Transfinite Recursion in Higher Reverse Mathematics

Noah Schweber

18 February 2014
Higher Reverse Mathematics

Splitting $\text{ATR}_0$

Clopen vs. Open Determinacy

Further Questions
From Lower to Higher
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- Reverse mathematics beyond reals: finite types
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  - Standard finite types: $0 = \omega$, $1 = \omega^\omega = \mathbb{R}$, $2 = \omega(\omega) = \omega^\mathbb{R}$, etc.
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  - Also mixed types: $1 \rightarrow 1$, etc.
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  - Also mixed types: 1 → 1, etc.

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- Higher-order robust systems?
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  - Is there a higher-type analogue of $ATR_0$?
Base Theories

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Kohlenbach ’01: base theory $\text{RCA}_0^\omega$ for arbitrary finite types
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Why ATR$_0$?
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- Negative result: separations – e.g. clopen determinacy for reals strictly weaker than open determinacy
- Positive result: principles linearly ordered (modulo choice)
- Choice principles also analyzed
Higher $\text{ATR}_0$, I/II
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- CWO$_\mathbb{R}$: comparability of well-orders of $\subseteq \mathbb{R}$;
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- Higher-type versions of ATR₀:
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  - BR₁[^R]: Σ₁¹ recursion along well-founded trees ⊆ R<ω;
  - Δ[^R]₁-Det: clopen determinacy on R;
  - Σ[^R]₁-Det: open determinacy on R;
Higher ATR\(_0\), I/II

- Higher-type versions of ATR\(_0\):
  - CWO\(^R\): comparability of well-orders of \(\subseteq \mathbb{R}\);
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  - \(\Sigma^2_1\)-Sep\(^R\): \(\Sigma^2_1\)-separation
Higher \( \text{ATR}_0 \), I/II

- Higher-type versions of \( \text{ATR}_0 \):
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  - \( \text{BR}_1(\mathbb{R}) \): \( \Sigma^1_1 \) recursion along well-founded trees \( \subseteq \mathbb{R}^{<\omega} \);
  - \( \Delta^\mathbb{R}_1 - \text{Det} \): clopen determinacy on \( \mathbb{R} \);
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- Choice principles:
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  - SF(\(\mathbb{R}\)): selection functions for collections of sets of reals (Quasi-strategies \(\rightarrow\) strategies)
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- Higher-type versions of ATR$_0$:
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  - $\text{BR}_1(\mathbb{R})$: $\Sigma^1_1$ recursion along well-founded trees $\subseteq \mathbb{R}^{<\omega}$;
  - $\Delta^\mathbb{R}_1\text{-Det}$: clopen determinacy on $\mathbb{R}$;
  - $\Sigma^\mathbb{R}_1\text{-Det}$: open determinacy on $\mathbb{R}$;
  - $\Sigma^2_1\text{-Sep}^\mathbb{R}$: $\Sigma^2_1$-separation

- Choice principles:
  - $\text{SF}(\mathbb{R})$: selection functions for collections of sets of reals (Quasi-strategies $\rightarrow$ strategies)
  - $\text{WO}(\mathbb{R})$: well-orderability of reals (Kleene-Brouwer: trees $\rightarrow$ ordinals)
Higher ATR$_0$, II/II

- $(S.)$

\[ \Sigma^2_1\text{-Sep}^\mathbb{R} + \text{SF}(\mathbb{R}) \]
\[ \downarrow \]
\[ \Sigma^\mathbb{R}_1\text{-Det} \]
\[ \downarrow \]
\[ \text{TR}_1(\mathbb{R}) + \text{WO}(\mathbb{R}) + \text{SF}(\mathbb{R}) \quad \Delta^\mathbb{R}_1\text{-Det} \quad \text{BR}_1(\mathbb{R}) + \text{SF}(\mathbb{R}) \]
\[ \downarrow \]
\[ \text{CWO}^\mathbb{R} \]
Higher ATR$_0$, II/II

- (S.)

\[
\begin{align*}
&\Sigma^2_1-\text{Sep}^\mathbb{R} + \text{SF}(\mathbb{R}) \\
&\downarrow \\
&\Sigma^\mathbb{R}_1-\text{Det} \\
&\downarrow \\
&\text{TR}_1(\mathbb{R}) + \text{WO}(\mathbb{R}) + \text{SF}(\mathbb{R}) \\
&\Delta^\mathbb{R}_1-\text{Det} \\
&\downarrow \\
&\text{BR}_1(\mathbb{R}) + \text{SF}(\mathbb{R}) \\
&\downarrow \\
&\text{CWO}^\mathbb{R}
\end{align*}
\]

