Separating the uniformly computably true from the computably true via strong Weihrauch reducibility

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Let's start with a favorite example

RT^n_k is the statement *for every* $f\colon [\mathbb{N}]^n \to k$ *there is an infinite* $H \subseteq \mathbb{N}$ *such that* f *is constant on* $[H]^n$.

(The H in the statement of RT^n_k is called homogeneous for f.)

$\mathsf{RT}^3_2 \to \mathsf{RT}^2_2$ by an easy proof:

- Let $f\colon [\mathbb{N}]^2 \to 2$.
- Define $g\colon [\mathbb{N}]^3 \to 2$ by $g(x, y, z) = f(x, y)$ for all $x < y < z$.
- Apply RT^3_2 to g to obtain a set H homogenous for g.
- Check that H is also homogeneous for f.

The easy proof is effective

Every set appearing in the easy proof is either given, computable from existing sets, or arises from an application of RT^3_2:

- Let $f\colon [\mathbb{N}]^2 \to 2$. f is given
- Define $g\colon [\mathbb{N}]^3 \to 2$ by $g(x,y,z) = f(x,y)$ for all $x < y < z$. $g \leq_{\mathsf{T}} f$
- Apply RT^3_2 to g to obtain a set H homogenous for g. RT^3_2
- Check that H is also homogeneous for f.

The proof is formalizable in the system RCA_0. So $\mathsf{RCA}_0 \vdash \mathsf{RT}^3_2 \to \mathsf{RT}^2_2$. We might say that the implication $\mathsf{RT}^3_2 \to \mathsf{RT}^2_2$ is computably true.

(RCA_0 essentially says that if sets $X_0, \ldots, X_{n-1}$ exist, then so do all the sets computable from $\bigoplus_{i<n} X_i$.

Formally, the axioms of RCA_0 are those of a discretely ordered commutative semi-ring with 1, the comprehension scheme for Δ^0_1 predicates, and the induction scheme for Σ^0_1 formulas.)

The easy proof is even more effective

We translated RT^2_2 instances f into RT^3_2 instances g via $g(x,y,z) = f(x,y)$, and we noticed that $g \leq_{\mathsf{T}} f$.

Now notice that the reduction witnessing $g \leq_{\mathsf{T}} f$ does not depend on f.

That is, there is a single Turing functional Φ such that $\Phi(f)(x,y,z) = f(x,y)$ is an RT^3_2 instance whenever f is an RT^2_2 instance.

There is also a single Turing functional Ψ such that $\Psi(H)$ is homogeneous for f whenever H is homogeneous for Φ^f: $\Psi(H) = H$.

So we can uniformly computably translate RT^2_2 instances f into RT^3_2 instances $\Phi(f)$, and then uniformly computably translate solutions H of $\Phi(F)$ back to solutions $\Psi(H)$ of the original instance f.

Thus we might say that the implication $\mathsf{RT}^3_2 \to \mathsf{RT}^2_2$ is uniformly computably true.

Strong Weihrauch reducibility

Consider a Π^1_2 statement $\forall X \exists Y \varphi(X, Y)$ in second-order arithmetic, such as RT^2_2, weak König's lemma (WKL), the extreme value theorem on $[0, 1]$, etc.

The statements we are interested in typically have a natural class of instances (colorings, trees, continuous functions), and a natural class of solutions (homogenous sets, paths, real numbers).

Here is today's key definition:

Definition (strong Weihrauch reducibility)

Let P and Q be Π^1_2 statements. Then P is strongly Weihrauch reducible to Q (P $\leq_{\mathsf{sW}}$ Q) if there are Turing functionals Φ and Ψ such that

- when I is an instance of P, $\Phi(I)$ is an instance of Q, and
- when S is a solution to $\Phi(I)$, $\Psi(S)$ is a solution to I.

Strong Weihrauch reducibility

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We can write $\mathsf{RT}^2_2 \leq_{\mathsf{sW}} \mathsf{RT}^3_2$.

Well-known results of Jockusch tell us that $\mathsf{RT}^3_2 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$:

- There is a computable instance of RT^3_2 with no Δ^0_3 solution.
- Every computable instance of RT^2_2 has a Δ^0_3 solution.

P $\leq_{sW}$ Q versus $\mathsf{RCA}_0 \vdash$ Q $\to$ P

Many proofs of Q $\to$ P in RCA_0 describe strong Weihrauch reductions:

- (Friedman, Simpson, Smith) Let P be the statement *every commutative ring with* 1 *has a prime ideal*. Then P $\leq_{sW}$ WKL.
- (Cholak, Jockusch, Slaman) COH $\leq_{sW}$ RT^2_2.

