quad \text{(for odd } \alpha)$$

for some $\psi(\beta, f) \in \Sigma^0_1$.



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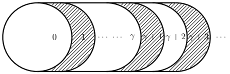
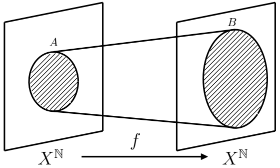


Theorem (Tanaka) In Π^0_1-CA$_0$, $\Delta^0_2 = \bigcup_{\text{w.o. } X}(\Sigma^0_1)_X$

Wadge hierarchy



- *Wadge classes* are classes of subsets of $X^{\mathbb{N}}$ closed under continuous pre-images.
- All reasonable classes (Σ^0_1, Δ^0_1,...) of formulae $\psi(f)$ must form Wadge classes because boolean operations and quantifiers are preserved under continuous pre-images.

Wadge hierarchy up to Σ^0_2

$$\Sigma^0_2$$
$$\Delta^0_2$$
$$\vdots$$
$$\Delta((\Sigma^0_1)_{\alpha+1}) = \text{Bisep}(\Delta^0_1, (\Sigma^0_1)_\alpha)$$
$$(\Sigma^0_1)_\alpha$$
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$$\vdots$$
$$(\Sigma^0_1)_2$$
$$\Delta((\Sigma^0_1)_2) = \text{Bisep}(\Delta^0_1, \Sigma^0_1)$$
$$\Sigma^0_1$$
$$\Delta^0_1$$

Between Π^0_1-CA$_0$ and Π^0_1-TR$_0$

Γ-TR$_0$ $\mathrm{WO}(Y) \to \exists X(X = H^Y_\theta)$ for $\theta \in \Gamma$, where
$\forall x \forall y(\langle x, y\rangle \in H^Y_\theta \leftrightarrow \theta(x, \{\langle z, w\rangle \in H^Y_\theta : w \prec_Y y\}))$.

$(\Gamma\text{-CA}_0)_\alpha$ $\exists X(X = H^\alpha_\theta)$ for $\theta \in \Gamma$.

Theorem The following equivalences hold over RCA$_0$

- Π^0_1-CA$_0 \leftrightarrow (\Sigma^0_1)_2$-Det*.
- $(\Sigma^0_1\text{-CA}_0)_{\omega^\alpha} \to (\Sigma^0_1)_{\omega^\alpha}$-Det*,
  $(\Sigma^0_1)_{\omega^\alpha}$-Det$^* + \mathrm{WO}(\omega^\alpha) \to (\Sigma^0_1\text{-CA}_0)_{\omega^\alpha}$.
- Π^0_1-TR$_0 \leftrightarrow \bigcup_{X:\text{w.o.}} (\Sigma^0_1)_X$-Det*($= \Delta^0_2$-Det*) $\leftrightarrow \Sigma^0_2$-Det*.

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Lemma (Flumini and Sato) $(\Pi^0_1\text{-CA}_0)_\alpha \vdash \mathrm{WO}(\alpha)$

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Question Does -Det* implies $\mathrm{WO}(\alpha)$

Proof theoretic ordinals

Proof theoretic ordinal $|\mathsf{S}|$ of system S

- $|\mathsf{S}| = \sup\{\beta : \mathsf{S} \vdash \mathrm{WO}(\beta)\}$.
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Famous proof theoretic ordinals

- (Gentzen) $|\Pi^0_1\text{-}\mathsf{CA}_0| = \varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots\}$
- Veblen function φ
  - $\varphi 0 \alpha = \omega^\alpha$
  - $\varphi\alpha\beta =$ the β-th simultaneous fixed point of the functions $\varphi\gamma$ for all $\gamma < \alpha$.

  (Friedman, MacAloon and Simpson)
  $|\Pi^0_1\text{-}\mathsf{TR}_0| = |(\Pi^{0-}_1\text{-}\mathsf{CA}_0)_{<\Gamma_0}| = \Gamma_0$
  $=$the least $\gamma > 0$ s.t. $\alpha, \beta < \gamma \to \varphi\alpha\beta < \gamma$

Proof theoretic strength and reverse mathematical strength

Proof theoretic strength

Let S and T be "usual" theories (all theories in this talk!).

