Reverse Mathematics and Isbell's Zig-Zag Theorem

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Main results of this research

1. Isbell's zig-zag theorem for countable monoids $\Longleftrightarrow_{\mathsf{RCA}_0} \mathsf{WKL}_0$.
2. The existence of dominions $\Longleftrightarrow_{\mathsf{RCA}_0} \mathsf{ACA}_0$.

Reverse Mathematics

Subsystems of second order arithmetic (Big 5)

- $\mathsf{RCA_0}$: Recursive comprehension, Σ^0_1 induction.
- $\mathsf{WKL_0}$: Weak König's Lemma.
- $\mathsf{ACA_0}$: Arithmetical comprehension.
- $\mathsf{ATR_0}$: Arithmetical transfinite recursion.
- $\Pi^1_1 - \mathsf{CA}_0$: Π^1_1 comprehension.

Reverse mathematics phenomenon

Very often, a formalized mathematical theorem can be proved in $\mathsf{RCA_0}$ or equivalent to one of $\mathsf{WKL_0}, \mathsf{ACA_0}, \mathsf{ATR_0}$ or $\Pi^1_1 - \mathsf{CA}_0$ over $\mathsf{RCA_0}$.

Isbell's zig-zag theorem

Definition 1 (countable monoids)**.** The following definitions are made in RCA_0.

- A *countable monoid* A consists of a set $|A| \subset \mathbb{N}$ together with binary operation $\cdot_A : A \times A \to A$ and an element $1_A \in |A|$ satisfying
  1. $(\forall a, b, c \in A)((a \cdot_A b) \cdot_A c = a \cdot_A (b \cdot_A c))$.
  2. $(\forall a \in A)(a \cdot_A 1_A = 1_A \cdot_A a = a)$.
- If B is a monoid and a subset A of B satisfies following then we say that A is a *submonoid* of B.
  1. $(\forall a_1, a_2 \in A)(a_1 \cdot_B a_2 \in A)$.
  2. $1_B \in A$.

  (We write "$A \subset B$" for a monoid B and a submonoid A of B.)
- For two monoids A and B, a *monoid homomorphism* $\alpha : A \to B$ is a function $\alpha : A \to B$ satisfying following.
  1. $\alpha(1_A) = 1_B$.
  2. $(\forall a_1, a_2 \in A)(\alpha(a_1 \cdot_A a_2) = \alpha(a_1) \cdot_B \alpha(a_2))$.

Definition 2 (dominions)**.** The following definitions are made in RCA_0.

Let $A \subset B$ be monoids and $b \in B$. b is *dominated* by A if for any monoid C and for any pair of homomorphisms $\alpha : B \to C$ and $\beta : B \to C$, if $(\forall a \in A)(\alpha(a) = \beta(a))$, then $\alpha(b) = \beta(b)$.

The *dominion* of A is a set of all elements of B that is dominated by A.

- Note that the assertion "b is dominated by A" is Π^1_1 and the dominion of A may not exist in RCA_0.

- Which set existence axiom is required to prove the existence of dominions?

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- Note that the assertion "b is dominated by A" is Π^1_1 and the dominion of A may not exist in RCA_0.
- Which set existence axiom is required to prove the existence of dominions?
- This research reveals that ACA_0 is sufficient and necessary to prove the existence of dominions.

Example 3. Let $A \subset B$ be monoids and $b \in B$. A tuple of elements

$$a_0, a_1, a_2, a_3, a_4 \in A, x_1, x_2, y_1, y_2 \in B$$

is a *zig-zag* of b over A if

$$\begin{aligned} b = & x_1 a_0 \\ & x_1 a_1 y_1 \\ & x_2 a_2 y_1 \\ & x_2 a_3 y_2 \\ & a_4 y_2. \end{aligned}$$

Definition 4 (zig-zags)**.** The following definitions are made in RCA_0.