Guideline:

- P $\leq_{sW}$ Q is stronger than $\mathsf{RCA}_0 \vdash$ Q $\to$ P.
- $\mathsf{RCA}_0 \nvdash$ Q $\to$ P is stronger than P $\nleq_{sW}$ Q.

(This is not strictly fact because $\leq_{sW}$ is over ω, while RCA_0 considers non-standard models.)

Examples:

- $\mathsf{RCA}_0 \vdash \mathsf{RT}^3_2 \leftrightarrow \mathsf{RT}^4_2$, but $\mathsf{RT}^4_2 \nleq_{sW} \mathsf{RT}^3_2$.
- $\mathsf{RT}^3_2 \nleq_{sW} \mathsf{RT}^2_2$ followed from Jockusch. $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2 \to \mathsf{RT}^3_2$ (Seetapun) was a major breakthrough.

(Aside: the interesting situation with DNR functions)

Let $\mathsf{DNR}(k)$ be the statement *for every set X there is a function f that is $\mathsf{DNR}(k)$ relative to X.*

$\mathsf{RCA}_0 \vdash \mathsf{DNR}(k) \leftrightarrow \mathsf{WKL}$ for every fixed, standard $k \geq 2$ (by classic results of Friedberg and Jockusch and Soare).

$\mathsf{WKL} \equiv_{\mathsf{sW}} \mathsf{DNR}(2)$.

$\mathsf{WKL} \not\leq_{\mathsf{sW}} \mathsf{DNR}(k)$ for $k > 2$ (by a classic result of Jockusch).

In fact, $\mathsf{DNR}(\ell) \not\leq_{\mathsf{sW}} \mathsf{DNR}(k)$ when $2 \leq \ell < k$.

The statement $(\forall k \geq 2)(\mathsf{DNR}(k) \rightarrow \mathsf{WKL})$ is not provable in RCA_0 (or in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$), but it is provable in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ (recent work of Dorais, Hirst, S).

$\mathsf{P} \leq_{\mathsf{sW}} \mathsf{Q}$ versus $\mathsf{RCA}_0 \vdash \mathsf{Q} \rightarrow \mathsf{P}$?

versus

preposition

(1) against (esp. in sports and legal use): *Penn versus Princeton*.

(2) as opposed to; in contrast to: *weighing the pros and cons of organic versus inorganic produce*.

We mean definition 2!

$\leq_{\mathsf{sW}}$ can detect differences between statements that are equivalent in RCA_0, so one might consider $\leq_{\mathsf{sW}}$ and provability in RCA_0 as operating on different scales.

$\leq_{\mathsf{sW}}$ is computability-theoretically motivated, and provability in RCA_0 is proof-theoretically motivated.

On to a more colorful Ramsey's theorem

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_3 \to \mathsf{RT}^2_2$ and $\mathsf{RT}^2_2 \leq_{\mathrm{sW}} \mathsf{RT}^2_3$ by trivial proofs.

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_3$ by an easy proof that has interesting features:

- Let $f\colon [\mathbb{N}]^2 \to 3$ be given.
- Define $g\colon [\mathbb{N}]^2 \to 2$ by $g(x,y) = 0$ if $f(x,y) = 0$ and $g(x,y) = 1$ if $f(x,y) > 0$.
- By RT^2_2, let H_0 be homogeneous for g. If H_0 is homogeneous for color 0, then H_0 is homogeneous for f.
- Otherwise, fix an order-preserving bijection $\iota\colon \mathbb{N} \to H_0$ and define $h\colon [\mathbb{N}]^2 \to 2$ by $h(x,y) = f(\iota(x), \iota(y)) - 1$.
- By RT^2_2, let H be homogeneous for h. Then $\iota(H)$ is homogeneous for f.

Again every set is given, computable from existing sets, or arises from an application of RT^2_2, but proof uses two applications of RT^2_2 and doesn't seem to describe an $\leq_{\mathrm{sW}}$-reduction. Does $\mathsf{RT}^2_3 \leq_{\mathrm{sW}} \mathsf{RT}^2_2$?

$\mathsf{RT}^2_3 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$

Theorem (DDHMS)

$\mathsf{RT}^2_3 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$. *In fact, fix* $n \geq 1$ *and* $2 \leq j < k$. *Then* $\mathsf{RT}^n_k \not\leq_{\mathsf{sW}} \mathsf{RT}^n_j$.

We will discuss $\mathsf{RT}^2_4 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$. The general result just needs some extra coding tricks.