- \[\text{WO}(\mathbb{R}) \leftrightarrow \text{SF}(\mathbb{R})\]
Separating Determinacy Principles
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- Over $\text{RCA}_0^3$, $\Delta^R_1\text{-Det}$ does not imply $\Sigma^R_1\text{-Det}$
Separating Determinacy Principles

- Over $\text{RCA}_0^3$, $\Delta^R_1$-Det $\not\rightarrow \Sigma^R_1$-Det
- Ground model $\mathcal{V} \models \text{ZFC} + \text{CH}$
Separating Determinacy Principles

- Over $\text{RCA}_0^3$, $\Delta^R_1$-Det $\not\rightarrow$ $\Sigma^R_1$-Det
- Ground model $V \models \text{ZFC+CH}$
- Force with (countably closed) $\mathbb{P}$ to add generic open game
Over $\text{RCA}_0^3$, $\Delta^R_1\text{-Det} \nRightarrow \Sigma^R_1\text{-Det}$

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- Get structure $(\omega, \mathbb{R}, \omega^\mathbb{R} \cap \mathcal{V}[G])$
Separating Determinacy Principles

- Over $\text{RCA}_0^3$, $\Delta^R_1$-Det $\not\rightarrow \Sigma^R_1$-Det
- Ground model $V \models \text{ZFC} + \text{CH}$
- Force with (countably closed) $\mathbb{P}$ to add generic open game
- Get structure $(\omega, R, \omega^R \cap V[G])$
- Take substructure $M = (\omega, R, \{ f \in \omega^R : f \text{ has “stable” name}\})$
The Game \( \mathcal{O} \)

\[ \omega^* \mathcal{2} = \omega \mathcal{2} \cup \{ \infty \} \]

Ordered by \( \infty > x \) for \( x \in \omega^* \mathcal{2} \)

Play elements of \( \omega^* \mathcal{2} \):

- Player 1 (Open): \( \alpha_0, \alpha_1, \ldots \)
- Player 2 (Closed): \( \beta_0, \beta_1, \ldots \)

Legal sequences:

- \( \alpha_i > \alpha_{i+1} \Rightarrow \beta_i > \beta_{i+1} \)

Player 2 wins unless illegal, or

- \( \exists i (\beta_i = 0) \)

Win for 2 (keep playing \( \infty \)), but complicated game tree
The Game $\mathcal{O}$

- $\omega^*_2 = \omega_2 \cup \{\infty\}$, ordered by $\infty > x$ for $x \in \omega^*_2$
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- $\omega^*_2 = \omega_2 \cup \{\infty\}$, ordered by $\infty > x$ for $x \in \omega^*_2$
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The Forcing

Want to create a game on $\mathbb{R}$ which classically is $\mathbb{O}$, but unlabelled

Force with $P =$ countable partial maps $p : \mathbb{R} < \omega \to (\omega^* \times 2)^2$ such that

1. $\text{dom}(p)$ a tree
2. $p_1(\langle \rangle) = p_2(\langle \rangle) = \infty$
3. $|\sigma| = 2^k \Rightarrow p_2(\sigma \downarrow a) = p_2(\sigma)$
4. $|\sigma| = 2^k + 1 \Rightarrow p_1(\sigma \downarrow a) = p_1(\sigma)$
5. $p_1(\sigma) > p_1(\sigma \downarrow a) \Rightarrow p_2(\sigma) > p_2(\sigma \downarrow a)$
6. $p_2(\sigma) = 0 \Rightarrow \sigma \downarrow a \notin \text{dom}(p)$

$P$ countably closed

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- Want to create a game on $\mathbb{R}$ which classically is $\mathcal{O}$, but unlabelled
- Force with $\mathbb{P}=\text{countable partial maps}$

$$p : \mathbb{R}^{<\omega} \to (\omega_2^*)^2$$

such that
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- Want to create a game on $\mathbb{R}$ which classically is $O$, but unlabelled
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- $p_1(\sigma) > p_1(\sigma \upharpoonright a) \implies p_2(\sigma) > p_2(\sigma \upharpoonright a \upharpoonright b)$
- $p_2(\sigma) = 0 \implies \sigma \upharpoonright a \notin dom(p)$
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- $\mathbb{P}$ countably closed
Building $M$