- $|\mathsf{S}| < |\mathsf{T}|$ iff $\mathsf{T} \vdash \mathsf{Con}(\mathsf{S})$.
- $\mathsf{T} \subsetneq \mathsf{S}$, i.e., $\{\psi\colon \mathsf{T} \vdash \psi\} \subsetneq \{\psi\colon \mathsf{S} \vdash \psi\}$ doesn't imply $|\mathsf{T}| < |\mathsf{S}|$.
  (Example: $\mathsf{RCA}_0 \subsetneq \mathsf{WKL}_0$ but $|\mathsf{RCA}_0| = |\mathsf{WKL}_0|$)
- In particular, $\mathsf{RCA}_0 \vdash \mathsf{A} \to \mathsf{B}$ and $\mathsf{RCA}_0 \nvdash \mathsf{B} \to \mathsf{A}$ does not imply $|\mathsf{RCA}_0 + \mathsf{B}| < |\mathsf{RCA}_0 + \mathsf{A}|$.

Removing $\mathrm{WO}(\alpha)$

Lemma

If $\alpha \prec |\Sigma^0_1\text{-}\mathsf{CnTR}_0|$, then $(\Sigma^0_1)_{1+\alpha}\text{-}\mathsf{Det}^* \vdash \mathrm{WO}(\alpha)$,
where $\Sigma^0_1\text{-}\mathsf{CnTR}_0$ states $\forall\beta(\mathrm{WO}(\beta) \rightarrow (\Sigma^0_1\text{-}\mathsf{CA})_\beta)$.

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- Then $|(\Sigma^0_1)_{1+\alpha}\text{-Det}^*| \succeq \min\{\alpha + 1, |\Sigma^0_1\text{-CnTR}_0|\}$.

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  For any $\beta \prec \min\{\alpha+1, |\Sigma^0_1\text{-}\mathsf{CnTR}_0|\}$, $(\Sigma^0_1)_{1+\alpha}\text{-}\mathsf{Det}^*$ proves:
  - $\mathrm{WO}(\alpha) \to \mathrm{WO}(\beta)$,
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Theorem

Let $(\star)_\alpha$ be $(\Sigma^0_1)_{1+\alpha}\text{-Det}^* \to (\Sigma^0_1\text{-CA})_\alpha$.

- $\mathsf{RCA}_0 \vdash (\star)_\alpha$ if $\alpha \prec |\Sigma^{0-}_1\text{-CnTR}_0|$.
- $\mathsf{RCA}_0 \nvdash (\star)_\alpha$ if $|(\Sigma^0_1\text{-CA}_0)_{<\alpha}| \succeq |\Sigma^0_1\text{-TR}_0|$,
  and $|\Sigma^0_1\text{-CnTR}_0| \preceq |(\Sigma^0_1)_{1+\alpha}\text{-Det}^*| \preceq |\Sigma^0_1\text{-TR}_0|$.

Comparing strength

Theorem

1. The following are equivalent
   - $\mathsf{RCA}_0 \nvdash (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^* \to (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^*$
   - $\mathsf{RCA}_0 + \mathrm{WO}(\alpha) + (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^* \vdash$ $\mathsf{Con}(\mathsf{RCA}_0^* + \mathrm{WO}(\beta) + (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^*)$
   - $\mathsf{RCA}_0^* \nvdash (\Pi^0_1\text{-}\mathsf{CA}_0)_\beta \to (\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha$
   - $(\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha \vdash \mathsf{Con}((\Pi^0_1\text{-}\mathsf{CA}_0)_{beta})$
   - $\beta \cdot \omega \leq \alpha$.

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   - $\mathsf{RCA}_0 + \mathrm{WO}(\alpha) + (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^* \vdash$
     $\mathsf{Con}(\mathsf{RCA}_0^* + \mathrm{WO}(\beta) + (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^*)$
   - $\mathsf{RCA}_0^* \nvdash (\Pi^0_1\text{-}\mathsf{CA}_0)_\beta \rightarrow (\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha$
   - $(\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha \vdash \mathsf{Con}((\Pi^0_1\text{-}\mathsf{CA}_0)_{beta})$
   - $\beta \cdot \omega \leq \alpha$.
2. In particular, $\alpha < \beta \cdot \omega$ and $\beta < \alpha \cdot \omega$,
   $\mathsf{RCA}_0 \vdash (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^* \leftrightarrow (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^*$

