Universal properties in higher-order Reverse Mathematics

Sam Sanders[1]

CTFM, Tokyo Institute of Technology

UNIVERSITEIT GENT

LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

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NO stars: ${}^*f$ vs f

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FAN is the classical contraposition of WKL:

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UNIVERSAL: Both for classical and intuitionistic principles, we obtain $UT^{st} \leftrightarrow T^*$ equivalences.

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$\mathsf{WKL}^* \equiv (\forall^{st} T^1 \leq_1 1)(\mathbb{T}^{st}_\infty(T) \to (\exists^{st}\alpha^1)(\forall x^0)(\overline{\alpha}x \in T)$.

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$$\Psi(T,M)(1) := \begin{cases} 0 & (\forall n^0 \leq M)(\exists \alpha^0)(|\alpha| = n \wedge (0 * \alpha) \in T) \\ 1 & \text{otherwise} \end{cases}$$

$$\Psi(T,M)(n+1) := \begin{cases} 0 & (\forall n^0 \leq M)(\exists \alpha^0)\big(|\alpha| = n \wedge (\Psi(T,M)(n) * 0 * \alpha) \in T\big) \\ 1 & \text{otherwise} \end{cases}$$

Let Ω be the set of infinite numbers. We have

$$(\forall^{st} T^1 \leq_1 1)(\forall^{st} n^0)(\forall N, M \in \Omega)(\Psi(T,M)(n) = \Psi(T,N)(n)).$$

i.e. $\Psi(T,M)$ is Ω-invariant, the NSA-version of 'being $(\Delta^0_1)^{st}$'.

By Δ^0_1-Standard Part Principle (in RCA^Ω_0), there is STANDARD Φ s.t.

$$(\forall^{st} T^1 \leq_1 1)(\forall^{st} n^0)(\forall N \in \Omega)(\Phi(n) = \Psi(T,N)(n)).$$

This Φ is as in UWKL^{st}, i.e. we have $\mathsf{WKL}^* \to \mathsf{UWKL}^{st}$.

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In other words: If Ξ behaves like the functional Φ from UWKL^{st}, then $\Xi(T)$ equals $\Psi(T, M)$ for any $M \in \Omega$, and vice versa.

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But PRA proves consistency of $(\mathsf{RCA}_0^\omega)^* + \mathsf{BASIC}$. Hence, finitistic reduction of Φ from UWKL (and hence TJ)!

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Any questions?