The plan:

- Assume for a contradiction that $\mathsf{RT}^2_4 \leq_{\mathsf{sW}} \mathsf{RT}^2_2$.
- Show that two simultaneous instances of RT^2_2 $\leq_{\mathsf{sW}}$-reduce to RT^2_4 and hence to RT^2_2.
- Show that then infinitely many simultaneous instances of RT^2_2 must $\leq_{\mathsf{sW}}$-reduce to RT^2_2.
- Show that the previous conclusion is false to get the contradiction.

Parallelization and sequentialization

Definition

Let P and Q be Π^1_2 statements.

- $\langle \mathsf{P}, \mathsf{Q} \rangle$ is the Π^1_2 statement whose instances are pairs $\langle I, J \rangle$, where I is an instance of P and J is an instance of Q, and whose solutions are pairs $\langle S, T \rangle$, where S is a solution to I and T is a solution to J.
- SeqP is the Π^1_2 statement whose instances are sequences $\langle I_i : i \in \omega \rangle$ of instances of P and whose solutions are sequences $\langle S_i : i \in \omega \rangle$, where S_i is a solution to I_i for each i.

The first step is to show that $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leq_{\mathsf{sW}} \mathsf{RT}^2_4$.

The contradiction will be that both $\mathsf{SeqRT}^2_2 \leq_{\mathsf{sW}} \mathsf{RT}^2_2$ and $\mathsf{SeqRT}^2_2 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$.

$\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leq_{\mathsf{sW}} \mathsf{RT}^2_4$ is pretty easy

Proposition

$\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leq_{\mathsf{sW}} \mathsf{RT}^2_4$

Let Φ and Ψ be

- $\Phi(\langle f, g \rangle) = 2f + g$
- $\Psi(H) = \langle H, H \rangle$.

If f and g are functions $[\mathbb{N}]^2 \to 2$, then $2f + g$ is a function $[\mathbb{N}]^2 \to 4$.

If H is homogeneous for $2f + g$, then H is homogeneous for both f and g.

$\mathsf{SeqRT}^2_2 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$ isn't so bad either

$\mathsf{SeqRT}^2_2 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$ follows from:

Proposition

There is a computable instance of SeqRT^2_2 such that every solution computes $0''$. (More generally, for every $n \geq 1$ there is a computable instance of SeqRT^n_2 such that every solution computes 0^n.)

The instance is $\langle f_e : e \in \omega \rangle$, where

$$f_e(x, y) = \begin{cases} 0 & \text{if } (\exists n < x)\Phi_{e,y}(n)\uparrow \\ 1 & \text{if } (\forall n < x)\Phi_{e,y}(n)\downarrow. \end{cases}$$

Given a solution $\langle H_e : e \in \omega \rangle$, determine whether or not Φ_e is total by checking whether or not H_e is homogenous for color 1.

$\mathsf{SeqRT}^2_2 \not\leq_{\mathsf{sW}} \mathsf{RT}^2_2$ isn't so bad either

Suppose $\mathsf{SeqRT}^2_2 \leq_{\mathsf{sW}} \mathsf{RT}^2_2$, and let Φ and Ψ witness the reduction.

Let $\langle f_e : e \in \omega \rangle$ be the computable SeqRT^2_2 instance from the proposition.

Then $\Phi(\langle f_e : e \in \omega \rangle)$ is a computable RT^2_2 instance.

By Jockusch, $\Phi(\langle f_e : e \in \omega \rangle)$ has a solution $H \not\geq_{\mathsf{T}} 0''$ (in fact, $H' \leq_{\mathsf{T}} 0''$).

Thus $\Psi(H)$ is a solution to $\langle f_e : e \in \omega \rangle$ that does not compute $0''$, a contradiction.

Where are we?

Reminder:

- The assumption was $\mathsf{RT}^2_4 \leq_{\mathsf{sW}} \mathsf{RT}^2_2$.
- We showed $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leq_{\mathsf{sW}} \mathsf{RT}^2_4$.
- We showed $\mathsf{SeqRT}^2_2 \nleq_{\mathsf{sW}} \mathsf{RT}^2_2$.

To finish the proof, we need the squashing theorem: if $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leq_{\mathsf{sW}} \mathsf{RT}^2_2$, then $\mathsf{SeqRT}^2_2 \leq_{\mathsf{sW}} \mathsf{RT}^2_2$.

The squashing theorem

Theorem (squashing theorem; DDHMS)

Let P *and* Q *be* Π^1_2 *statements, where* P *and* Q *are total and* P *has finite tolerance. Then* $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathsf{sW}} \mathsf{P} \to \mathsf{SeqQ} \leq_{\mathsf{sW}} \mathsf{P}$.

P *is total* means that every set is an instance of P.