Let $A \subset B$ be monoids and $b \in B$. A *zig-zag* of b over A is a triple of sequences $\langle\langle a_0, a_1, \ldots, a_{2m}\rangle, \langle x_1, x_2, \ldots, x_m\rangle, \langle y_1, y_2, \ldots, y_m\rangle\rangle$ such that

1. $a_i \in A$ and $x_j, y_j \in B$ $(0 \le i \le 2m, 1 \le j \le m)$,
2. $b = x_1 a_0 = a_{2m} y_m$,
3. $a_0 = a_1 y_1$, $a_{2i} y_i = a_{2i+1} y_{i+1} (1 \le i < m)$,
4. $x_i a_{2i-1} = x_{i+1} a_{2i} (1 \le i < m)$, $x_m a_{2m-1} = a_{2m}$.

- Note that the assertion "b has a zig-zag over A" is Σ^0_1.

Theorem 5 (Isbell's zig-zag theorem for countable monoids, [3])**.** If A is a submonoid of a monoid B, then $b \in B$ is dominated by A if and only if b has a zig-zag over A.

- First stated by Isbell [3] (1966), and Philip [4] (1974) completed the proof.
- Many simpler proofs have been published including those of Storre [8] (1976), Higgins [1] (1990), Renshaw [5] (2002) or Hoffman [2] (2008).

Example 6. Let $A \subset B$ be monoids and $b \in B$. If b has a zig-zag

$$\begin{aligned} b &= x_1 a_0 \\ &= x_1 a_1 y_1 \\ &= \quad a_2 y_1, \end{aligned}$$

then

$$
\begin{aligned}
\alpha(b) &= \alpha(x_1 a_0) \\
&= \alpha(x_1)\alpha(a_0) \\
&= \alpha(x_1)\beta(a_0) \\
&= \alpha(x_1)\beta(a_1 y_1) \\
&= \alpha(x_1)\beta(a_1)\beta(y_1) \\
&= \alpha(x_1)\alpha(a_1)\beta(y_1) \\
&= \alpha(x_1 a_1)\beta(y_1) \\
&= \alpha(a_2)\beta(y_1) \\
&= \beta(a_2)\beta(y_1) \\
&= \beta(a_2 y_1) \\
&= \beta(b).
\end{aligned}
$$

for any monoid C and two homomorphisms $\alpha, \beta : B \to C$ such that $(\forall a \in A)(\alpha(a) = \beta(a))$.

Namely b is dominated by A.

Standard proofs of the zig-zag theorem

($\Longleftarrow$)(easy direction): By Δ^0_1 induction on the length of the zig-zag.

($\Longrightarrow$):

- Show the contraposition.
- Let $A \subset B$ be monoids and $b \in B$, suppose that b has no zig-zag.
- Construct a monoid C and two homomorphisms $\alpha, \beta : B \to C$ such that $(\forall a \in A)(\alpha(a) = \beta(a))$ and $\alpha(b) \neq \beta(b)$.
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- Typical proofs involve construction of an algebraic structure with a Σ^0_1 definable equality relation which can be formalized in ACA_0.
- This research reveals that WKL_0 is sufficient and necessary to prove the zig-zag theorem.

Key lemma

Lemma 7 (S. [6])**.** The following is provable in WKL_0.

Let S be a set. For any symmetric relation $R \subset S \times S$ and elements $s, s' \in S$, if there is no sequence of elements of S such that

$$s = s_1 \wedge s_1 R s_2 \wedge s_2 R s_3 \wedge \cdots \wedge s_{n-1} R s_n \wedge s_n = s' \ (2 \leq n, s_i \in S),$$

then R can be extended to an equivalence relation R' such that $\neg s R' s'$.

Main theorems of this research

Theorem 8 (S. [6])**.** The following is equivalent over $\mathsf{RCA_0}$.

1. $\mathsf{WKL_0}$.
2. Isbell's zig-zag theorem for countable monoids.

Theorem 9 (S. [6])**.** The following are equivalent over $\mathsf{RCA_0}$.

1. $\mathsf{ACA_0}$.
2. If A is a submonoid of B, the dominion of A exists.

References

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