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     $\mathsf{Con}(\mathsf{RCA}_0^* + \mathrm{WO}(\beta) + (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^*)$
   - $\mathsf{RCA}_0^* \nvdash (\Pi^0_1\text{-}\mathsf{CA}_0)_\beta \to (\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha$
   - $(\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha \vdash \mathsf{Con}((\Pi^0_1\text{-}\mathsf{CA}_0)_{beta})$
   - $\beta \cdot \omega \leq \alpha$.
2. In particular, $\alpha < \beta \cdot \omega$ and $\beta < \alpha \cdot \omega$,
   $\mathsf{RCA}_0 \vdash (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^* \leftrightarrow (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^*$
3. $\mathsf{RCA}_0 \nvdash (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^* \to \Delta^0_2\text{-}\mathsf{Det}^*$

Comparing strength

Theorem

1. The following are equivalent
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     $\mathsf{Con}(\mathsf{RCA}^*_0 + \mathrm{WO}(\beta) + (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^*)$
   - $\mathsf{RCA}^*_0 \nvdash (\Pi^0_1\text{-}\mathsf{CA}_0)_\beta \to (\Pi^0_1\text{-}\mathsf{CA}_0)_\alpha$
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3. $\mathsf{RCA}_0 \nvdash (\Sigma^0_1)_\alpha\text{-}\mathsf{Det}^* \to \Delta^0_2\text{-}\mathsf{Det}^*$
4. Even if $\mathit{\Gamma}_0 \leq \beta$, $|\mathsf{RCA}_0 + (\Sigma^0_1)_\beta\text{-}\mathsf{Det}^*| = \mathit{\Gamma}_0$.

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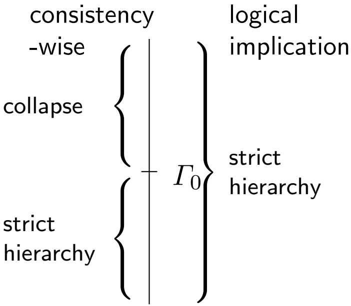
Thus, the hierarchy of $(\Sigma^0_1)_\beta\text{-}\mathsf{Det}^*$ for $\beta \geq \mathit{\Gamma}_0$ collapses proof theoretically, but not reverse mathematically.

Highlights

- Complete description of the strengths of all the "reasonably defined" determinacy schemata below Δ_2^0



Highlights

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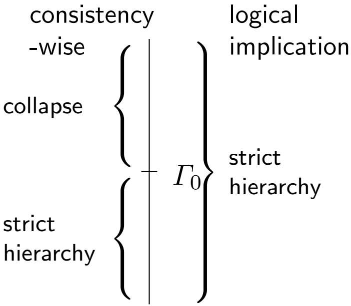
- Γ_0 is the "critical point" of a Phase Transition:



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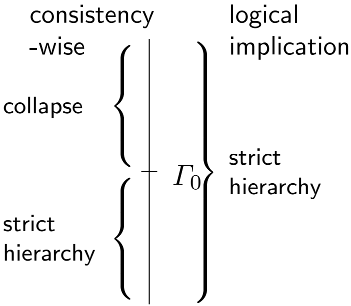

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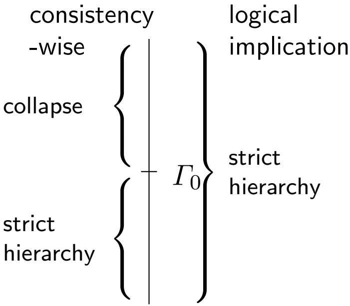

- Γ_0 is the “critical point” of a Phase Transition:
- The hierarchy $\langle (\Sigma^0_1)_{\omega^\beta}\text{-Det}^* : \beta \geq \Gamma_0 \rangle$
  - strict in the sense of logical implication
  - but collapses consistency-wise.
- The hierarchy of determinacy statements might be “better” than that of transfinite recursion (jump statements), as a measure:
  - $(\Sigma^0_1)_\alpha$-Det* is always below Δ^0_2-Det* , whereas
  - $(\Sigma^0_1\text{-CA}_0)_\alpha$ is sometimes beyond Σ^1_1-CA$_0$ and more.

