Transfinite Recursion in Higher Reverse Mathematics

Noah Schweber

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  - Is there a higher-type analogue of ATR_0?

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  - ω-models determined by type-$\mathbf{1}$ and $\mathbf{2}$ parts

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- Choice principles also analyzed

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- Choice principles:
  - $\mathsf{SF}(\mathbb{R})$: selection functions for collections of sets of reals (Quasi-strategies $\rightarrow$ strategies)
  - $\mathsf{WO}(\mathbb{R})$: well-orderability of reals (Kleene-Brouwer: trees $\rightarrow$ ordinals)

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$$
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 & & \Sigma^2_1\text{-Sep}^{\mathbb{R}} + SF(\mathbb{R}) & & \\
 & & \downarrow & & \\
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TR_1(\mathbb{R}) + WO(\mathbb{R}) + SF(\mathbb{R}) & \longrightarrow & \Delta^{\mathbb{R}}_1\text{-Det} & \longleftrightarrow & BR_1(\mathbb{R}) + SF(\mathbb{R}) \\
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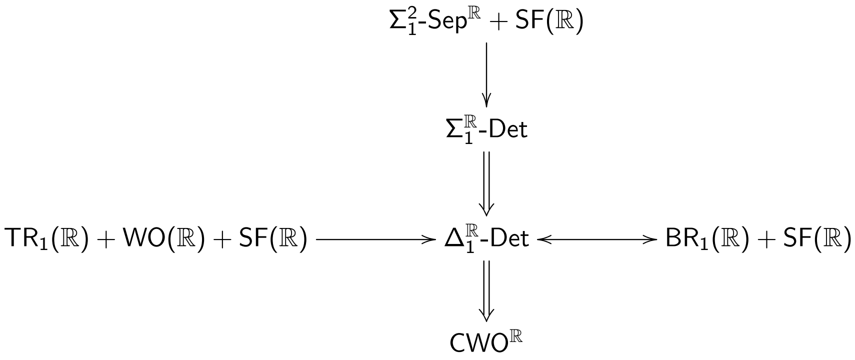

- $\mathsf{WO}(\mathbb{R}) \not\leftrightarrow \mathsf{SF}(\mathbb{R})$

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- Take substructure
  $M = (\omega, \mathbb{R}, \{f \in \omega^{\mathbb{R}} : \ f$ has “stable” name$\})$

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- Win for 2 (keep playing ∞), but complicated game tree

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