Name $\nu$ is $\alpha$-stable if $\begin{align*}
(p \approx \alpha q, \nu(p)(a) = n) = \Rightarrow (\nu(q)(a) = n)
\end{align*}$

Name $\nu$ is stable if $\alpha$-stable for some $\alpha$

Separating model is $M = (\omega, R, \{\nu[G] : \nu \text{ stable}\})$
Building $M$

“Name” = appropriate map: $\mathbb{P} \rightarrow \{\text{partial maps } \mathbb{R} \rightarrow \omega\}$
Building $M$

- “Name” = appropriate map: $\mathbb{P} \rightarrow \{\text{partial maps } \mathbb{R} \rightarrow \omega\}$
- For $\alpha < \omega_2$, $p, q \in \mathbb{P}$: Set $p \approx_\alpha q$ if

\[\text{dom}(p) = \text{dom}(q) \land p(\sigma) \neq q(\sigma) \Rightarrow p(\sigma) \geq \alpha \]

- Name $\nu$ is $\alpha$-stable if $[p \approx_\alpha q, \nu(p)(a) = n] \Rightarrow [\nu(q)(a) = n]$.

- Name $\nu$ is stable if $\alpha$-stable for some $\alpha$. 

Separating model is $M = (\omega, R, \{\nu[G] : \nu \text{ is stable}\})$. 

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- If $\nu$ is $\alpha$-stable name for Closed-strategy, $\nu$ can be “tricked”: 

Let $p \Vdash \nu(\langle \alpha \rangle) = \gamma$

$\gamma \geq \alpha$ if $\nu$ winning

Get $p' \approx \alpha p$ with $p' \Vdash \nu(\langle \gamma + 1 \rangle) = \gamma$

So $M \models \text{“}T$ not win for Closed”
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$M \models \Delta^R_1$-Det, I/II: Short games
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- $\mathbb{P}$ has retagging property:

$$p \approx_{\alpha+\omega_1} q \text{ and } r \leq p \quad \implies \quad \exists s (r \approx_\alpha s \text{ and } s \leq q)$$
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  - Ex: $\nu$ is $(\alpha + \omega_1 \cdot 2)$-stable $\implies$ name for characteristic function of $\{x : \exists y (\nu(x \oplus y) = 1)\}$ is $\alpha$ stable
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- Winning clopen games of rank $< \omega_2$: iterated retagging
$\mathcal{M} \models \Delta^R_1$-Det, II/II: No long games
Claim: All games in $M$ have rank $< \omega_2$
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$\nu$ an $\alpha$-stable name for well-ordering; show $\nu[G] < \omega_2$
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Tree $T$ of pairs $\langle p, a \rangle$ with $\text{ran}(p) \subseteq (\alpha \cup \{\infty\})$ and $p \models \"a\"$ is a descending sequence in $\nu$.

$\langle p, a \rangle \leq \langle q, b \rangle$ if $p \leq q$ and $b \prec a$.

$T$ is wellfounded.

$|T| = \aleph_1$

$T$ embeds tree of descending sequences in $\nu[G]$. 

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Claim: All games in $M$ have rank $< \omega_2$

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Claim: All games in $M$ have rank $< \omega_2$

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Tree $T$ of pairs $\langle p, a \rangle$ with

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Tree $T$ of pairs $\langle p, \bar{a} \rangle$ with

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Claim: All games in $M$ have rank $< \omega_2$

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- Countable closure: functional defined by $\varphi$ is $\alpha$-stable
Further Questions

▶ Do canonical models of $\Delta^R_1$-Det satisfy $\Sigma^R_1$-Det?

▶ $(\omega, R \cap L_\alpha, \omega) R \cap L_\alpha$

▶ $\Sigma^R_1$-Det $\Rightarrow \Sigma^2_1$-Sep $R$ $\Rightarrow$ $\Delta^R_1$-Det $\Rightarrow$ WO($R$)?

▶ Restrict games based on topological complexity of game tree coded as set of reals

▶ Is RCA$^3_0$/RCA$^{\omega_0}$ the “right” base theory?

▶ Ex: existence of jump operator $J$ does not imply existence of $0_\omega$ (Avigad/Feferman '98; Hunter '08, conservativity over ACA$^0_0$)

▶ Pluralism: may be right “family” of base theories

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$\Sigma^R_1$-Det $\implies$ $\Sigma^2_1$-Sep$^R$? $\Delta^R_1$-Det $\implies$ WO($\mathbb{R}$)?
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