Reverse mathematical results of determinacy

We had the following equivalences over RCA_0^* (except †: $+\Sigma^1_3$-Ind.):

	Systems	determinacy in $2^{\mathbb{N}}$ (-Det*)	determinacy in $\mathbb{N}^{\mathbb{N}}$(-Det)	
strong	Π^1_3-CA_0			
$\Uparrow$		Σ^0_3	Σ^0_3	
	$[\Sigma^1_1]^{\mathrm{TR}}$-$\mathsf{ID}_0$	Δ^0_3	Δ^0_3	†
	$[\Sigma^1_1]^2$-ID_0	$(\Sigma^0_2)_3$	$(\Sigma^0_2)_2$	
	Σ^1_1-ID_0	$(\Sigma^0_2)_2$	Σ^0_2	
	Π^1_1-TR_0	Bisep(Δ^0_2, Σ^0_2)	Δ^0_2	
	Π^1_1-CA_0	Bisep(Σ^0_1, Σ^0_2)	$(\Sigma^0_1)_2$	
	Π^0_1-TR_0	Δ^0_2, Σ^0_2	Δ^0_1, Σ^0_1	
	$\vdots$			
	$(\Sigma^0_1$-$\mathsf{CA}_0)_{\omega^\alpha}$	$(\Pi^0_1)_{\omega^\alpha}$		
	$\vdots$			
	ACA_0^+	$(\Sigma^0_1)_\omega$		
	ACA_0'	$(\Sigma^0_1)_{<\omega}$		
$\Downarrow$	Π^0_1-CA_0	$(\Pi^0_1)_2$		
weak	WKL_0^*	Δ^0_1, Σ^0_1		

(Steel, Tanaka, MedSalem, Welch and N)

For parameter free version

Language $\mathcal{L}_2'$ of Input/Output second order arithmetic

- 3 kinds of 2nd order variables:

  Input: I_0, I_1...; Output: O_0, O_1,...; Normal: X_0, X_1,...

- Usual langulage of 1st order arithmetic $\mathcal{L}_1$ and $\in$

Class Γ^- of formulas

For a class Γ of arithmetical formula in $\mathcal{L}_2$,

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Γ-CA in $\mathcal{L}_2$ and Γ^--CA in $\mathcal{L}_2'$

Γ-CA $\exists X(\psi(x, Y) \leftrightarrow x \in X)$ for $\psi \in \Gamma$

Γ^--CA $\exists O_0(\psi(x, I_0) \leftrightarrow x \in O_0)$ for $\psi \in \Gamma^-$

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$(\Gamma\text{-CA})_\alpha$ in $\mathcal{L}_2$ and $((\Gamma)^-\text{-CA})_\alpha$ in $\mathcal{L}_2'$

$(\Gamma\text{-CA})_\alpha$ $\exists X_0(X_0 = H_\psi^\alpha)$, where $\psi \in \Gamma$

$(\Gamma^-\text{-CA})_\alpha$ $\exists O_0(O_0 = Be\psi^\alpha)$, where $\psi \in \Gamma^-$

Iinput/Output Second Order Arithmetic

Definition

- (RbD): $\forall I_0 \exists O_0 \exists X_0 (I_0 = O_0 = X_0) \land \forall O_1 \exists X_1 (O_1 = X_1)$

Iinput/Output Second Order Arithmetic

Definition

- (RbD): $\forall I_0 \exists O_0 \exists X_0 (I_0 = O_0 = X_0) \wedge \forall O_1 \exists X_1 (O_1 = X_1)$
- Positive Π^0_0-CA is $\forall Y_0!, ... Y_{k-1} \forall \vec{x} \, \exists Y_k \, \forall z (z \in Y_k \leftrightarrow \varphi(z, \vec{x}, Y_0, ..., Y_{k-1}))$; where $\vec{Y}$ is $\vec{I}$, $\vec{O}$ or $\vec{X}$ and where $\varphi(z, \vec{x}, Y_0, ..., Y_{k-1}) \in \Pi^0_0$ is positive in $Y_0, ..., Y_{k-1}$ and without other parameters.