P *has finite tolerance* means that if you make a finite change to a P-instance, then you only need to make finite changes to its solutions.

Formally: there is a Turing functional Θ such that when I and J are P-instances with $(\forall x > m)(I(x) = J(x))$ and S is a solution to I, then $\Theta(S, m)$ is a solution to J.

Ramsey theorems are total and have finite tolerance

Proposition

RT^2_2 *(in general,* RT^n_k*) is total and has finite tolerance.*

It's easy to see every set as coding a function $[\mathbb{N}]^2 \to 2$ (and in such a way that all such functions are coded).

Assume our coding of tuples is such that always $x, y \leq \langle x, y \rangle$.

- Let $\Theta(H, m) = \{x \in H : x > m\}$.
- Suppose $f, g : [\mathbb{N}]^2 \to 2$ are such that $(\forall \langle x, y \rangle > m)(f(x, y) = g(x, y))$.
- Let H be homogeneous for f with color c.
- If $x, y \in \Theta(H, m)$, then $x, y \in H$ and $\langle x, y \rangle > m$, so $g(x, y) = f(x, y) = c$.
- Thus $\Theta(H, m)$ is homogeneous for g.

This finishes the proof

Reminder:

Theorem (squashing theorem; DDHMS)

Let P *and* Q *be* Π^1_2 *statements, where* P *and* Q *are total and* P *has finite tolerance. Then* $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathsf{sW}} \mathsf{P} \rightarrow \mathsf{SeqQ} \leq_{\mathsf{sW}} \mathsf{P}$.

The squashing theorem applies to RT^2_2.

Thus if $\langle \mathsf{RT}^2_2, \mathsf{RT}^2_2 \rangle \leq_{\mathsf{sW}} \mathsf{RT}^2_2$, then $\mathsf{SeqRT}^2_2 \leq_{\mathsf{sW}} \mathsf{RT}^2_2$, giving the contradiction.

Some words on the proof of the squashing theorem

Reminder:

- Have Q total; P total, finite tolerance such that $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathsf{sW}} \mathsf{P}$.
- Want SeqQ $\leq_{\mathsf{sW}}$ P.

The basic plan: Uniformly fold Q-instances into a single P-instance.

Given a SeqQ-instance $\langle I_i : I \in \omega \rangle$. Compute a sequence $\langle J_i : i \in \omega \rangle$ of P-instances such that, for all $i \in \omega$:

$$J_i = (C \restriction m_i)^\frown \Phi(I_i, J_{i+1}).$$

The P-instance we really want is J_0.

- C is some fixed, computable P-instance.
- $\langle m_i : i \in \omega \rangle$ is a cleverly chose computable sequence that helps make the folding work.

Special today only: $\sigma^\frown\tau$ means replace the first $|\sigma|$ bits of τ by σ!!!

Some words on the proof of the squashing theorem

The unfolding

For all $i \in \omega$:

$$J_i = (C \upharpoonright m_i)^\frown \Phi(I_i, J_{i+1}).$$

From a solution S_0 to J_0 we can recover a solution $\langle T_i : i \in \omega \rangle$ to $\langle I_i : i \in \omega \rangle$:

- P has finite tolerance, so from S_0 we get a solution to $\Phi(I_0, J_1)$.
- The solution to $\Phi(I_0, J_1)$ produces a solution $\langle T_0, S_1 \rangle$ to $\langle I_0, J_1 \rangle$.
- P has finite tolerance, so from S_1 we get a solution to $\Phi(I_1, J_2)$.
- The solution to $\Phi(I_1, J_2)$ produces a solution $\langle T_1, S_2 \rangle$ to $\langle I_1, J_2 \rangle$.
- Et cetera.

A few more results

For a rational $p \in (0,1)$, let p-WWKL denote WKL for trees of measure $\geq p$.

Theorem (DDHMS)

If $0 < p < q < 1$, then p-WWKL $\not\leq_{\mathsf{sW}}$ q-WWKL

TS^n_k is the statement *for every $f\colon [\mathbb{N}]^n \to k$ there is an infinite $H \subseteq \mathbb{N}$ such that $|f([H]^n)| < k$.*

Theorem (DDHMS)

- *Let $n \geq 1$ and $j, k \geq 2$. Then $\langle \mathsf{TS}^n_k, \mathsf{TS}^n_j \rangle \not\leq_{\mathsf{sW}} \mathsf{TS}^n_j$.*
- *If $2 \leq j < k$, then $\mathsf{TS}^1_j \not\leq_{\mathsf{sW}} \mathsf{TS}^1_k$. Improved by Hirschfeldt and Jockusch to all exponents.*