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Proposition

Let $\psi_0(X, Y)$ and $\psi_1(X, Y)$ are essentially Σ_1 formulas in $\mathcal{L}_2$ without any 2nd order varables other than X and Y.
If $(\Pi^{0-}_1\text{-}\mathsf{CA}_0)_\alpha \vdash \forall I_0 \exists X_0 \psi_0(I_0, X_0)$ and $(\Pi^{0-}_1\text{-}\mathsf{CA}_0)_\beta \vdash \forall I_1 \exists X_1 \psi_1(I_1, X_1)$, then $(\Pi^{0-}_1\text{-}\mathsf{CA}_0)_{\alpha+\beta} \vdash \forall I_0 \exists X_0, X_1 (\psi_0(I_0, X_0) \wedge \psi_1(X_0, X_1))$.

Examples of Models of I/O SOA

- $\mathfrak{M}_0$: ω-Model of Π^{0-}_1-CA_0
  - Input part: all Π^0_0 sets
  - Output part: all Π^0_1 sets
  - Normal part: $\mathcal{P}(\mathbb{N})$

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  - Normal part: $\mathcal{P}(\mathbb{N})$
- Arith: ω-model of Π^0_1-CA_0 (with parameters!)
  - 2nd order part: all arithmetical sets.

Determinacy in I/O SOA

Consider the statement Ψ: "player I has a winning strategy in ψ"

- "$\exists I_0$: I's strategy $\forall I_1$: II's strategy $\psi(I_0 \otimes I_1)$" means "there is a strategy for I in the input part which wins against all II's strategies in the input part."
- "$\exists O_0$: I's strategy $\forall X_0$: II's strategy $\psi(O_0 \otimes X_0)$" means "there is a strategy for I in the output part which wins against all II's strategies in the normal part."

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Recall ω-models of Π^0_1-CA_0.

- "$\mathrm{Arith} \models \Psi$" means "there is an arithmetical strategy for I which wins only against II's arithmetical strategies," so we have no information about "real winning strategy".
- "$\mathcal{P}(\mathbb{N}) \models \Psi$" tells the existence of the "real" winning strategies, but no information about their complexity.

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"$\mathfrak{M}_2 \models \exists O_0$: I's strategy $\forall X_0$: II's strategy $\psi(O_0 \otimes X_0)$" tells the existence and complexity of "real" winning strategies!

We formalize determinacy as follows:

- $\psi(f)$ is determinate:
  ($\exists O_0$: I's strategy $\forall X_0$: II's strategy $\psi(O_0 \otimes X_0)$)$\vee$
  ($\exists O_1$: II's strategy $\forall X_1$: I's strategy $\psi(X_1 \otimes O_1)$)

Input/Output SOA and Determinacy

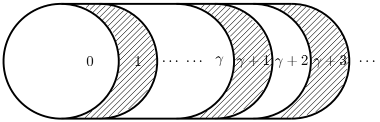
I/OSOA	complexity	ordinal	-Det*	S(F)OA
⋮	⋮	⋮	⋮	
$(\Pi^{0-}_1\text{-CA}_0)_{<\omega^2}$	$\Delta^0_{<\omega^2}$	$\varphi 20$	$(\Sigma^{0-}_1)_{<\omega^2}$	ACA_0^+
⋮	⋮	⋮	⋮	
$(\Pi^{0-}_1\text{-CA}_0)_{\alpha+1} \otimes \text{WKL}^-$	$\text{Low}(\Delta^0_{\alpha+1})$	$(\alpha+1)^\bullet$	$(\Sigma^{0-}_1)_{\alpha+1}$	
$(\Pi^{0-}_1\text{-CA}_0)_\alpha \otimes (\Pi^{0-}_1\text{-bCA}) \otimes \text{WKL}^-$	$\text{Low}(\Delta^0_\alpha)$		$\Delta((\Sigma^{0-}_1)_\alpha)$	
$(\Pi^{0-}_1\text{-CA}_0)_\alpha \otimes \text{WKL}^-$		$\alpha^\bullet$	$(\Sigma^{0-}_1)_\alpha$	
⋮	⋮	⋮	⋮	
$(\Pi^{0-}_1\text{-CA}_0)_\omega \otimes \text{WKL}^-$	$\text{Low}(\Delta^0_\omega)$	ε_ω	$(\Sigma^{0-}_1)_\omega$	
$(\Pi^{0-}_1\text{-CA}_0)_{\prec\omega}$	$\Delta^0_{<\omega}$		$(\Sigma^{0-}_1)_{\prec\omega}$	ACA_0'
$(\Pi^{0-}_1\text{-CA}_0)_{<\omega}$		ε_0	$(\Sigma^{0-}_1)_{<\omega}$	$\Pi^0_1\text{-CA}_0$
⋮	⋮	⋮	⋮	⋮
$(\Pi^{0-}_1\text{-CA}_0)_{k+1} \otimes \text{WKL}^-$	$\text{Low}(\Delta^0_{k+2})$	$\omega_{k+3}(0)$	$(\Sigma^{0-}_1)_{k+1}$	$(\text{B}\Sigma_{k+2})$
$(\Pi^{0-}_1\text{-CA}_0)_k \otimes (\Pi^{0-}_1\text{-bCA}) \otimes \text{WKL}^-$	$\text{Low}(\Delta^0_{k+1})$		$\Delta((\Sigma^{0-}_1)_{k+1})$	
$(\Pi^{0-}_1\text{-CA}_0)_k \otimes \text{WKL}^-$		$\omega_{k+2}(0)$	$(\Sigma^{0-}_1)_{k+1}$	$(\text{B}\Sigma_{k+1})$
⋮	⋮	⋮	⋮	⋮
$\text{WKL}^- \otimes (\Pi^{0-}_1\text{-bCA}) \otimes \text{WKL}^-$	$\text{Low}(\Delta^0_1)$	ω^ω	$\Delta((\Sigma^{0-}_1)_2)$	WKL_0
WKL^-		ω^2	Π^{0-}_1	WKL_0^*

$(\Pi^0_1\text{-CA}_0)_\alpha + \mathrm{WO}(\alpha) \to (\Sigma^0_1)_\alpha\text{-Det}^*$

Idea for a proof

Iterating the following proof of Π^0_1-CA$_0 \to (\Sigma^0_1)_2$-Det*:

- Write a $(\Sigma^0_1)_2$ game in a form of $\exists m\theta(f[m]) \wedge \psi(f)$, where $\theta \in \Pi^0_0$ and $\psi \in \Pi^0_1$.
- Π^0_1-CA$_0$ provides the Π^0_1 set $W = \{s \in 2^{\mathbb{N}}$: player I has a w.s. in $\psi(f)$ at $s\}$.
- Then, player I wins $\exists m\theta(f[m]) \wedge \psi(f)$ at each $s \in W' = \{s \in \mathbb{N}^{\mathbb{N}} : s \in W \wedge \theta(s)\}$.
- So, the game $\exists m\theta(f[m]) \wedge \psi(f)$ can be reduced to a Σ^0_1 game $\exists m(f[m] \in W')$.



$(\Sigma^0_1)_\alpha$-Det$^* + \mathrm{WO}(\alpha) \to (\Pi^0_1$-CA$)_\alpha$

Idea for a proof

Modifying the following proof of $(\Sigma^0_1)_2$-Det$^* \to \Pi^0_1$-CA$_0$:

- Let $\forall m \theta(x, m)$ be a Π^0_1 formula.

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- Consider the following game.

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Idea for a proof

Modifying the following proof of $(\Sigma^0_1)_2$-Det$^* \to \Pi^0_1$-CA$_0$:

- Let $\forall m \theta(x, m)$ be a Π^0_1 formula.
- Consider the following game.
  - player I asks if $\forall m \theta(n, m)$ or not.
  - player II answers yes or no.
  - If no, II wins by giving m s. t. $\neg\theta(n, m)$.
  - If yes, I wins by giving m s. t. $\neg\theta(n, m)$.

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- In the above game, player I has no w.s..
- By determinacy, player II has a w.s..
- II's w.s. yields the set $\{n : \forall m\theta(n, m)\}$.

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Modifying the following proof of $(\Sigma^0_1)_2$-Det$^* \to \Pi^0_1$-CA$_0$:

- Let $\forall m \theta(x, m)$ be a Π^0_1 formula.
- Consider the following game. (Iterate!)
  - player I asks if $\forall m \theta(n, m)$ or not.
  - player II answers yes or no.
  - If no, II wins by giving m s. t. $\neg\theta(n, m)$.
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- II's w.s. yields the set $\{n : \forall m \theta(n, m)\}$.